- These notes will combine what is in the lectures, possibly slightly re-organised, and with a few extra words.
- Anything new will be in green!

Algebra 4 - 2021

1) Homological algebra
2) Commutative algebra (rel. to alg. geo)
3) Representation theory of finite groups.

Course details

- Weekly lecture uploaded to IS
- Exercise class on Tuesday
- 3 marked assignments (50%) + oral exam

Homological algebra - overview of topics covered

- Ex. sequences, ch. complexes, long ex. seq.
  - Homology of ch. comp.
  - Simpl. & sing. homology
  - Cat of ch. comp & homology a functor
  - Chain homotopies - induce equal maps on homology
  - Cochain complexes & cohomology (singular cohomology)
- Pre-additive & additive cats
  - biproducts
  - Kernels, cokernels & abelian categories
  - Additive & exact functors
  - Freyd-Mitchell embedding → to prove something in a gen abelian cat, suff. to prove it in a Mod R
- Snake lemma & long ex. seq. of homology
  - Homology groups of spheres (in ex. class)
- Homology & cohomology of alg. structures
  - proj & inj. obs
  - proj., inj resolution
  - left & right derived functors L_nF, R_nF
  - functoriality & their props: L_0F, L_1F & les.
  - Def of Ext & Tor via der. functors
  - Ext via extensions
  - Mentioned group cohomology (very briefly)
- Proj & inj. dimension
  - proj, inj dim. of modules
  - Rel. to Ext
  - Calc for abelian groups
  - P.d. of a ring, global dim.
  - Hilb. S y theorem (no proof)

Homological algebra

Exact sequences

- R a ring
- Mod_R cat of left R-modules.

- Consider
$$A \xrightarrow{f} B \xrightarrow{g} C \quad \in Mod_R$$
in which $g \circ f = 0$, the zero homomorphism.

- Then $g(f(a)) = 0$ all $a \in A$ so that $im(f) \leq Ker(g)$.

Def) - The sequence is said to be exact at B if $im(f) = Ker(g)$.

Example

- $0 := \{0\} \longrightarrow A \xrightarrow{f} B$ is exact $\Leftrightarrow$ $kerf = 0$ $\Leftrightarrow$ f is injective (mono)
- $A \xrightarrow{f} B \longrightarrow 0$ is exact $\Leftrightarrow$ $imf = B$ $\Leftrightarrow$ f is surjective (epi - we will see later)

- A short exact sequence is a sequence
$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$
which is exact in each position:

- ex@A : f is injective

- ex@C : g is surjective
- ex@B : $\operatorname{im} f = \ker g$, but since g is surjective $C \cong B/\ker g = B/\operatorname{im} f$.

Exercise: Using this, show that each ses is of the form

$$0 \hookrightarrow A \hookrightarrow B \longrightarrow B/A \longrightarrow 0$$

for A a submodule of B, (up to isomorphism of sequences)

$\boxed{\text{Def}^n}$. A chain complex $\mathbf{A}$ is a sequence

$$\cdots \; A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$$

of R-modules (for $n \in \mathbb{Z}$) where $d_n \circ d_{n+1} = 0 \quad \forall n$.

- The elts of $Z_n := \ker d_n \subseteq A_n$ are called n-cycles
- $B_n := \operatorname{im} d_{n+1} \leq \ker d_n$ are called n-boundaries

Def^n) The n-th homology of A is the

R-module $H_n(A) = \frac{\ker d_n}{\operatorname{im} d_{n+1}} = \frac{Z_n}{B_n}$.

Remark). Observe that

A is exact @ $A_n \iff H_n(A) = 0$.

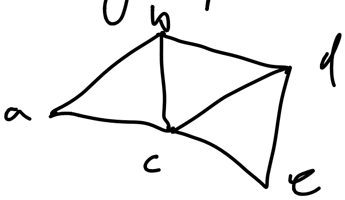
Thus the n-th homology measures the failure of A to be exact at A_n.

- If A is exact at all n, it is called a long exact sequence.

Examples

Simpl. homology

① K a geom. simp. complex - eg. triang. space



- K_n set of n-simplices. Each has n+1 faces.
- If set of vertices is ordered, so are these maps (omit i'th vertex)

so $K_n \xrightarrow{d_i} K_{n-1}$ for $i = 0, \dots, n$.

- Forming free R-modules, gives

$$C_n \xrightarrow{d_i} C_{n-1}$$ applying to basis elts

& then .. $C_n \xrightarrow{\sum_{i=0}^{n}(-1)^i d_i} C_{n-1}$ -- is a chain complex C_K,

with $H_n(C_K)$ = simplicial homology of K.

② Singular homology:

X a top. space,

$S_n(X)$ = Free R-module on set of cts maps $\Delta_n \longrightarrow X$ where Δ_n is the stand. n-simplex.

- As before get Face maps, & ch. comp $S_k(X) \xrightarrow{d = \Sigma(-1)^i d_i} S_{k-1}(X)$ whose homology is singular homology of X. (same as simp h. - -)

- A chain map $f : A \longrightarrow B$ of chain complexes consists of maps $f_n : A_n \longrightarrow B_n$ such that

$$\begin{tikzcd} A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] \arrow[dr, phantom, "="] & A_n \arrow[d, "f_n"] \\ B_{n+1} \arrow[r, "d_{n+1}"'] & B_n \end{tikzcd} \quad \forall n .$$

Notation : One often just writes $d : B_{n+1} \to B_n$ when the context is clear.

- Chain complexes and chain maps form a category $Ch(Mod_R)$.

Proposition

The n-th homology determines a functor

$$H_n : Ch(Mod_R) \longrightarrow Mod_R .$$

Proof) It sends $A \longmapsto H_n(A)$.

At $f : A \longrightarrow B$, then given $x \in Z_n(A) = \ker(d : A_n \to A_{n-1})$ we have $dfx = fdx = 0$ so $fx \in Z_n(B)$; similarly if $x \in B_n(A)$ then $f(x) \in B_n(B)$; hence we obtain

$$\begin{tikzcd} H_n(A) = \dfrac{Z_n A}{B_n A} \arrow[r] & \dfrac{Z_n B}{B_n B} = H_n(B) \\ {[x]} \arrow[r, mapsto] & {[fx]}. \end{tikzcd}$$

This is clearly functorial. □

Homotopy & homology

Def^n) Let $f, g : A \rightrightarrows B$ be chain maps. A chain htpy s from f to g (written $s : f \sim g$) is a sequence of homomorphisms $s_n : A_n \longrightarrow B_{n+1}$ as in the picture below :

$$\begin{tikzcd}
\cdots A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}-g_{n+1}"'] & A_n \arrow[r, "d_n"] \arrow[dl, "s_n"'] \arrow[d, "f_n-g_n"] & A_{n-1} \cdots \arrow[dl, "s_{n-1}"'] \arrow[d, "f_{n-1}-g_{n-1}"] \\
\cdots B_{n+1} \arrow[r, "d_{n+1}"'] & B_n \arrow[r, "d_n"'] & B_{n-1} \cdots
\end{tikzcd}$$

such that

$$d_{n+1} s_n + s_{n-1} d_n = g_n - f_n \quad \forall n \in \mathbb{Z} .$$

- A chain map $f : A \to B$ is null homotopic if $f \sim 0$.

- It is a htpy equivalence if $\exists g : B \to A$ such that $fg \sim 1_B$ & $gf \sim 1_A$.

Lemma

Homotopy is an e-rel, compatible with composition.

$\boxed{\text{Proof}}$ - Taking $s_n = 0$ shows $f \sim f$.

- If $s : f \sim g$ & $t : g \sim h$ take $s+t : f \sim h$; indeed
$$h - f = (h-g)+(g-f) = (ds+sd)+(dt+td)$$
$$= d(s+t)+(s+t)d.$$

- If $s : f \sim g$ then $-s : g \sim f$ where
$$(-s)_n = -s_n.$$

- If
$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "s \wr"] & B \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "s' \wr"] & C \end{tikzcd}$$

must show $f'f \sim g'g$.

- Suff. to show $f'f \sim f'g \sim g'g$ by above.

- Consider $f's$ with $(f's)_n = f'_{n+1} s_n$.

- Then $d(f's) + (f's)d = f'ds + f'sd$

$$= f'(ds + sd)$$
$$= f'(g - f)$$
$$= f'g - f'f$$

Sim. $d(s'g) + (s'g)d = ds'g + s'dg$

$$= (ds' + s'd)g$$
$$= (f' - g')g = f'g - g'g \, .$$

□

Lemma

If $f \sim g$, they induce the same map $H_n f = H_n g$ on homology.

Proof - Consider $x \in Z_n(A) = \ker(d : A_n \to A_{n-1})$

- We must show $f(x), g(x) \in Z_n(B)$ coincide modulo $B_n(B) := \mathrm{Im}(d_{n+1} : B_{n+1} \to B_n)$.
- But $g(x) - f(x) = sdx + dsx = dsx$ $\in \mathrm{Im}(d_{n+1}) = B_n B$ as $sdx = s0 = 0$.
- Hence $H_n(f) = H_n(g)$.

