Representation Theory

Final 4 weeks:

group representations:

- basic theory of complex reps of finite groups including Maschke's theorem, Schur's lemma, decomposition results. (2 wks)
- Representations of symmetric group (1 wk)
- Tensor products & Hopf algebras. (1 wk)

Lecture 10 - Representation Theory of groups

Next 4 weeks - 2 weeks : basics, theory
- 1 week : symmetric group
- 1 week : Hopf algebras

Basic def^{n}s

- let k be a field & Vect denote the category of k-vector spaces & linear transf.
- Given U a vect. space, $End(U) = Vect(U,U)$ is a monoid with composition given by comp. of lin. transformations.
- When U is n-dim vect. space, $End(U) \cong Mat(n,k)$, the monoid of $n \times n$-matrices w/ values in k.

Def^n

(monoid would suffice)

Let G be a group. A G-module / G-repres. is a monoid homomorphism

$$\rho : G \longrightarrow End(U) :$$

that is, for each $g \in G$ an invertible lin. transformation $\rho(g): U \to U$ such that

$$\rho(gh) = \rho(g)\rho(h) \ \& \ \rho(e) = Id.$$

Remark

Equivalently,

- for each $g \in G$, $v \in U$ an elt $\rho(g)(v)$, which we write as gv:

  such that

- $g(v+w) = gv + gw$ & $g(\lambda v) = \lambda(gv)$
- $gh(v) = g(hv)$ & $ev = v$ where $e \in G$ is id.

Remark

- When U is n-dim vect. space, $\mathrm{End}(U) \cong \mathrm{Mat}(n, K)$, the monoid of $n \times n$-matrices w/ values in K.

Hence a G-module str. on U is specified by a homomorphism

$$\rho : G \longrightarrow \mathrm{Mat}(n, K),$$

which is often called an <u>n-dimensional matrix representation</u>.

Examples

① (<u>Trivial representation</u>)

$$G \longrightarrow \mathrm{End}(K, K) : g \longmapsto 1 : K \to K.$$

② (Some more 1-d representations)

$C_n = \langle g \mid g^n = 1 \rangle$ <u>cyclic group</u> of order n

<u>$K = \mathbb{C}$</u>. Then a 1-d rep. of C_n is a homomorph $C_n \longrightarrow \mathrm{End}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$: such is specified by an $\sigma(g) \in \mathbb{C}$ st $\sigma(g)^n = 1$ - hence $\sigma(g)$ is a n'th root of unity.

There are n such roots:

$$\sigma(g) = \cos(2k\pi/n) + i\sin(2k\pi/n) \text{ for}$$

$k = 0, \dots, n-1$. Eg. for C_4 these are $\{1, i, -1, -i\}$ & so n 1-d representations.

③ $D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ the <u>dihedral group</u>. This captures symmetries of the square, generated by a rotation of $90°$ & a reflection.

Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be the matrices for such a rot. & refl.

Defining $a, b \longmapsto A, B$ gives a 2-d representation since $A^4 = B^2 = 1$ & $B^{-1}AB = A^{-1}$.

Def^n) Let V, W be G-modules. A homomorph. of G-modules $\theta : V \longrightarrow W$ is a linear transformation such that $\theta(gv) = g\,\theta(v)$ all $g \in G, v \in V$.

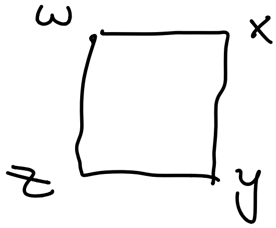
G-modules & homomorphisms form a category G-Mod.

Examples from group actions

- Let G act on a set X: we have bijections
  $$g.- : X \longrightarrow X \quad \text{st} \quad g(hx) = (gh)x \ \& \ ex = x.$$
- Let FX be the free vector space on X, with basis elements of X — we obtain
  $$g.- := F(g.-) : FX \longrightarrow FX$$
  $$\lambda_1 x_1 + \dots + \lambda_n x_n \longmapsto \lambda_1 g x_1 + \dots + \lambda_n g x_n.$$
  by linear extension.

- This is clearly a G-module, & such G-modules arising from actions are called permutation representation.

Ex ①: $D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ acts on corners $\{w, x, y, z\}$ of square



where a rotates by $90°$ & b swaps w & y.

