Homework 2—Differential Geometry

Due date: 20.4. 2021

1. Show that the cross-product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ defines a Lie bracket on $\mathbb{R}^3$ and that the Lie algebra $(\mathbb{R}^3, \times)$ is isomorphic to the Lie algebra $(\mathfrak{so}(3,\mathbb{R}), [\cdot,\cdot])$.

2. Consider the Lie group $\mathrm{SL}(2,\mathbb{K})$ and its Lie algebra $\mathfrak{sl}(3,\mathbb{K})$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

   - Show that $\langle\cdot,\cdot\rangle : \mathfrak{sl}(2,\mathbb{K}) \times \mathfrak{sl}(2,\mathbb{K}) \to \mathbb{K}$ defined by
   $$\langle X, Y\rangle = \frac{1}{2}\mathrm{trace}(XY)$$
   defines a symmetric non-degenerate $\mathbb{K}$-bilinear form on the 3-dimensional $\mathbb{K}$-vector space $\mathfrak{sl}(2,\mathbb{K})$. Moreover, show that over $\mathbb{R}$ it has signature $(2,1)$.
   - Show that the adjoint representation $\mathrm{Ad} : \mathrm{SL}(2,\mathbb{K}) \to \mathrm{GL}(\mathfrak{sl}(2,\mathbb{K})) \cong \mathrm{GL}(3,\mathbb{K})$ induces covering maps
   $$\mathrm{Ad} : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}((\mathfrak{sl}(2,\mathbb{C}), \langle\cdot,\cdot\rangle) \cong \mathrm{SO}(3,\mathbb{C})$$
   $$\mathrm{Ad} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}((\mathfrak{sl}(2,\mathbb{R}), \langle\cdot,\cdot\rangle) \cong \mathrm{SO}_o(2,1).$$
   What are the kernels of these group homomorphisms?

3. Consider the upper-half plane $\mathcal{H} = \{(x,y) \in \mathbb{R}^2 : y > 0\} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

   - Show that
   $$\mathrm{SL}(2,\mathbb{R}) \times \mathcal{H} \to \mathcal{H}$$
   $$(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) \mapsto \frac{az+b}{cz+d}$$
   defines a smooth transitive left action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{H}$. What is the isotropy group of $i \in \mathcal{H} \subset \mathbb{C}$?
   - Consider the following Riemannian metric on $\mathcal{H}$:
   $$g = \frac{dx^2 + dy^2}{y^2} = \frac{4|dz|^2}{|z-\bar{z}|^2} \qquad (z = x + iy, dz = dx + idy).$$
   Show that $\mathrm{SL}(2,\mathbb{R})$ acts by isometries on $(\mathcal{H}, g)$, that is, $z \mapsto \frac{az+b}{cz+d}$ is an isometry for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$.

4. Suppose G is a connected Lie group. Let $\phi : G \to \mathrm{GL}(V)$ be a representation on a finite-dimensional vector space V and let $\phi' : \mathfrak{g} \to \mathfrak{gl}(V)$ be the induced representation of the Lie algebra $\mathfrak{g}$ of G.

- Show that a subspace $W \subset V$ is G-invariant $\iff$ it is $\mathfrak{g}$-invariant.
- Show that V is unitary as G-representation $\iff$ it is unitary as $\mathfrak{g}$-representation.

5. Suppose G is a compact Lie group of dimension n. Choose a nonzero element $\omega \in \Lambda^n \mathfrak{g}^*$ (i.e. a volume form on the vector space $\mathfrak{g}$). Then via left-multiplication this gives rise to a volume form on G:

$$\mathrm{vol}(g)(\xi_1, ...\xi_n) = \omega(T_g\lambda_{g^{-1}}\xi_1, ..., T_g\lambda_{g^{-1}}\xi_n), \quad \text{for } \xi_1, ..., \xi_n \in T_gG.$$

Hence, we can integrate smooth functions $f : G \to \mathbb{K} = \mathbb{R}, \mathbb{C}$ by setting

$$\int_G f := \int_G f\mathrm{vol}.$$

- Show that vol is left-invariant (i.e. $\lambda_g^*\mathrm{vol} = \mathrm{vol}$ for all $g \in G$) and deduce that $\int_G f = \int_G f \circ \lambda_g$ for all $g \in G$.
- Let V be a real or complex representation of G and let $b(\cdot, \cdot) : V \times V \to \mathbb{K}$ be an arbitrary positive definite (Hermitian in the complex case) inner product on V. For two vectors $v, w \in V$ set
$$\langle v, w\rangle := \int_G f_{v,w},$$
where $f_{v.w} : G \to \mathbb{K}$ is the smooth function defined by $f_{v.w}(g) = b(g^{-1}v, g^{-1}w)$. Show that $\langle \cdot, \cdot\rangle$ defines a G-invariant positive definite inner product on V (Hermitian in the complex case), i.e. V is a unitary representation.