Corollary

If $f : A \longrightarrow B$ is htpy equiv,
then $H_n f : H_n A \longrightarrow H_n B$ is invertible.

Proof

If $gf \sim 1_A$ & $fh \sim 1_B$ then

$$
\begin{aligned}
H_n(g)H_n(f) &= \text{ by funct.} \\
H_n(gf) &= \text{ by previous prop} \\
H_n(1_A) &= \text{ by funct.} \\
1_{H_n(A)}&.
\end{aligned}
$$

Sim $H_n(f)H_n(g) = 1_{H_n(B)}$.

Cohomology

- So far we have talked about chain complexes & homology

- Dually we have cochain complexes & cohomology
- A cochain complex in $\mathcal{C}$ is a diagram $\cdots X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \cdots$ st. $d^n \circ d^{n-1} = 0$ all $n \in \mathbb{Z}$.

Remark: Equally, a chain complex in the opposite abelian cat, in the sense of what follows shortly.

Everything about chain complexes in an abelian cat thus has dual version for cochain complexes

- The cohomology of a cochain complex X is defined as

$$H^n X := \ker d^n / \operatorname{im} d^{n-1}$$

Examples

① If X is a ch. complex of ab. groups & A an abelian group

then $Ab(X^n, A) \xrightarrow{Ab(d^{n+1}, A)} Ab(X^{n+1}, A) \dots$ is a cochain complex in Ab.

② Recall if X a top space, can form a chain complex in Ab $S_n X$ whose homology is the singular homology of X.

If A is an abelian group, the cohomology of $Ab(S_n X, A)$ is called

singular cohomology of X

of X with coefficients in A.

Lecture 2 - Abelian categories

- What is correct categorical context for homological algebra?
  - Need to talk about zero maps, add & subtract morphisms & form kernels, images & quotients & these should behave like in Mod_R
- Resulting notion: abelian category

Def^n) A pre-additive category (Ab-enriched category) $\mathcal{C}$ is a cat in which hom-sets $\mathcal{C}(a,b)$ has structure of an abelian gp.

$$\left(\text{ie. } a \overset{f}{\underset{g}{\rightrightarrows}} b \longmapsto a \xrightarrow{f+g} b,\ a \xrightarrow{-f} b,\ a \xrightarrow{0} b\right)$$

& moreover pre & postcomposition preserves ab. group str:

$$\left(x \xrightarrow{r} a \overset{f}{\underset{g}{\rightrightarrows}} b \xrightarrow{s} y \quad \text{we have } (f+g)r = fr+gr,\ 0_{a,b}r = 0_{x,b}\right)$$

$$\& \quad s(f+g) = sf+sg$$
$$s0 = 0.$$

Example

- Mod_R is pre-additive.

Given $M \overset{f}{\underset{g}{\rightrightarrows}} N$, $f+g : M \to N$ is defined by $x \longmapsto fx+gx$

which is ab. gp. hom. by commutativity of $+$.

$$(f+g)(rx) = f(rx)+g(rx) = rf(x)+rg(x)$$
$$= r(f(x)+g(x)) =$$
$$r.\ f+g\,(x).$$

$-f : M \to N : x \mapsto -(fx).$

$0 : M \to N : x \longmapsto 0.$

- $s(f+g) = sf+sg$ holds as s a homom. & $(f+g)r = fr+gr$ is trivial.

Remark

$\mathcal{C}$ is pre-additive $\iff$ $\mathcal{C}^{op}$ is .

Prop^n

Let $\mathcal{C}$ be pre-additive.

① Then $\mathcal{C}$ has term. ob. $\iff$ $\mathcal{C}$ has init. ob.

② $\mathcal{C}$ has binary prods. $\iff$ $\mathcal{C}$ has binary coproducts.

~~Proof~~

① - Let t be terminal.

- Then $t \xrightarrow{0} t = 1_t$.
- Now given x, we have $0 : t \to x$ & must show it is unique, so consider $f : t \to x$. Then $f = f \circ 1_t = f \circ 0 = 0$, so t initial.
- Converse is dual.

② Let

$$\begin{tikzcd} & a \\ c \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & b \end{tikzcd}$$

be a product diagram.

We have

$$\begin{tikzcd} & a \\ a \arrow[ur, "id"] \arrow[dr, "0"'] & \\ & b \end{tikzcd}$$

induces ! $a \xrightarrow{i_1 = \langle id, 0\rangle} c$.

Similarly

$$\begin{tikzcd} & a \\ b \arrow[ur, "0"] \arrow[dr, "id"'] & \\ & b \end{tikzcd}$$

ind. ! $b \xrightarrow{i_2 = \langle 0, id \rangle} c$

& we have

$$\begin{tikzcd} a \arrow[dr, "i_1"] & \\ & c \\ b \arrow[ur, "i_2"'] & \end{tikzcd}$$

, which we will show is coprod. diagram.

Key point: we have diagram

$$\begin{tikzcd} a \arrow[r, shift left, "i_1"] & c \arrow[l, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & b \arrow[l, shift right, "i_2"'] \end{tikzcd}$$

satisfying

- $p_1 i_1 = 1_A$, $p_2 i_1 = 0$
- $p_1 i_2 = 0$, $p_2 i_2 = 1_B$

- $i_1 p_1 + i_2 p_2 = 1_C$.

Such a diag. is called a biproduct diagram.

To see last equation,

$$p_1(i_1p_1+i_2p_2) = p_1i_1p_1 + p_1i_2p_2$$
$$= 1p_1 + 0p_2 = p_1(1)$$

& sim for p_2, as required.

- Any biprod. diag. is both product & coproduct - let's show coproduct.
- Indeed given

$$\begin{tikzcd} a \arrow[dr, "f"] & \\ & d \\ b \arrow[ur, "g"'] & \end{tikzcd}$$

must find $!k$ st

$$\begin{tikzcd} a \arrow[dr, "i_1"'] \arrow[drr, bend left, "f"] & & \\ & c \arrow[r, "k"] \arrow[ur, phantom, "="] \arrow[dr, phantom, "="] & d \\ b \arrow[ur, "i_2"] \arrow[urr, bend right, "g"'] & & \end{tikzcd}$$

But if we have such a k then

$$k = k.1_c = k(i_1p_1+i_2p_2) = ki_1p_1 + ki_2p_2$$
$$= fp_1 + gp_2,$$

so must have $k = fp_1 + gp_2$.

Let's check that fp_1+gp_2 has required props:

$$(fp_1+gp_2)i_1 = fp_1i_1 + gp_2i_1 = f + 0 = f$$

- - - - - i_2 $= g$

as required.

Converse is dual □

Defⁿ) A preadd. cat is additive if it has finite products (equivalently, by above, finite coproducts).

Notation: In additive cat, write 0 for the terminal = initial object.

Example Mod$_R$ is additive.

Prop 3 generalises fact from Alg. 3 that in Mod$_R$ binary prods & coproducts coincide.

Kernels & quotients

Consider $a \xrightarrow{f} b \in \mathcal{C}$ preadditive.

- The kernel of f $\ker f$ comes with

$$\ker f \xrightarrow{i} a \xrightarrow{f} b \quad \& \quad f \circ i = 0 \quad \&$$

is universal with this property:

(ie. given $g: c \to a$ & $fg = 0$ $\exists! \bar{g}: c \to \ker f$ st $i \circ \bar{g} = g$.)

- The cokernel of f is dual: we have

$$a \xrightarrow{f} b \xrightarrow{p} \operatorname{coker} f \quad \text{s.t. } pf = 0$$

& is universal in dual sense.

Remark: • $\ker f$ & $\operatorname{coker} f$ are equaliser & coequaliser of

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & b \end{tikzcd}.$$

This implies $\ker f \xrightarrow{i} a$ is mono

& $\quad b \xrightarrow{f} \operatorname{coker} f$ is epi.

- Sometimes I'll write $kf = \ker f$
  $cf = \operatorname{coker} f$.
- If $i : a \hookrightarrow b$ is mono,
  write $\operatorname{coker}(i) := b/a$.

Example

- In Mod_R, given $f : A \longrightarrow B$,
  $\ker f = \{ a \in A : fa = 0 \}$
  $\operatorname{coker} f = B/\operatorname{im} f$.

Lemma

In an additive cat.,

① $A \xrightarrow{f} B$ is mono $\iff \ker f = 0$

② $A \xrightarrow{F} B$ is epi $\iff \operatorname{coker} f = 0$

Proof

- By duality, it suffices to prove ①.
- Let $f : A \longrightarrow B$ mono & consider

$$0 \xrightarrow{0} A \xrightarrow{f} B.$$

- Given $g : C \longrightarrow A$ s.t. $f \circ g = 0$ then
  $f \circ g = 0 = f \circ 0$ so $g = 0$; then

$$\begin{tikzcd} C \arrow[rr, "g=0"] \arrow[dr, "!0"'] & \arrow[d, phantom, "||"] & A \\ & 0 \arrow[ur, "0"'] & \end{tikzcd}$$

as 0 terminal.