Then we obtain permutation D_8-module w/ basis w, x, y, z. The matrix for a is then

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

② The regular G-module KG is the permutation module associated to the (left) action of G on itself: ie. KG has basis elts of G,

$$g(\lambda_1 g_1 + \dots + \lambda_n g_n) = \lambda_1 g g_1 + \dots + \lambda_n g g_n.$$

G-modules as representations over a ring / algebra

- Observe that KG is K-algebra (non-comm.):
- cert. KG is a K-vector space;
- multiplication

$$\left(\sum_{i=1}^{n}\lambda_i g_i\right)\left(\sum_{j=1}^{m}\mu_j g_j\right)=\sum_{i=1}^{n}\sum_{j=1}^{m}(\lambda_i\mu_j)(g_ig_j)$$

& multiplicative unit e.

- If V is a K-vector space, then $\mathrm{End}(V)$ is a K-algebra where K-vector space str. is componentwise as in V.

By def^n, a representation of a K-algebra A is a K-alg. hom. $A \longrightarrow \mathrm{End}(V)$.

Prop^n There is a bij^n between

1. Representations of G;
2. Representations of K-algebra KG;
3. Modules over the ring KG.

Proof

- A rep of G is equally a monoid hom $G \xrightarrow{\sigma} \mathrm{End}(V)$.

This admits a ! extension along $G \to KG : g \mapsto g$ to a k-alg hom. $kG \xrightarrow{\sigma} \mathrm{End}(U)$ given by

$$\lambda_1 g_1 + \dots + \lambda_n g_n \longmapsto \lambda_1 \sigma(g_1) + \dots + \lambda_n \sigma(g_n),$$

& this gives the bij^n between ① & ②.

(Exercise: show $G \longmapsto kG$ provides left adjoint to forg. functor $k\text{-Alg} \longrightarrow \text{Mon}$.)

– Given ②, we have a kG-module str. on underlying abelian group of U:

$$kG \longrightarrow \mathrm{Vect}(U,U) \longrightarrow \mathrm{Ab}(U,U).$$

Conversely, given a ring hom

$\sigma : kG \longrightarrow \mathrm{Ab}(A,A)$ (ie. kG-module),

then restriction along

$k \longrightarrow kG : \lambda \longmapsto \lambda.e$ gives a ring hom $k \longrightarrow kG \longrightarrow \mathrm{Ab}(A,A)$

making A a k-vector space, in such a way that σ lifts to an alg map

$kG \longrightarrow \mathrm{Vect}(A,A)$. These constructions are inverse. $\square$

· In particular, $G\text{-Mod} \simeq \mathrm{Mod}_{kG}$, so everything we know about module cats (kernels, quotients, direct sums etc) holds for G-Mod.

A few Facts & Terminology for G-modules

- Given $\theta : V \to W \in G\text{-Mod}$, we can form $\ker\theta \leq V$ & $\operatorname{im}\theta \leq W$ which are G-submodules.

Direct sums (of submodules)

- If W a vector space, $U, V \leq W$ are subspaces then $U + V = \{u+v : u \in U, v \in V\} \leq W$ is a subspace.
- If given $w \in W$ $\exists!\, (u,v) \in U \times V$ st $w = u+v$, we say W is (internal) direct sum of U & V and write $W = U \oplus V$.
  This is equiv. to saying $W = U+V$ & $U \cap V = 0$.
- Of course, then $U \times V \cong U \oplus V$, so this is direct sum in usual sense.
- If W is a G-module, & $U, V \leq W$ submodule s.t. $W = U \oplus V$ as above, say W is direct sum $U \oplus V$ of G-submodules.

Projections

- A projection $p : V \to V$ of G-modules is a homom. s.t. $p^2 = p$.

$\boxed{\text{Prop}^n}$ (Note: holds for modules over any ring.)

If p is a proj^n, then $V = \operatorname{im} p \oplus \ker p$ & each direct sum arises from a proj^n in this way.

$\boxed{\text{Proof}}$ - Write $v = \overset{\in \operatorname{im} p}{pv} + \overset{\in \ker p}{(v - pv)}$

This is unique since if $v \in \operatorname{im} p \cap \ker p$, then $pv = 0$ but as $v = px$, $0 = pv = ppx = px = v$.

- If $W = U \oplus V$, define $p: W \to W : u + v \mapsto u$. This is proj^n with $\operatorname{im} p = U$ & $\ker p = V$. $\square$

Decomposing G-modules

Def^n) A G-module V is reducible if it contains a non-trivial submodule. A non-triv. G-module is irreducible if it is not reducible.

Theorem (Maschke)

Let G be a finite group & suppose char(K) does not divide order of G, $|G|$. (eg. if $k = \mathbb{R}$ or $\mathbb{C}$)

If V is a G-module & $U \leq V$ a proper submodule, then G-submodule W st $V = U \oplus W$.