Hence $0 = \ker f$.

- Conversely, let $0 \xrightarrow{0} A \xrightarrow{f} B$ be kernel
  & consider $C \underset{h}{\overset{g}{\rightrightarrows}} A$ s.t. $fg = fh$.
  Equiv. $f(g-h) = 0$ so $\exists!\ C \xrightarrow{t} 0$ s.t.

$g - h = 0.t = 0$ so $g = h$ & f is mono. □

- In an additive cat with kernels & cokernels, we can factor each morphism in two ways:

①
$$\begin{tikzcd} kf \arrow[r, "i"] & a \arrow[rr, "f"] \arrow[dr, "p"'] & & b \\ & & ckf \arrow[ur, "\exists ! t"'] & \end{tikzcd}$$
where t is induced by $f \circ i = 0$.

②
$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "\exists ! l"'] & & b \arrow[r, "q"] & cf \\ & kcf \arrow[ur, "m"'] & & \end{tikzcd}$$
where l is induced by $q \circ f = 0$

As $0 = qf = qtp$ & p is epi, we have $qt = 0$ so we get ! map $ckf \xrightarrow{\alpha} kcf$ such that

$$\begin{tikzcd} a \arrow[rrr, "f"] \arrow[dr, "p"'] & & & b \\ & ckf \arrow[r, "\alpha"'] \arrow[rr, dashed, bend left, "t"] & kcf \arrow[ur, "m"'] & \end{tikzcd}$$

$\boxed{\text{Def}^n}$ An add. cat w/ kernels & cokernels is abelian if $\alpha : ckf \to kcf$ is invertible.

- In an abelian category, the two ways of factoring f coincide (up to ! iso).

we write

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "e"'] & & b \\ & \operatorname{im} f \arrow[ur, "t"'] & \end{tikzcd}$$

epi ~ $e = p$ t ~ mono as $= ma$.

$\operatorname{im} f = ckf \cong kcf$.

<u>Example</u>

In Mod$_R$, given $f : A \to B$ we have $A \to ckf = A \to A/\ker f$ & $B \to cf = B \twoheadrightarrow B/\operatorname{im} f$

so $kcf \hookrightarrow A = \operatorname{im} f$ & the induced map $ckf \xrightarrow{\alpha} kcf$ is the hom.

$$\begin{aligned} A/\ker f &\longrightarrow \operatorname{im} f \\ [a] &\longmapsto fa \end{aligned}$$

which is an iso by first iso Theorem.

In particular Mod$_R$ is abelian.

Another characterisation of abelian cats is as follows.

$\boxed{\text{Prop}^n}$ $\mathcal{C}$ is abelian $\Leftrightarrow$

① Each mono is kernel of its cokernel

② Each epi is cokernel of its kernel.

Remark: ② shows that epis are

coequaliser maps i.e. regular epi, so in an ab. cat

epi ≡ regular epi.

- In particular, this is true in Mod_R.
- Remember in Rng, $\mathbb{Z} \longrightarrow \mathbb{Q}$ is epi but not reg. epi.

So Rng is not abelian.

Proof of prop.

- We will only show $\Rightarrow$ & leave $\Leftarrow$ for interested.
- Let $A \xrightarrow{f} B$ be mono, so $\ker f = 0$.
- Then consider

$$\begin{tikzcd}
kf=0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "1"'] & B \arrow[r] & cf \\
 & ckf = A \arrow[r, "\cong", "\alpha"'] \arrow[ur, "f"] & Kcf \arrow[u, "m"'] &
\end{tikzcd}$$

where $A \xrightarrow{1} A$ is ckf as $kf = 0$.
Therefore f is iso to $kcf \Rightarrow B$, as required.

- To show each epi is cokernel of kernel is dual.

$\square$

Examples

① Mod_R

② If A is abelian so is A^{op}!
So can apply duality to abelian cats.

③ If A is abelian, we can consider chain complexes in A:

$$\begin{tikzcd}
\cdots & A_{n+1} \arrow[r, "d_{n+1}"] \arrow[rr, bend right, "0"', "\|"] & A_n \arrow[r, "d_n"] & A_{n-1} & \cdots
\end{tikzcd}$$

We have

$$\begin{tikzcd}
\operatorname{im}(d_{n+1}) \arrow[rr, hook] \arrow[dr, hook] & & \operatorname{Ker}(d_n) \arrow[dl, hook'] \\
& A_n &
\end{tikzcd}$$

& so can form

$$H_n(A) := \operatorname{Ker}(d_n) / \operatorname{im}(d_{n+1}).$$

- Thus can speak of homology of chain complexes in abelian cat.
- We can also consider exact sequences, chain maps, chain homotopies in A &

the results of last week hold in this setting.

Moreover we can form the category $Ch(A)$ of chain complexes in A & it is again abelian: indeed $(f+g)_n = f_n + g_n$ for chain maps $(-f)_n = -f_n$ .. & $(\ker f)_n = \ker(f_n)$, $(\operatorname{coker} f)_n = \operatorname{coker}(f_n)$ are constructed componentwise.

As before,

$$H_n : Ch(A) \longrightarrow A$$

is a functor.

④ If $\mathcal{C}$ is a small cat, then $[C, A]$ the functor cat is abelian with componentwise structure.

⑤ X a top. space, the poset

of open sets $O(X)$ is a cat,
& cat of presheaves valued
in Mod_R $[O(X)^{op}, \text{Mod}_R]$
is abelian, as is its
full subcat of sheaves.
important in alg. geometry.

lecture 3

Additive & exact functors

Def^n) let A, B be abelian cats. A functor $F: A \longrightarrow B$ is additive if each function $F_{X,Y}: A(X,Y) \longrightarrow B(FX, FY)$ is a homom. of abelian groups (ie. $F(f+g) = Ff + Fg$ & $F0_{X,Y} = 0_{FX,FY}$.

- From last week,
  - term/in. ob are char. by diags $a \xrightarrow{0=id} a$ which we call zero object diags & a a zero ob., denoted by 0.
  - bin. products/coproducts are characterised by diagrams of form

$$\begin{tikzcd} a \arrow[r, shift left, "i_1"] & c \arrow[l, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & b \arrow[l, shift right, "i_2"'] \end{tikzcd}$$

sat.

- $p_1 i_1 = id$, $p_2 i_2 = id$ $i_1 p_1 + i_2 p_2 = Id_c$
- $p_1 i_2 = 0$, $p_2 i_1 = 0$,

which are called biproduct diagrams, and often denote biprod. by $a \oplus b$.

Proposition

Any additive functor F preserves finite prods & finite coproducts - ie. zero ob. & biproducts.

Proof

F preserves zero ob. diags / biproduct diagrams □

Def^n) An additive functor $F : A \to B$ bet. abelian cats is
- left exact if it pres. kernels (lex)
- right exact if it pres. cokernels (rex)
- exact if it preserves both. (ex)

Remark : Lex functors are those preserving finite limits (fin. prods + equalises ≡ kernels). Rex functors preserve finite colimits & ex functors preserve both.

<u>Lemma</u>

① F is lex $\Leftrightarrow$ it preserves exactness of sequences $0 \to A \xrightarrow{f} B \xrightarrow{g} C$

② F is rex $\Leftrightarrow$ it preserves exactness of sequences $A \xrightarrow{f} B \xrightarrow{g} C \to 0$

③ F is ex $\Longleftrightarrow$ it preserves exactness everywhere $\Longleftrightarrow$ it pres ses $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$.

<u>Proof</u>

First we prove ①. Observe $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is exact $\Longleftrightarrow$ $\ker f = 0$ (ie f is mono) & $\ker g = f$ ($A \to \operatorname{im} f$ is an iso)

Hence if F pres. kernels, it pres ex. of such sequences.

Conversely, consider the exact sequence

$0 \to \ker f \xhookrightarrow{i} A \xrightarrow{f} B$ which F preserves

so $0 \to F\ker f \xrightarrow{Fi} FA \xrightarrow{Ff} FB$ is exact

$\Rightarrow F\ker F = \ker Ff$. □

② is dual to ①.

For ③, if F is ex. it preserves kernels & cokers & so images. Therefore it preserves exactness everywhere & so in particular ses.

Suppose F pres ses. We will show F pres. monos & epis. If f is mono then $0 \to A \xrightarrow{f} B \to \operatorname{coker} f \to 0$ is ses which F preserves, so by ex ⓐ Fa, Ff is mono.

Dually F preserves epis.

Now show F preserves kernels (cokernels dual.)

At general $f: A \to B$ consider ses on hor. row.

$$\begin{tikzcd}
 & & & B & \\
0 \arrow[r] & \ker f \arrow[r, hook, "q"] & A \arrow[r, two heads, "p"] \arrow[ur, "f"] & \operatorname{im} f \arrow[u, hook, "i"] \arrow[r] & 0
\end{tikzcd}$$

Then Fq, Fi mono, Fp epi & hor. sequence exact after applying F, so

$\ker(Ff)$ = as Fi mono

$\ker(Fp)$ = as F pres ses

$\operatorname{im}(Fq)$ = as Fq mono

$F(\ker F)$, so F pres. kernels. □

Examples

① If $\mathcal{C}$ abelian cat & $A \in \mathcal{C}$ the functor

$$\begin{tikzcd}
\mathcal{C}(A,-) : \mathcal{C} \arrow[rr] & & Ab & \\
X \arrow[d, "f"'] & & \mathcal{C}(A,X) \arrow[d, "f_*"'] & g \arrow[d, mapsto] \\
Y & & \mathcal{C}(A,Y) & fg
\end{tikzcd}$$

is additive & preserves all limits - therefore it is lex.

② If $\mathcal{C} = Mod_R$, we have $Mod_R(A,-) : Mod_R \longrightarrow Ab$ & this has a left adjoint $A \otimes_{\mathbb{Z}} - : Ab \to Mod_R$.

- Here $A \otimes_{\mathbb{Z}} B$ classifies functions $K : A \times B \longrightarrow \boxed{C}$ $\sim$ R-module & $K(a,-) : B \longrightarrow C$ is hom. of ab. groups & $K(-,b) : A \longrightarrow C$ is hom. of R-modules, & $A \otimes_{\mathbb{Z}} B$ is constructed as a quotient sim. to the tensor prod. of R-modules in Alg. 3.

In particular, as a left adjoint it preserves colimits & so is right exact.

More gen, if right adjoint is lex then left adjoint is rex & conversely.

③ The forgetful functor
$U : \mathrm{Mod}_R \longrightarrow \mathrm{Ab}$ is left exact,
since $U \cong \mathrm{Mod}_R(R, -)$ which is
lex by ① (or directly).

④ If $\mathcal{C}$ is abelian, so is $\mathrm{Ch}(\mathcal{C})$
& then

$$\begin{aligned} \mathrm{Ch}(\mathcal{C}) &\xrightarrow{(-)_n} \mathcal{C} \\ X &\longmapsto X_n \end{aligned}$$

is exact since kernels & cokernels are componentwise in $\mathrm{Ch}(\mathcal{C})$.

Theorem (Freyd - Mitchell)

If $\mathcal{C}$ is a small abelian cat., $\exists$ a ring R & an exact fully faithful embedding

$$F : \mathcal{C} \longrightarrow \mathrm{Mod}_R .$$

- We will not prove it.

Consequence :- When proving things about diagrams in an abelian cat it suffices to supp. we are working in a cat. of R-modules, since the theorem lets us view our category as a full subcat. of Mod_R such that the inclusion pres all structure - kernels, cokernels etc.

- For instance, suffices to prove snake lemma in Mod_R.

The snake lemma & long exact sequence of homology

- The main reason that homology can be easily computed is the long exact sequence of homology, which we turn to now.

The snake lemma & long exact sequence of homology

Snake lemma

Given a diagram

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' & \end{tikzcd}$$

with exact rows in an abelian cat., there is an exact sequence

$$\ker(\alpha) \xrightarrow{f} \ker(\beta) \xrightarrow{g} \ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha) \xrightarrow{f'} \operatorname{coker}(\beta) \xrightarrow{g'} \operatorname{coker}(\gamma).$$

$\boxed{\text{Proof}}$ - Maps except for δ are ind. by u.p.'s.

- Exactness @ $\ker(\beta)$ & $\operatorname{coker}(\beta)$ are dual. So prove ex @ $\ker(\beta)$. Enough to prove it in Mod_R by FM-thm, which assume for rest of proof.
- let $b \in \ker(\beta)$ & suppose $gb=0$. By ex. $\exists a$ st $fa=gb$. Then $f'\alpha a = \beta f a = \beta b = 0$ & as f' is mono (since bottom row exact) therefore $\alpha a = 0$ so $a \in \ker(\alpha)$ w/ $fa=b$. Hence ex@ $\ker(\beta)$.
- Now we construct $\delta : \ker\gamma \to \operatorname{coker}\alpha$. Consider $x \in \ker\gamma$. As g epi $\exists x' \in B$ st $gx' = x$. Consider $\beta x'$. Then $g'\beta x' = \gamma g x' = \gamma x = 0$ so $\beta x' \in \ker(g') = \operatorname{im}(f')$ so $\exists! x'' \in A'$ as f' is mono s.t. $f'(x'') = x'$.
Define $\delta(x) = x'' + \operatorname{im}(\alpha) \in \operatorname{coker}(\alpha)$.
- To show δ well defined, let $gy' = x$.
- Then $g(y'-x') = 0$ so by ex. of top row, $\exists a \in A$ st $f(a) = y' - x'$, so $\alpha a \in A'$. Then $f'\alpha a = \beta f a = \beta y' - \beta x' = f'y'' - f'x''$ so as f' mono $\alpha a = y'' - x''$.
Hence $y'' + \operatorname{im}\alpha = x'' + \operatorname{im}\alpha$.

Therefore δ is well defined.
Easy to see it is a homomorphism
& exactness @ $\ker(\gamma)$, $\operatorname{coker}(\alpha)$
are left an exercise. $\square$

Theorem

let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a ses of chain complexes in an ab. cat. Then we obtain a long exact sequence of homology

$$\cdots H_{n+1}(A) \xrightarrow[H_{n+1}(f)]{} H_{n+1}(B) \xrightarrow[H_{n+1}(g)]{} H_{n+1}(C) \xrightarrow{\delta} H_n(A) \xrightarrow[H_n(f)]{} H_n(B) \xrightarrow[H_n(g)]{} H_n(C) \cdots$$

Proof For a chain complex A

$$\cdots A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots\cdots$$

we have an ind. homomorphism

$$A_n/\mathrm{im}(d_{n+1}) \xrightarrow{d} \ker(d_{n-1}) = Z_{n-1}(A)$$

$$x + \mathrm{im}(d_{n+1}) \longmapsto d_n x$$

- whose kernel contains elements $x + \mathrm{im}(d_{n+1})$ with $d_n(x) = 0$ & so is precisely $\ker(d_n)/\mathrm{im}(d_{n+1}) = H_n(A)$.
- whose cokernel is $\ker(d_{n-1})/\mathrm{im}(d_n) = H_{n-1}(A)$

Then we obtain a comm. diag.

$$\begin{tikzcd}
 & A_n/\mathrm{im}(d_{n+1}) \arrow[r, "f_n"] \arrow[d, "d"'] & B_n/\mathrm{im}(d_{n+1}) \arrow[r, "g_n"] \arrow[d, "d"'] & C_n/\mathrm{im}(d_{n+1}) \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r, "f_{n-1}"'] & Z_{n-1}(B) \arrow[r, "g_{n-1}"'] & Z_{n-1}(C) &
\end{tikzcd}$$

& we will show these rows are

exact, & then by snake lemma we obtain les of homology. It remains to check rows are exact.

- Ex @ $C_n/\mathrm{im}(d_n)$, $Z_{n-1}(A)$ is easy as f_{n-1} is mono & g_n is epi.
- At $x \in Z_{n-1}(B)$, suppose $gx = 0$. Then $\exists y \in A_{n-1}$ st $Fa = x$. Must show $d_{n-1}a = 0$, but as F is inj, $Fd_{n-1}a = d_{n-1}fa = d_{n-1}x = 0$ implies $d_{n-1}a = 0$ so $a \in Z_{n-1}(A)$, as required.
- At $B_n/\mathrm{im}(d_{n+1})$ let $g(x + \mathrm{im}\, d_{n+1}) = 0$. Then $gx \in \mathrm{im}\, d_{n+1}$ so $gx = dc$ some $c \in C_{n+1}$. Then $\exists y \in B_{n+1}$ st $gy = c$. Now consider $x - dy \in B_n$. Then

$$g(x-dy) = gx - gdy = gx - dgy = gx - dc$$

$= 0$, so $\exists a \in A_n$ st $fa = x - dy$. Then

$$f(a + \mathrm{im}\, d_{n+1}) = x - dy + \mathrm{im}\, d_{n+1}$$
$$= x + \mathrm{im}\, d_{n+1},$$

as required. □

Stop here for today.

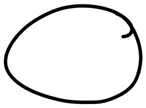
- See exercises for application to homology groups of spheres.

Homology of algebraic structures

- Homology and cohomology provide useful invariants of topological spaces.
- Can we define them for algebraic structures?