~~Proof~~

- Firstly, as U subspace of V, can find linearly independent vectors giving subspace W_0 s.t. $V = U \oplus W_0$ as a vector space.
- This gives a projection of vector spaces $p: V \to V : u + w \mapsto u$ with image U & kernel W_0, but p need not be a G-module map.
- We will modify p to a G-module map

$q : V \to V$ st $q^2 = q$ & $\operatorname{im} q = U$;
then $V = \operatorname{im} q \oplus \ker q = U \oplus \ker q$, a dir. sum of G-submodules, as required.

- For $g \in G$, let $p_g : V \to V : v \mapsto g^{-1}(p(gv))$.
  As a composite of 3 linear maps, p_g is linear map.
- Define $q = \frac{1}{|G|} \sum_{g \in G} p_g$ as the "average" of these maps, which is again linear as it is a linear comb. of linear maps.
- To check q a G-module map:

$q(hv) = \frac{1}{|G|} \sum_{g \in G} g^{-1} p(g.hv)$.

Since each elt of G is uniquely of form gh^{-1} for some $g \in G$,

$$\frac{1}{|G|} \sum_{g \in G} g^{-1} p(g.hv) =$$

$$\frac{1}{|G|} \sum_{g \in G} (gh^{-1})^{-1} p(gh^{-1}.hv) \quad \text{using } U \text{ a } G\text{-mod.}$$

$$= \frac{1}{|G|} \sum_{g \in G} hg^{-1} p(gv)$$

$$= h.\frac{1}{|G|} \sum_{g \in G} g^{-1} p(gv) = h.q(v).$$

- Remains to show $\operatorname{im} q = U$.
  Let $u \in U$. Then

$q(u) = \frac{1}{|G|} \sum_g g^{-1} p(gu)$ (as $gu \in U$ so $p(gu) = gu$)

$= \frac{1}{|G|} \sum_g g^{-1} g u$

$= \frac{1}{|G|} \sum_{g \in G} u = \frac{|G|}{|G|} u = u.$

- Hence $u \leq \operatorname{im} q$. To see $\operatorname{im} q \leq U$, observe that since p takes it's image in U, each $g^{-1} p(gv) \in U$; hence $q(v) \in U$ all $v \in V$.

Therefore $\operatorname{im} q = U$.

- Since $qv \in U$ & $qu = u$ all $u \in U$, we get $q q v = q v$ all $v \in V$, as required. $\square$

Theorem

Let G be a finite group & suppose $\operatorname{char}(K)$ does not divide order of G, $|G|$. (eg. if $K = \mathbb{R}$ or $\mathbb{C}$).

Then each non-zero finite dim. G-module V admits a decomposition

$V = V_1 \oplus \dots \oplus V_n$ as direct sum of irreducible G-submodules.

Proof) - By ind. on dimension of V.

- If $\dim V = 1$, trivial as each 1-d G-module is irreducible.
- Else, suppose it is true for all W st. $\dim(W) < \dim(V)$.
- Suppose $U \leq V$ is a proper G-submodule. Then by Mashke's Theorem,

$V = U \oplus W$ for U, W proper submodules

Then $\dim(U), \dim(W) < \dim(V)$ so

$$V = U \oplus W = (U_1 \oplus \cdots \oplus U_m) \oplus (V_1 \oplus \cdots \oplus V_\ell)$$

where all the U_i & V_j are irreducible.

$\square$

Lecture 11 - Group representations ctd.

Last time: decomposition into irred.

Today: finer decomposition results,
in partic. when $k = \mathbb{C}$.

Prop $k[G]$ is the free G-module on 1.

Proof Follows from fact (last time) that

G-modules $\equiv$ $k[G]$-modules for $k[G]$ ring,

& free $k[G]$-module on 1 is of course $k[G]$.

Explicitly,

$$\begin{tikzcd} k[G] \arrow[dr, dashed, "\exists! f\ G\text{-mod map}"] & \\ 1 \arrow[u, "e"] \arrow[r, "a"'] & U \end{tikzcd}$$

Must have $f(e) = a$.

Then need $f(g) = f(g.e) = g.f(e) = g.a$

& for linearity

$$f\Big(\sum_{i\in I} \lambda_i g_i\Big) = \sum_{i\in I} \lambda_i g_i . a \,. \qquad \square$$

Schur's lemma

Let $\theta : U \longrightarrow W$ be a morph. of irred. G-modules.

(1) Then $\theta = 0$ or θ is an isomorph.

(2) If K is algebraically closed, & $U = W$ of finite dimension then $\exists \lambda \in K$ st
$$\theta(v) = \lambda v \text{ all } v \in V.$$

(1) As U irr., $\ker\theta = 0$ or V

As W irr, $\operatorname{im}\theta = 0$ or W.

- If $\theta \neq 0$, then $\ker\theta \neq V$ & $\operatorname{im}\theta \neq 0$

so $\ker\theta = 0$ & $\operatorname{im}\theta = W$.

Hence θ is inj & surj $\Rightarrow$ an iso.

(2) Since K is alg. closed,

the poly. in λ, $\det(\theta - \lambda . \mathrm{Id}) = 0$, has a solution (eigenvalue).

But then $\theta - \lambda . \mathrm{Id} : U \longrightarrow U$ has non-zero kernel (the eigenvector)

& so kernel $= U$ - hence $\theta = \lambda . \mathrm{Id}$. $\square$

Prop

Let G be finite gp st. char(K) does not divide $|G|$.

Let $K[G] = U_1 \oplus \dots \oplus U_n$ be a decomp. into irreducibles (last time).

Then every irreducible G-module U is iso. to one of the U_i.

Proof

- Let $v \in U$ be non-zero.
- By freeness of $K[G]$, $\exists! \, \theta : K[G] \to U$ st $\theta(e) = v$.
- Then $\mathrm{im}\,\theta \leq U$; since non-zero & U irr. $\mathrm{im}\,\theta = U$.
- Consider $\ker\theta \leq K[G]$. By Maschke's Theorem, $K[G] = \ker\theta \oplus V$; consider composite

$$\begin{tikzcd} V \arrow[r, hook, "i"] \arrow[rr, bend right, "\bar{\theta}"'] & \ker\theta \oplus V \arrow[r, "\theta"] & U \end{tikzcd}.$$

Will show $\bar{\theta}$ is an iso; then

$$K[G] = \ker\theta \oplus V \cong \ker\theta \oplus U$$

so that U is iso. to a submodule

of $K[G]$.

Now $\bar{\theta}(v) = \theta(v)$.

- Let's show $\ker\bar{\theta} = 0$.

If $\bar{\theta}(v) = 0$ then $\theta(v) = 0$ so $v \in \ker\theta \cap V = 0$; hence $v = 0$.

- Since θ is surj., given $u \in U$

$\exists\ a+b \in \ker\theta \oplus V$ st. $\theta(a+b) = u$.

But $\theta(a+b) = \theta a + \theta b = \theta b$ as $a \in \ker\theta$; hence $\bar{\theta}(b) = u$, as required, so $\bar{\theta}$ surj.

Hence $\bar{\theta}$ an iso.

Therefore suffices to prove Theorem when U is a submodule of $K[G]$.

- For $K[G] = U_1 \oplus \cdots \oplus U_n$, consider

$$K[G] \xrightarrow{p_i} U_i$$
$$u_1 + \cdots + u_n \longmapsto u_i.$$

- Since U is non-zero, one of the composites

$$U \hookrightarrow K[G] \longrightarrow U_i$$ must be non-zero, & so invertible by Schur's lemma Part i. $\square$

Cor) Let G be finite gp st. char(K) does not divide $|G|$.
Then there are only finitely many irreducible G-modules up to iso.

Def^n) $\{U_1, \dots, U_n\}$ is a complete set of irred. G-modules if no two are iso. & every irreduc. G-module is iso to one of them.

Def^n) Let U, W be G-modules. Write
$\mathrm{Hom}_{K[G]}(U, W)$ for vector space of G-module maps from U to W with pointwise operations.

Remark: $\mathrm{Hom}_{K[G]}(U, W)$ need not be a G-module unless G is commutative!

Finer results when $k = \mathbb{C}$

In this subsection, assume G is finite & $k = \mathbb{C}$.

Prop^n) Let V, W be irreducible finite-dimensional G-modules. Then

$$\dim \mathrm{Hom}_{\mathbb{C}(G)}(U, W) = \left\{ \begin{matrix} 1 & \text{if } U \cong W \\ 0 & \text{otherwise} \end{matrix} \right\}$$

Proof

- $\dim \mathrm{Hom}_{\mathbb{C}(G)}(U, W) = 0 \iff$ only homomorphism $U \to W$ is zero $\iff$ (by Schur) $U \not\cong W$.
- If $\dim \mathrm{Hom}_{\mathbb{C}(G)}(U, W) = 1$, $\exists$ non-zero hom. $U \to W$, which is an iso (by Schur)
- If $U \cong W$, we obtain iso of vector spaces

$$\mathrm{Hom}_{\mathbb{C}(G)}(U, U) \cong \mathrm{Hom}_{\mathbb{C}(G)}(U, W)$$

so it suffices to show lhs has dim 1.