For groups, we certainly can. Indeed, a group G gives rise to a topological space BG, which is path connected, has fundamental group $\pi_1(BG) = G$, and all higher htpy groups equal $\pi_n(BG) = 0$.

(These properties in fact characterise BG up to homotopy equivalence.)

- Indeed, when $G = \mathbb{Z}$, $BG =$ 

& when G is the free group on n elts,

$BG =$  a wedge of n circles.

- Then we can define the homology of G as the homology of BG, and the cohomology of G w/ coefficients in $A \in Ab$ as the cohomology of BG w/ coefficients in A.

- Our goal now is to describe an approach which does not use topological spaces, but is purely algebraic -
the most general approach uses the language of derived functors.

Lecture 4

Today: projectives, proj. resolutions & derived functors.

Notation: In ab. cat, $\twoheadrightarrow$ for epi, $\rightarrowtail$ or $\hookrightarrow$ for mono.

Projectives (see Alg 3 - we will do it quickly here)

Def^n) An ob. $A \in \mathcal{C}$ in an abelian cat is projective if given any diagram

$$\begin{tikzcd} & A \arrow[d, "\alpha"] \\ B \arrow[r, two heads, "f"'] & C \end{tikzcd} \quad \exists\, A \xrightarrow{\alpha'} B \text{ st } \begin{tikzcd} & A \arrow[dl, "\alpha'"'] \arrow[d, "\alpha"] \arrow[dl, phantom, shift left=3, "="] \\ B \arrow[r, two heads, "f"'] & C \end{tikzcd} .$$

Remark: This says $\mathcal{C}(A,B) \xrightarrow{f_*} \mathcal{C}(A,C) \in Ab$

$$g \longmapsto fg$$

is surjective/epi, so

A is proj. $\Longleftrightarrow$ $\mathcal{C}(A,-) : \mathcal{A} \longrightarrow Ab$ preserves epis.

Def^n) $\mathcal{C}$ has enough projectives if for each $A \in \mathcal{A}$ $\exists$ X projective & epi $X \twoheadrightarrow A$.

Prop^n) Mod_R has enough projectives. The projectives are the retracts of free modules.

Proof) See Alg. 3. Given A we take counit map $FUA \xrightarrow{\varepsilon_A} A$ relative to adj $\begin{tikzcd} Mod_R \arrow[r, phantom, "\perp"] & Set \arrow[l, bend right, "F"'] \arrow[l, leftarrow, bend left, "U"] \end{tikzcd}$, which takes formal sums to actual sums. □

Proposition (Properties of projectives)

① Direct sums/biproducts and retracts of projectives are projective.

② $A \in \mathcal{C}$ is proj. $\Longleftrightarrow$ $\mathcal{C}(A,-) : \mathcal{C} \to Ab$ is exact.

Proof.

① is straightforward. Consider A, B proj & direct sum $A \oplus B$.

Given
$$\begin{tikzcd} & A\oplus B \arrow[d, "\alpha"] \\ C \arrow[r, two heads, "f"'] & D \end{tikzcd}$$
we have
$$\begin{tikzcd} & A \arrow[d, "i_1"] \\ B \arrow[r, "i_2"'] & A\oplus B \end{tikzcd}$$
& so

$$\begin{tikzcd} & A \arrow[dl, dashed, "\exists \theta_1"'] \arrow[d, "\alpha i_1"] \\ C \arrow[r, two heads, "f"'] & D \end{tikzcd}$$
&
$$\begin{tikzcd} & B \arrow[dl, dashed, "\exists \theta_2"'] \arrow[d, "\alpha i_2"] \\ C \arrow[r, two heads, "f"'] & D \end{tikzcd}$$
& then by u.p. of coprod. $A \oplus B$

$\exists !\ \theta : A \oplus B \to C$ s.t. $\theta i_1 = \theta_1$, $\theta i_2 = \theta_2$.

Then
$$\begin{tikzcd} & A\oplus B \arrow[dl, "\theta"'] \arrow[dr, "\alpha"] \arrow[d, phantom, "="] & \\ C \arrow[rr, two heads, "f"'] & & D \end{tikzcd}$$
commutes using u.p. of coproduct.

Hence $A \oplus B$ projective.

- For retracts - easy: did in proof of "Proj. Mods = retracts of frees" in Alg. 3.

② If $\mathcal{C}(A,-) : \mathcal{C} \to Ab$ is exact, let $B \xrightarrow{f} C$ be epi, i.e. $B \xrightarrow{f} C \to 0$ is exact.

Then let $F = \mathcal{C}(A,-)$. Then as exact functors pres. exactness,

$$FB \xrightarrow{Ff} FC \to F0 = 0$$ is exact, so Ff is epi.

Conversely, let F pres epis. We must show F preserves ses

Each ses is of form $0 \to \ker f \hookrightarrow A \xrightarrow{f} B \to 0$

& as F is lex it pres. kernels so above is sent to

$$0 \hookrightarrow \ker Ff \hookrightarrow FA \xrightarrow[S]{Ff} FB \to 0$$

as F preserves epis

& this is a ses as Ff is epi.
Hence F preserves ses, and so is exact. $\square$

Injectives

- For $\mathcal{C}$ abelian, $A \in \mathcal{C}$ injective $\Longleftrightarrow$ $A \in \mathcal{C}^{op}$ is projective.

So anything we can prove about projectives in a gen. ab. cat has dual for injectives.

- $A \in \mathcal{C}$ is injective if

$$\begin{tikzcd} B \arrow[r, tail, "\text{mono}"] \arrow[d, "f"'] & C \\ A & \end{tikzcd} \quad \exists \quad \begin{tikzcd} B \arrow[r, tail, "\text{mono}"] \arrow[dr, "f"'] & C \arrow[d, "f'"] \\ & A \end{tikzcd}.$$

- In Ab, injective $\equiv$ divisible.
- $\mathcal{C}$ has enough injectives if $\forall A \in \mathcal{C}$ $\exists$ mono $A \rightarrowtail A'$ w/ A' inj.
- Mod_R has enough injectives - follows from Baer criterion in Alg 3 (relatively hard).

Projective resolutions

Notation: Ch. complex X st $X_n = 0$ for $n < 0$ can be id. w' a positive chain complex

$$\cdots \; X_2 \xrightarrow{d} X_1 \xrightarrow{d} X_0.$$ write

$Ch(\mathcal{C})_{\geq 0}$ for cat of positive chain complexes.

Def) let $A \in \mathcal{C}$ abelian cat. A proj. resolution of A is a chain complex $C \in Ch(\mathcal{C})_{\geq 0}$ with a map $C_0 \xrightarrow{\varepsilon} A$ such that:

① $\cdots \; C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{\varepsilon} A \to 0$ is exact

② each C_i is projective.

Can give more conceptual definition, as below.

Def^n) A chain map $f : A \to B \in Ch(\mathcal{C})$ quasi-isomorphism if $H_n f : H_n A \to H_n B$ is an iso $\forall n \in \mathbb{Z}$.

- For $A \in \mathcal{C}$, let $A[0]$ be chain complex

$$\cdots 0 \cdots 0 \to 0 \to \underset{\text{degree } 0}{A}$$

$\in Ch(\mathcal{C})_{\geq 0}$

Proposition

Projective resolution C of A

$$\equiv$$

quasi-iso $C \longrightarrow A[0]$ st. each C_i is projective.

~~Proof~~ - A chain map $C \to A[0]$ is a diag.

$$\begin{tikzcd}
\vdots \arrow[d, dash, dotted] & \vdots \arrow[d, dash, dotted] \\
C_2 \arrow[r] \arrow[d, "d"'] & 0 \arrow[d] \\
C_1 \arrow[r] \arrow[d, "d"'] & 0 \arrow[d] \\
C_0 \arrow[r, "\varepsilon"] & A
\end{tikzcd}$$

is specified by a single morphism $\varepsilon : C_0 \to A$ st. $\varepsilon d = 0$.

- To ~~say~~ it is a quasi-iso is to ~~say~~

a) $H_n(C) \xrightarrow{\simeq} H_n(A[0]) = 0$ for $n \geq 1$ &

$$\begin{tikzcd}
H_0(C) \arrow[r, "H_0(\varepsilon)"] \arrow[d, equal] & H_0(A[0]) \arrow[d, equal] \\
C_0/\mathrm{im}\, d \arrow[r, "\bar{\varepsilon}"] & A \\
x + \mathrm{im}\, d \arrow[r, mapsto] & \varepsilon x
\end{tikzcd}$$

induced map is invertible.

Now $\bar{\varepsilon}$ is invertible $\iff$

b) $\bar{\varepsilon}$ is inj. (its kernel $\ker \varepsilon / \mathrm{im}\, d = 0$)

c) $\bar{\varepsilon}$ is surj. (equiv, ε is surj) so

(a), (b), (c) $\iff$

$$\cdots C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{\varepsilon} A \longrightarrow 0 \text{ is exact}$$

where - (c) corr. to ex@A

- (b) - - - - ex@C_0

(c) ex @ other positions. □

Proposition

If $\mathcal{C}$ has enough projectives, then each object has a projective resolution.