But by Schur's lemma Part 2, each $f : U \to U$ equals $\lambda . \mathrm{Id}$ - thus lhs has basis $\{ \mathrm{Id} : U \to U \}$, & so has dim. 1 $\square$

Theorem

Let $V \neq 0$ be a f.d. G-module.
Then

1. $V = V_1 \oplus \cdots \oplus V_n$ where the V_i are irreducible.
2. Each irreducible G-module W appears in decomp., up to iso, $\dim(\mathrm{Hom}_{\mathbb{C}G}(V, W))$ times.
3. In particular, let $U_1, \ldots, U_m$ is a complete set of irreducible G-modules. Then
$V \cong U_1^{d_1} \oplus \cdots \oplus U_m^{d_m}$ where
$d_i = \dim(\mathrm{Hom}_{\mathbb{C}G}(V, U_i))$.

Proof

- Proved (1) last week.
- For (2), we have

$$\mathrm{Hom}_{\mathbb{C}G}(V, W) = \mathrm{Hom}_{\mathbb{C}G}(V_1 \oplus \cdots \oplus V_n, W)$$
$$\cong \mathrm{Hom}_{\mathbb{C}G}(V_1, W) \oplus \cdots \oplus \mathrm{Hom}(V_n, W)$$

since direct sum is a coproduct & restriction along each $V_i \hookrightarrow V$ is linear.

Taking dimensions,

$$\dim(\mathrm{Hom}_{\mathbb{C}G}(V, W)) = \sum_{i=1}^{n} \dim(\mathrm{Hom}_{\mathbb{C}G}(V_i, W))$$
$$= \sum_{i : V_i \cong W} 1$$ by prev. proposition, i.e.

the number of i st. $V_i \cong W$.

For 3, by 2,

$$U_1 \oplus \cdots \oplus U_n =$$

$$\underbrace{(U_{11} \oplus \cdots \oplus U_{1n_1})}_{\substack{\text{those } U_i \text{ iso to } U_1 \text{ -} \\ \text{by 2 there are} \\ d_1 \text{ of these}}} \oplus \cdots \oplus \underbrace{(U_{m1} \oplus \cdots \oplus U_{mn_m})}_{\substack{\text{those } U_i \text{ iso to } U_m, \\ \text{of which there} \\ \text{are } d_m}}$$

$$\cong U_1^{d_1} \oplus \cdots \oplus U_m^{d_m}. \quad \square$$

Corollary

Let $U_1, \ldots, U_m$ be complete set of irreducibles. Then $\mathbb{C}[G] \cong U_1^{\dim(U_1)} \oplus \cdots \oplus U_m^{\dim(U_m)}$.

Proof

By Part 3 of previous result, we must show

$$d_i := \dim(\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U_i)) = \dim(U_i)$$

In fact, since $\mathbb{C}[G]$ is free G-module on 1, we have bij^n

$$\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U_i) \cong U_i$$

$$F \longmapsto F(e)$$

& this evaluation map is clearly linear, hence an iso. of vector spaces.

Therefore lhs & rhs have same dimension. $\square$

Cor

$$|G| = \sum_{U_1,\dots,U_m} \dim(U_i)^2 .$$

Proof

Since

$$\mathbb{C}[G] \cong U_1^{\dim(U_1)} \oplus \dots \oplus U_m^{\dim(U_m)}$$

Taking dimensions of lhs & rhs proves claim as

$$\dim(\mathbb{C}[G]) = |G| . \qquad \square$$

Remark: Above formula relating order of G with number of its irreducible reps is very useful in calculating all irreps. of a finite group.

Example

- Consider $D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$.

By prev. Thm,

$$|D_8| = 8 = \sum_{U_1, \dots, U_m} \dim(U_i)^2$$

$$= 2^2 + 4 \cdot 1^2$$

$$= 8 \cdot 1^2$$

so 1 2-d irrep & 4 1d irreps

or 8 1d irreps.

- Recall 2-d real rep:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \&$$

view as a complex 2-d rep — ie. rep. on $\mathbb{C}^2$.

- As a 2-d rep, a non-Triv. submodule must be 1-d subspace $\langle v \rangle$ st. $gv = \lambda v \in \langle v \rangle$ for each $g \in V$:

ie. v should be eigenvector for

both A & B.

- Can calc. eigenvectors of A which are $(1, i)$ & $(1, -i)$
& of B $(1,0)$ & $(0,1)$
but they have none in common.
Hence this is irreducible 2-d rep.

Therefore D_8 has one 2-d irrep & 4 1-d irreps: ie 4 1-d reps.