Proof

- Let $A \in \mathcal{C}$ & consider $C_0 \xrightarrow{\varepsilon} A$ w/ C_0 proj. Then $C_0 \xrightarrow{\varepsilon} A \to 0$ exact.
- Now form

$$\begin{tikzcd}
 & \mathrm{Ker}\,\varepsilon \arrow[dr, "i"] & & & \\
C_1 \arrow[ur, two heads, "p_0"] \arrow[rr, "d"] & & C_0 \arrow[r, two heads, "\varepsilon"] & A \arrow[r] & 0
\end{tikzcd}$$

with C_1 proj, p_0 epi.

- Then $\mathrm{Ker}\,\varepsilon = \mathrm{im}\,p_0 \leq \mathrm{im}\,d \leq \mathrm{Ker}\,\varepsilon$ so $\mathrm{im}\,d = \mathrm{Ker}\,\varepsilon$.
- Now we continue in this way

$$\begin{tikzcd}
 & & & & \mathrm{Ker}\,\varepsilon \arrow[dr, "i"] & & & \\
\cdots & C_2 \arrow[rr, "d"] \arrow[dr, two heads] & & C_1 \arrow[ur, two heads, "p_0"] \arrow[rr, "d"] & & C_0 \arrow[r, two heads, "\varepsilon"] & A \arrow[r] & 0 \\
 & & \mathrm{Ker}\,p_0 \arrow[ur] & & & & &
\end{tikzcd}$$

obtaining a projective resolution. □

Derived Functors

let $F : A \to B$ be a right exact functor be abelian categories.

The n'th left derived functor $L_nF : A \to B$ is defined as follows:

@ $X \in A$, let

stand. terminology $X_\bullet \xrightarrow{d'} X[0]$ be a proj. resolution of X so

$X_\bullet = \dots X_2 \xrightarrow{d} X_1 \xrightarrow{d} X_0$, so can form

$FX_\bullet = \dots \to FX_2 \xrightarrow{Fd} FX_1 \xrightarrow{Fd} FX_0$.

Then we set

$$L_nF(X) = H_n(FX_\bullet).$$

Still have to define L_nF on morphisms.

Remark: Strange definition, since pr. resol. not functorial nor unique up to iso. However they are so, up to homotopy, & we will use this to define L_nF on morphisms.

Lemma

Consider $f : A \to B \in \mathcal{C}$ & proj. resolutions $A_\bullet \xrightarrow{d} A[0]$ & $B_\bullet \xrightarrow{d} B[0]$ of A & B.

Then $\exists$ a chain map $f_\bullet$ s.t. square

*
$$\begin{tikzcd} A_\bullet \arrow[r, "f_\bullet"] \arrow[d, "d"'] & B_\bullet \arrow[d, "d"] \\ A[0] \arrow[r, "f{[0]}"] & B[0] \end{tikzcd}$$

f in deg-0, else 0.

commutes, & it is unique w/ this prop. up to chain homotopy.

Proof) – Will constr. f_0 inductively.

$$\begin{tikzcd}
\cdots \arrow[r, dash] & A_1 \arrow[rr, "d"] \arrow[dd, "③ f_1"'] \arrow[dr, "② \exists"] & & A_0 \arrow[r, "d"] \arrow[dd, "① f_0"] \arrow[r, phantom, shift right=6ex, "="] & A \arrow[r] \arrow[dd, "f"] & 0 \\
& \arrow[r, phantom, "="] & \ker d \arrow[dr] & & & \\
\cdots \arrow[r, dash] & B_1 \arrow[ur] \arrow[rr, "d"'] & & B_0 \arrow[r, two heads, "d"'] & B \arrow[r] & 0
\end{tikzcd}$$

Since rows ex., $d : B_0 \to B$ surj. As A_0 proj., $\exists f_0$ as in ①. Next, $df_0d = fdd = 0$ so f_0d factors through $\ker d \to B_0$. As row is ex., $\mathrm{im}(B_1) = \ker(d)$ so $B_1 \twoheadrightarrow \ker d$ surj., so as A_1 is proj., ② factors through B_1 as a map f_1, as in ③. Then continue in same way.

- For uniqueness, suppose we have $g_0 : A_0 \to B_0$ making * commute.

Then we have

$$\begin{tikzcd}
\cdots \arrow[r, dash] & A_2 \arrow[rr, "d"] & & A_1 \arrow[rr, "d"] \arrow[dl, "④"'] \arrow[dd, "f_1-g_1"] \arrow[ddll, bend right, "h_1"'] & & A_0 \arrow[r, "d"] \arrow[dl, "②"'] \arrow[dd, "①"', "f_0-g_0"] \arrow[ddll, bend right, "③ h_0"'] & A \arrow[r] \arrow[dd, "f"] & 0 \\
& & \ker d \arrow[dr] & & \ker d \arrow[dr] & \arrow[r, phantom, "\neq"] & & \\
\cdots \arrow[r, dash] & B_2 \arrow[ur] \arrow[rr, "d"'] & & B_1 \arrow[ur] \arrow[rr, "d"'] & & B_0 \arrow[r, two heads, "d"'] & B \arrow[r] & 0
\end{tikzcd}$$

Then $d(f_0 - g_0) = fd - fd = 0$, so ① factors as ② through kerd & as $B_1 \to \ker d$ is surj & A_0 proj, we obtain ③ h_0 satisfying

$$dh_0 = f_0 - g_0 .$$

Next we need a map $h_1 : A_1 \to B_2$ s.t.

$dh_1 + h_0d = f_1 - g_1$ or equiv.

$dh_1 = f_1 - g_1 - h_0d := k$. Now $dk = 0$ as

$df_1 - dg_1 - dh_0d = f_0d - g_0d - (f_0 - g_0)d = 0$, so

k factors through kerd as in ④, & now as

A_1 proj, $B_2 \twoheadrightarrow \ker d$ epi, obtain ⑤ h_1 sat $dh_1 = K$, as required.
Then continue inductively to constr $h_2, \dots$.

$\square$

With this in place, we can define L_nF on morphisms:
@ $f : A \to B$ obtain $f_\bullet : A_\bullet \to B_\bullet$ sat. *
so $Ff_\bullet : FA_\bullet \to FB_\bullet$ inducing

$$\begin{tikzcd}
L_nF(A) \arrow[r, "L_nF(f)"] \arrow[d, phantom, "\parallel"] & L_nF(B) \arrow[d, phantom, "\parallel"] \\
H_n(FA_\bullet) \arrow[r, "H_n(Ff_\bullet)"] & H_n(FB_\bullet)
\end{tikzcd}$$

Proposition

L_nF is a functor.

Proof

Firstly observe that if

$$\begin{tikzcd}
A_\bullet \arrow[r, shift left, "g_\bullet"] \arrow[r, shift right, "f_\bullet"'] \arrow[d, "d"'] & B_\bullet \arrow[d, "d"] \\
A[0] \arrow[r, "F[0]"] & B[0]
\end{tikzcd}$$

as in *, then by lemma $f_\bullet \overset{h}{\simeq} g_\bullet$.

Then $Ff_\bullet \overset{Fh}{\simeq} Fg_\bullet$ so

$$H_n(Ff_\bullet) = H_n(Fg_\bullet)$$ as homology identifies homotopic maps.

Hence L_nF is

well defined on morphisms.

Consider $A \xrightarrow{f} B \xrightarrow{g} C$. Then

$$\begin{tikzcd}
A_\bullet \arrow[r, "f_\bullet"] \arrow[d, "d"'] & B_\bullet \arrow[r, "g_\bullet"] \arrow[d, "d"] & C_\bullet \arrow[d, "d"] \\
A[0] \arrow[r, "f[0]"] \arrow[rr, bend right, "gf[0]"'] & B[0] \arrow[r, "g[0]"] & C[0]
\end{tikzcd}$$

so

by lemma, have $g_\bullet f_\bullet \simeq (gf)_\bullet$,

so $F(g_\bullet)F(f_\bullet) \simeq F(gf_\bullet)$ so

$$\begin{array}{ccc}
H_nF(g_\bullet)H_nF(f_\bullet) & = & H_nF(gf_\bullet) \\
\| & & \| \\
L_nF(g)L_nF(f) & & L_nF(gf).
\end{array}$$

Similarly, L_nF pres. identities & so is a functor. □

Next time, more theory of der. functors & examples - Ext, Tor.

Lecture 5

Today :- finish left derived functors
- dual cohomology & right derived functors
- examples : Ext, Tor

Recap : For $F : \mathcal{A} \to \mathcal{B}$ right exact, & $\mathcal{A}$ having enough projectives, defined

missing in video

$$L_n F : \mathcal{A} \to \mathcal{B} : X \longmapsto H_n(F\underbrace{X_\bullet}_{\text{proj resolution of } X})$$

Proposition
For $F : \mathcal{A} \to \mathcal{B}$ right exact functor (between abelian categories) as above, we have a natural isomorphism $L_0 F \cong F$.