A 1-d rep is simply a homomorphism

$$D_8 \longrightarrow (\mathbb{C}, \cdot, 1)$$

& it is easy to see these are given by

$$a, b \longrightarrow (\pm 1, \pm 1)$$

so these are 1-d irreps.
Hence we have calc.

all complex irreps of D_8.

Lecture 12 - The symmetric group

Goal: glance at irreps of symmetric group S_n.

Omitted: characters of groups

Theorem: For G a finite group,
no of iso classes of complex irreps of G
= no of conjugacy classes of G

Recall $a, b \in G$ are conjugate $(a \sim b)$ if
$\exists g \in G$ st $g^{-1} a g = b$.
E-classes of $\sim$ are called conjugacy classes.

- The symmetric group S_n is the group of permutations of the set $\{1, \dots, n\}$.
Each $g \in S_n$ can be written as a product of disjoint cycles:
eg. $(45)(132)(6) \in S_6$ & its cycle type is the list of orders of its cycles
in this example $\{2, 3, 1\}$.

- Moreover $g, h \in S_n$ are conjugate $\iff$ they have the same cycle type.

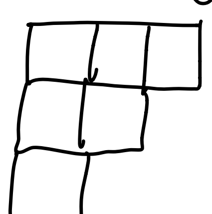
So cycle types $\sim$ conjugacy classes

- Cycle types are parametrised by partitions of n:

sequences $(\lambda_1, \ldots, \lambda_t)$ with $\lambda_i \geq \lambda_{i+1}$ st $\sum_{i=1}^{t} \lambda_i = n$.

Eg. $(3,2,1)$

- We write $\lambda \vdash n$ to indicate λ is a partition of n.
- By theorem, irreps of S_n are parametrised by partitions $\lambda \vdash n$.
- Partition $\lambda = (\lambda_1, \ldots, \lambda_t) \vdash n$ can be represented by an array with t rows where i'th row has length λ_i.

E.g. $(3,2,1) \sim$ 

Array called shape of the partition λ.

- A Young tableau t of shape $\lambda \vdash n$ (or λ-tableau) is an array of shape λ whose entries are bij. Filled with $\{1, \ldots, n\}$.

  - E.g.

4	6	5
3	2	
1		

    is λ-tableau for $\lambda = (3,2,1)$

- Observe there is bij^n between λ-tableau & elements of S_n.

  E.g. above λ-tableaux $\sim$
  $1,2,3,4,5,6 \mapsto 4,6,5,3,2,1$
  so $n!$ λ-tableaux for each $\lambda \vdash n$.

- S_n acts on the set of λ-tableaux in obvious way:
  $(gt)_{i,j} = g(t_{i,j})$ by applying permutations g to entries of tableaux:

  eg.

$$(1\ 3)\ \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

- Two λ-tableaux s, t are <u>row equivalent</u> if entries of each row of s, t coincide.

E.g. $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$ & $\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$ are <u>row equiv</u>.

- Row equivalence classes $\{t\}$ are called <u>λ-tabloids</u> : diagramatically remove boxes from rows

eg. $\begin{array}{c c}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$, $\begin{array}{c c}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$ each represent the above two λ-tableau.

<u>Lemma</u>

The action of S_n on λ-tableaux respects row equivalence & so induces an action of S_n on <u>set of λ-tabloids</u>.

Proof/ Let s & t be row equiv. ($s \sim t$). Must show for $g \in S_n$

that $gs \sim gt$ & suffices to do this for generators – transp, $\sigma = (i\ j) \in S_n$.

Suppose $i \in \text{row}_n$ of s & t

-- $j \in \text{row}_m$ of s & t

Then $i \in \text{row}_m$ of $\sigma s, \sigma t$

$j \in \text{row}_n$ of $\sigma s, \sigma t$

which are otherwise unchanged; hence $\sigma s \sim \sigma t$. $\square$

Let $\{t_1\}, \ldots, \{t_m\}$ be the complete set of λ-tabloids.

Def^n) We define

$$M^\lambda = \mathbb{C}(\{t_1\}, \ldots, \{t_m\})$$ to

be the corresponding permutation representation (i.e. w/ basis elements $\{t_1\}, \ldots, \{t_m\}$.

Typical element of M^λ:

e.g. $$7\,\begin{array}{|cc|}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array} + \begin{array}{|cc|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}.$$

Examples

(1) $\lambda = (n)$, only one λ-tabloid $\overline{\underline{1\,2 \cdots\cdots n}}$ so $M^{(n)} = \mathbb{C}(\overline{\underline{1\,2\cdots n}})$ with trivial action of S_n – ie. trivial rep of S_n.

(2) $\lambda = (1,1,\ldots,1)$ no 2 tableaux are row equiv. as rows have length 1, so a tabloid $\sim$ tableau $\sim$ elt of S_n; hence $M^{(1,1,\ldots,1)} \cong \mathbb{C}\{S_n\}$ the regular representation (ie. free S_n-module on 1 ε.

(3) $\lambda = (n-1,1)$:

λ-tabloid $\sim$ choice of elt on second row.

Write $\bar{i} = \begin{array}{|l|}\hline - - - - - - - \\ \hline i \\ \cline{1-1}\end{array}$. Hence

$M^\lambda = \mathbb{C}\{\bar{1},\ldots,\bar{n}\}$ which is iso to perm. rep. ind. by action of S_n on $\{1,\ldots,n\}$.

Polytabloids & Specht modules

Def^n) Let t be a λ-tableau.
The column stabiliser $C_t \leq S_n$ consists of those $g \in S_n$ which permute elements within each column of t.

- If t has columns $c_1, \ldots, c_k$ then
$$C_t = S_{c_1} \times \cdots \times S_{c_k}:$$

eg. $t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 & \\ \cline{1-2}\end{array}$ then

$$C_t = S_{4,3} \times S_{1,5} \times S_2 = \{(4\,3), (1\,5), (4\,3)(1\,5), e\}$$

For t a λ-tableau, the <u>associated polytabloid</u> $e_t \in M^\lambda$ is the element
$$e_t = \sum_{g \in C_t} \operatorname{sign}(g) . g\{t\} \in M^\lambda,$$
where $\{t\}$ is λ-tabloid associated to t.

<u>Example</u>

In above case, $e_t =$

$$\begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 & \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline 4 & 5 & \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 4 & 5 & 2 \\ \hline 3 & 1 & \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 3 & 5 & 2 \\ \hline 4 & 1 & \\ \cline{1-2}\end{array}.$$

Def^n The Specht module S^λ is S_n-submodule
$\langle e_t : t \text{ a } \lambda\text{-tableau} \rangle \leq M^\lambda$

Remark

One can show $g e_t = e_{gt}$: hence S^λ consists of linear combinations of the ass. polytabs e_t.

Theorem (see eg. notes on my webpage)

The Specht modules S^λ are irreducible & form a complete set of irreducible S_n-modules for $\lambda \vdash n$.

Examples

① $\lambda = (n)$, one λ-tabloid $\overline{\underline{1\ 2 \cdots n}}$.

For each tableau t, C_t is trivial, hence $e_t = \overline{\underline{1\ 2 \cdots n}}$, the unique λ-tabloid.
Then $S^{(n)} = M^{(n)} = \mathbb{C}(\overline{\underline{1\ 2 \cdots n}})$ the trivial S_n-module.

② $\lambda = (1, 1, \ldots, 1)$. $M^\lambda \cong \mathbb{C}\{S_n\}$.

Let t $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline \end{array}$. Then $C_t = S_n$. Will show
$e_{\pi t} = \operatorname{sgn}(\pi) e_t$ - hence
$S^\lambda = \langle e_t \rangle \leq M^\lambda$ so
$S^\lambda \cong \mathbb{C}$ with so-called

sign representation $\quad g\cdot a = \mathrm{sgn}(g)\cdot a$.

Proof of claim:

$$e_{\pi t} = \pi e_t = \pi \sum_{\theta} \mathrm{sign}(\theta)\,\theta\{t\}$$

not prove, but true for all λ, t

$$= \sum_{\theta} \mathrm{sign}(\theta)\,\pi\theta\{t\}$$

$$= \sum_{\theta} \mathrm{sign}(\pi^{-1}\theta)\,\pi(\pi^{-1}\theta\{t\})$$

$$= \mathrm{sign}\,\pi^{-1} \sum_{\theta} \mathrm{sign}\,\theta\;\theta\{t\}$$

$$= \mathrm{sign}(\pi)\, e_t.$$

The above are two irred. 1-d reps.

(3) $\lambda = (n-1,1)$, $M^\lambda = \mathbb{C}\{\bar{1}, \bar{2}, \dots, \bar{n}\}$

let $t = \begin{array}{|c|c|c|c|c|c|}\hline i & - & - & - & - & \cdot \\ \hline k \\ \cline{1-1}\end{array}$ so $\{t\} = \bar{k}$.

Then $C_t = \{e, (ik)\}$ so $e_t = \bar{k} - \bar{i}$.