~~Proof~~ At $X \in \mathcal{A}$, we have $X_\bullet \xrightarrow{d} X[0]$ & so $L_0 F X = H_0 F X_\bullet \xrightarrow{H_0 F(d)} H_0 F X[0] = X$ gives the components of a natural transformation

$$L_0 F \longrightarrow F.$$

For invertibility, as F is rex,

$$FX_1 \xrightarrow{Fd_1} FX_0 \xrightarrow{Fd} FX \longrightarrow 0 \text{ is exact,}$$

so $\ker Fd = \operatorname{im} Fd_0$ so

$$\begin{aligned} H_0 F X_\bullet &= FX_0 / \operatorname{im} Fd_0 \\ &\cong FX_0 / \ker Fd \\ &\cong FX \quad \text{as } Fd \text{ epi.} \ \square \end{aligned}$$

Further properties of left derived functors

① Horseshoe lemma (without proof)

- If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \in \mathcal{A}$ is ses we can lift it to a ses of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & X_\bullet \arrow[r, "f_\bullet"] \arrow[d, phantom, "\perp"] \arrow[rd, phantom, "''"] & Y_\bullet \arrow[r, "g_\bullet"] \arrow[d, phantom, "\perp"] \arrow[rd, phantom, "''"] & Z_\bullet \arrow[r] \arrow[d, phantom, "\perp"] & 0 \\
0 \arrow[r] & X[0] \arrow[r, "{f[0]}"'] & Y[0] \arrow[r, "{g[0]}"'] & Z[0] \arrow[r] & 0
\end{tikzcd}$$

as on top row, which is (moreover) componentwise split exact: for each $n \geq 0$, of form

$$0 \longrightarrow A \xrightarrow{i_A} A \oplus B \xrightarrow{p_B} B \longrightarrow 0$$

for a biproduct diagram (such is always short exact).

② If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \in \mathcal{A}$ is ses & $F : \mathcal{A} \longrightarrow \mathcal{B}$ we obtain a les

$$\dots L_2FZ \to L_1FX \to L_1FY \to LFZ \to FX \to FY \to FZ \to 0$$

of left derived functors:

- Indeed, F takes split exact sequences to split exact sequences (as it preserves biproducts)
- Then apply les of homology to ses

$$0 \longrightarrow FX_\bullet \xrightarrow{Ff_\bullet} FY_\bullet \xrightarrow{Fg_\bullet} FZ_\bullet \longrightarrow 0 .$$

③ The right exact F is exact $\Longleftrightarrow L_nF = 0 \ \forall n \geq 1$

$\Longleftrightarrow L_1F = 0$:

- If F is exact, $FX_\bullet$ is exact sequence, so $H_nFX_\bullet \simeq 0$ all $n \geq 1$, so $L_nF = 0$ all $n \geq 1$.
- Therefore $L_1F = 0$.

Ass. $L_1F = 0$, at ses $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \in \mathcal{A}$ obtain les

$$\begin{tikzcd}
\cdots \arrow[r] & L_1FZ \arrow[r] \arrow[d, equal] & FX \arrow[r, "Ff"] & FY \arrow[r, "Fg"] & FZ \arrow[r] & 0 \\
& 0 & & & &
\end{tikzcd}$$

so $0 \to FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \to 0$

is a ses.

Conclusion

The higher L_nF for $n \geq 1$ measure the failure of F is to be exact.

Right derived Functors

- An inj resolution of $X \in \mathcal{A}$ is an exact sequence
  $0 \to X \to X^0 \to X^1 \to \dots$ with each X^i injective.
  This gives a positive cochain comp. $X^\bullet$ & morphism $X[0] \to X^\bullet$ which induces an iso. on cohomology.
- If $\mathcal{A}$ has enough injectives, each ob X has an injective resolution.
- If $F : \mathcal{A} \to \mathcal{B}$ is lex functor & $\mathcal{A}$ has enough injectives, we can form its n'th right derived functor $R^nF : \mathcal{A} \to \mathcal{B}$, which has value $R^nF(X) = H^n(FX^\bullet)$.

We have dual props to those from before:

- $R^0F \cong F$
- For each ses $0 \to X \to Y \to Z \to$
  obtain les
  $0 \to FX \to FY \to FZ \to R^1FX \to R^2FY \to \dots$

- F is exact $\Leftrightarrow$
  $R^n F = 0$ all $n \geq 1$ $\Leftrightarrow$
  $R^1 F = 0$.

Ext & Tor (no proofs!)

- Begin with Tor, but focus on Ext.
- For A right R-mod, B left R-module, can form their t.p. $A \otimes_R B \in Ab$ w/ universal property:

$A \otimes_R B \longrightarrow C \in Ab$ is in bij^n w/ function
$f : A \times B \longrightarrow C$ which is Ab-gp hom. in each var.

- In this way, we obtain a functor

$$-\otimes_R - : R\text{-Mod} \times \text{Mod}R \longrightarrow Ab$$
$$(A\ ,B) \longmapsto A \otimes_R B$$

, & so functors $A \otimes_R - : \text{Mod}_R \to Ab$ & $-\otimes_R B : R\text{-Mod} \to Ab$, and these are $\underline{\text{rex}}$.

Then $\underline{\text{Tor}_n^A}(A,B) := L_n(-\otimes_R B)(A)$, which is given by n'th homology of

$$\cdots\ A_2 \otimes_R B \longrightarrow A_1 \otimes_R B \longrightarrow A_0 \otimes_R B$$

where $A_\bullet$ is proj. resolution of $A \in R\text{-Mod}$, so $\underline{H_n(A_\bullet \otimes_R B)}$.

Equiv., it can be calc as

$L_n(A \otimes_R -)(B)$, so take proj. resolution of B, tensor by A & calc. homology,

then $\underline{H_n(A_\bullet \otimes_R B)} \cong \underline{H_n(A \otimes_R B_\bullet)}$.

Ext

- Given $A \in \text{Mod}_R$, can form $\overset{\text{Hom}(A,-)}{\text{Mod}_R(A,-)} : \text{Mod}_R \to \text{Ab}$ which is rex & as Mod_R has enough injs, can form right der. functor $\text{Ext}_n(A,-) := R^n\text{Hom}(A,-)$, so $\text{Ext}_n(A,B) := H^n\text{Hom}(A,B^\bullet)$ for $B^\bullet$ a inj. resolution $B_0 \to B_1 \to B_2 \cdots$ of B.

- Note also have rex functor

$$\text{Hom}(-,B) : \text{Mod}_R^{op} \longrightarrow \text{Ab}$$

& so can calculate

$$R^n\text{Hom}(-,B) : \text{Mod}_R^{op} \longrightarrow \text{Ab} \text{ as}$$

$R^n\text{Hom}(-,B)(A)$ as Hn app to

$$\text{Hom}(A_0,B) \to \text{Hom}(A_1,B) \to \text{Hom}(A_2,B) \cdots$$

for $A_\bullet$ a proj resolution of A (ie. inj. resolution in Mod_R^{op}.)

In fact, $\text{Ext}^n(A,B) =$

$$H^n\text{Hom}(A,B^\bullet) \cong H^n\text{Hom}(A_\bullet,B)$$

so 2 ways of calc. $\text{Ext}^n(A,B)$.

- Ext can be understood more explicitly using extensions :

an extension of A by B is a ses

$$0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0 .$$

- Two extensions of A by B are equiv ($\simeq$) if $\exists$ iso of ses of form

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "1"'] & X \arrow[r] \arrow[d, "\wr"] & A \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & B \arrow[r] & Y \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

- Then $\mathrm{Ext}^1(A,B) \equiv \{\text{extensions of } A \text{ by } B\}/\simeq$

modulo equivalence

- From earlier work,

A is proj. $\Longleftrightarrow$ $\hom(A,-)$ is exact
$\Longleftrightarrow$ $\mathrm{Ext}^1(A,B) = 0$ all B :
this says that each extension as above of A by B is iso to trivial ext.

$$0 \longrightarrow B \longrightarrow A \oplus B \longrightarrow A \longrightarrow 0,$$

or split exact.

- For higher n, elts of $\mathrm{Ext}^n(A,B)$ are exact sequences of length n

$$0 \to B \to \cdots \to A \to 0$$ modulo equivalence.

Group cohomology

- G a group, can form group (co)homology using above:

- consider category Mod_G of G-modules: such consists of an ab. group A with an assoc. action $G \times A \xrightarrow{\cdot} A : (g,a) \mapsto g.a$, equiv. a hom. $G \longrightarrow \mathrm{Ab}(A,A)$.

(will look at these more closely in group representation theory!!)

- These are $\mathbb{Z}G$-modules where $\mathbb{Z}G$ the group ring, whose elements are sums

$$n_1 g_1 + \dots + n_k g_k \;, n_i \in \mathbb{Z}, g_i \in G.$$

- The inclusion

$$\mathrm{Ab} \xrightarrow{\;\; i \;\;} G\text{-Mod}$$ sends A to same ab. group A w/ trivial action $g.a = a$.

- We have adjunctions $\mathrm{Coinv} \dashv i \dashv \mathrm{Invar}$.
- Here $\mathrm{Invar}(M) = \{ x \in M : gx = x \text{ all } g \in G \}$ or equally the limit of the diagram

1 ob. cat.

$$\begin{tikzcd} \Sigma G \arrow[r, "M"] & \mathrm{Ab} \\ \cdot \circlearrowleft g \arrow[r, mapsto] & M \circlearrowleft g\cdot - \end{tikzcd}$$

ie.

$$\begin{tikzcd} \mathrm{Invar}(M) \arrow[r, hook] \arrow[dr, bend right] \arrow[dr, phantom, "''"] & M \arrow[d, "g\cdot -"] & \forall g. \\ & M & \end{tikzcd}$$

- It's colimit is Coinvar$(M) =$
  $M/\langle gx - x : x \in M, g \in G \rangle$.
- In partic.,
  Invar : $\mathbb{Z}G$-Mod $\longrightarrow$ Ab is lex,
  & the group cohomology of G w' coefficients in M is defined as $R^n(\text{Invar}(-))(M)$. Note, however that $\text{Invar}(A) \cong \mathbb{Z}G\text{-Mod}(i\mathbb{Z}, A)$, so $R^n(\text{Invar}(-))(M) \cong \text{Ext}_n(i\mathbb{Z}, M)$, which can be calc. using proj. resolution of $i\mathbb{Z}$ called Bar resolution.
  Group cohomology described in this way captures extensions, connections with crossed homomorphisms, Factor systems ... arising in group theory.
- When M is an ab. group, viewed as triv. G-mod. iM, then this coincides w' cohomology of top. space BG w' coefficients in M described earlier.

Projective dimension

Defⁿ) R a ring & $A \in \mathrm{Mod}_R$. A proj. resolution of form

$$\cdots 0 \to 0 \to P_n \cdots\cdots P_1 \to P_0 \to A \to 0$$

with all $P_i \neq 0$ for $i \leq n$ is said to have length n.

- The projective dimension, $pd(A)$, is the smallest n for which A has such a projective resolution.
- If no such $n \in \mathbb{N}$ exists, we say $pd(A) = \infty$.

Notation : For a proj. res. $P_\cdot$ of A as above, $K_n = \ker(P_n \to P_{n-1})$ are called n-syzygies.

Examples

① For A an abelian group, consider

$$0 \to \ker(\varepsilon) \hookrightarrow F_u A \xrightarrow{\varepsilon} A \to 0$$

free ab. group

-Then as each subgroup of free ab. group is free, $\ker(\varepsilon)$ is free, so A has proj. resolution of length 1.

② In the ex. , we saw that

when $R = \mathbb{Z}/4$, $\mathbb{Z}/2$ has a resolution

$$- - - \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

of infinite length.

- The following shows close relationship between projective dimension & Ext.

$\boxed{\text{Prop}}$ TFAE

① $\operatorname{pd} A \leqslant n$

② $\operatorname{Ext}^k(A,B) = 0$ all B & $k \geqslant n+1$.

③ $\operatorname{Ext}^k(A,B) = 0$ all B & $k = n+1$

④ For each proj. res. P of A, we have K_{n-1} is projective.

<u>Proof</u>

-Assume ① & let P be proj. resolution. Then $\operatorname{Ext}^k(A,B)$ is k'th cohomology of

$$\ldots \operatorname{Hom}(P_{k-1}, B) \longrightarrow \operatorname{Hom}(P_k, B) \longrightarrow \operatorname{Hom}(P_{k+1}, B).$$

but as $P_k = 0$, $\mathrm{Hom}(P_k, B) = 0$

so $\mathrm{Ext}^k(A,B) = 0$, proving ②.

② ⇒ ③ trivially.

- For ③ ⇒ ④, observe

$$\boxed{P_{n+2} \to P_{n+1} \to P_n \to K_{n-1}} \to P_{n-1}$$

is proj. res. of K_{n-1}

Hence $\mathrm{Ext}^1(K_{n-1}, B)$ is cohomology of

$$\mathrm{Hom}(P_n, B) \to \mathrm{Hom}(P_{n+1}, B) \to \mathrm{Hom}(P_{n+2}, B)$$

so $\mathrm{Ext}^1(K_{n-1}, B) = \mathrm{Ext}^{n+1}(A,B) = 0$;

hence K_{n-1} is projective.

For ④ ⇒ ③, take proj. resolution

$$0 \to K_{n-1} \hookrightarrow P_{n-1} \to P_{n-2} \to \cdots P_1 \to A.$$

Cor $\mathrm{pd}(A) = n$ if n is the largest number st ∃ non-zero $\mathrm{Ext}^n(A,B)$.

Remark • In the resolution

$$\cdots \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

$K_n = \mathbb{Z}/2$, which is not proj. as $\mathbb{Z}/4$-module (check.)

- Hence $\mathbb{Z}/2$ inf. proj d.

Calulation of Ext for abelian groups

- Of course $\mathrm{Ext}^0_{\mathbb{Z}}(A,B) = \mathrm{Ab}(A,B)$
- By the first, ex., since $\mathrm{pd}(A) \leq 1$, $\mathrm{Ext}^n_{\mathbb{Z}}(A,B) = 0$ all $n \geqslant 2$.
- $\mathrm{Ext}^1_{\mathbb{Z}}(A,B)$ is more complicated but we can easily calculate it if A is f.g. abelian group, since these are of form $A = \mathbb{Z}^m \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_k}$.
- Not hard to see $\mathrm{Ext}^k(-,B)$ pres direct. sums & since $\mathbb{Z}^m$ is free (so proj)

$$\mathrm{Ext}^1_{\mathbb{Z}}(A,B) = \mathrm{Ext}^1(\mathbb{Z}_{p_1},B) \oplus \cdots \mathrm{Ext}^1(\mathbb{Z}_{p_k},B).$$

- Now $0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \longrightarrow \mathbb{Z}/p \longrightarrow 0$ is ses giving proj. res of $\mathbb{Z}/p$, so $\mathrm{Ext}^1(\mathbb{Z}/p,B)$ is cohomology of

$$\begin{tikzcd}
\hom(\mathbb{Z},B) \arrow[r, "(\cdot p)^*"] \arrow[d, phantom, sloped, "\cong"] & \hom(\mathbb{Z},B) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & 0 \\
B \arrow[r, "p\cdot"] & B \arrow[r] & 0
\end{tikzcd}$$

ie.

$$\underline{B/pB}.$$

Hence $\mathrm{Ext}^1_{\mathbb{Z}}(A,B) = B/p_1B \oplus \cdots \oplus B/p_nB$.

- If A is not fin. gen, more complex.

Def^n The left projective dimension of a ring R is defined by $\text{lpd}(R) = \sup\{\text{pd}(A) : A \text{ a } \underline{\text{left}} \ R\text{-mod}\}$

Remark
- Dually, one can consider injective dimension, $\text{id}(A)$, the shortest length of an injective resolution of A.
- Since $\text{Ext}^k(A,B)$ can be calc. using a proj. res. of A or an inj res. of B, the above result has a dual version for inj. dimension.
- Again, one can consider left inj. dimension $\text{lid}(R)$. In fact,

Prop $\text{lpd}(R) = \text{lid}(R)$

Proof $\text{lpd}(R) \leq n \iff$

$\forall A \in \text{Mod}_R, \ \text{pd}(A) \leq n \iff$

$\forall A, B$ $\mathrm{Ext}^k(A,B) = 0$ all A, B, $k > n$

$\Longleftrightarrow$ $\forall B \in \mathrm{Mod}_R$, $\mathrm{id}(B) \leq n$

$\Longleftrightarrow$ $\mathrm{lid}(R) \leq n$

The common value $\mathrm{lgd} := \mathrm{lpd}(R) = \mathrm{lid}(R)$ is called the left global dimension of R.

- Of course, there is also right global dimension $\mathrm{rgd}(R)$, concerning right R-modules.

Of course, if R is commutative, they coincide, & we just speak of global dimension $\mathrm{gd}(R)$.

Hilbert's Syzygy Theorem

If R is a field, $\mathrm{gd}\, R[x_1, \ldots, x_n] = n$.

Exercise

- Consider the poly. ring $R = \mathbb{C}(x, y)$ length & let M be the max. ideal of polys with no zero term, and consider R/M as an R-module
- Consider the projection $R \longrightarrow R/M$. Extend this to a proj. resolution of length 2.