Thus $S^\lambda = \langle \bar{i} - \bar{j} : i \neq j \rangle \leq M^\lambda$ & this spans subspace

$$\{c_1\bar{1} + \dots c_n\bar{n} : \sum_{i=1}^{n} c_i = 0\}$$

& has basis the vectors $\{\bar{i} - \bar{1} : i \neq 1\}$ & so is of dim. $n-1$.

- Saw this example for S_3 as
  $\langle 3-2, 2-1 \rangle \leq \mathbb{C}\{1,2,3\}$ in exercises.

Final note on basis

A λ-tableau is standard if its rows & columns form increasing sequences:

eg
$$\begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array} \quad \text{but not} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 4 & 3 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

Theorem

The set $\{e_t : t \text{ standard } \lambda\text{-tableau}\}$ form a basis for S^λ.

Remark:

Lots of connections between reps. of symmetric group & other areas:

- combinatorics, probability (eg. card shuffling)

...

eg. see "The symm group: reps, combinatorial algo & symm. functions".

Lecture 13 - A glance at Hopf algebras

(Not examinable!)

Part 1 - Tensor products & internal homs

- Familiar with tensor product $U \otimes W$ of k-vector spaces (eg. Algebra 3) which classifies bilinear maps.
- Elements of $U \otimes W$ are lin. combos of $v \otimes w$ where $v \in U$ & $w \in W$.
- We have isos of vector spaces

$$\alpha : (U \otimes W) \otimes X \cong U \otimes (W \otimes X),$$
$$s : U \otimes W \cong W \otimes U,$$
$$l : k \otimes U \cong U \quad \& \quad r : U \otimes k \cong U,$$

making $(\mathrm{Vect}, \otimes, k)$ into a symmetric monoidal cat.

- We have bijections

$$\mathrm{Vect}(U \otimes W, A) \cong \underbrace{\mathrm{Bilin}(U, W; A)}_{\text{set of bilin. maps}} \cong \mathrm{Vect}(U, [W, A])$$

where $[W,A] = \mathrm{Vect}(W,A)$ equipped w/ pointwise vector space structure.

Hence $- \otimes W \dashv [W, -]$ are adjoint, making $(\mathrm{Vect}, \otimes, k, [-,-])$ into a symmetric mon. closed category - & call $[W,A]$ the internal hom.

Example

$(\underbrace{\mathrm{Set}}, \underbrace{\times}_{\text{cart. prod.}}, \underbrace{1}_{\text{1 elt set}})$ is a symmetric monoidal closed category.

Here $[X, Y] = \mathrm{Set}(X, Y)$

- We have forgetful functor $U: G\text{-Mod} \longrightarrow \text{Vect}$ & (in fact) the symmetric monoidal closed structure lifts along U to G-Mod:

- if V, W are G-modules, then $V \otimes W$ becomes a G-module on defining
$$g(v \otimes w) = gv \otimes gw;$$
more abstractly, $g.- : V \otimes W \longrightarrow V \otimes W$ is the unique linear map st

$$\begin{tikzcd} V \times W \arrow[r, "g.- \times g.-"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "="] & V \times W \arrow[d, "\otimes"] \\ V \otimes W \arrow[r, "\exists ! g.-"] & V \otimes W \end{tikzcd}$$

univ. bil. map

Follows from either description that $V \otimes W$ is a G-module.

In fact, $V \otimes W$ classifies G-bilinear maps: bilinear maps $V \times W \xrightarrow{F} A$ st.
$$g.F(v,w) = F(gv, gw).$$

- The unit is k equipped w/ trivial G-mod. structure.

- The isos α, s, l, r lift to isos of G-modules – so G-Mod has s.mon str. pres. by U.

- The internal hom $[V,W] = \text{Vect}(V,W)$ becomes a G-module if we define at $f \in [V,W]$:
$$(g.f)v = g(f(g^{-1}v)):$$

equiv.
$$\begin{tikzcd} V \arrow[r, "g.f"] \arrow[d, "g^{-1}.-"'] \arrow[dr, phantom, "="] & W \\ V \arrow[r, "f"] & W \arrow[u, "g.-"'] \end{tikzcd}$$

& so is linear, as a comb of 3 linear maps,

& is easy to see this makes $[U,W]$ a G-module, and that

$$G\text{-Mod}(U \otimes W, A) \cong G\text{-Bilin}(U,W;A) \cong G\text{-Mod}(U,[W,A])$$

so $- \otimes W \dashv [W,-]$ again making G-Mod s-mon closed.

- Can express the above story in much greater generality, using bialgebras & Hopf algebras.

Part 2 - Bialgebras & Hopf algebras

- Let $(\mathcal{C}, \otimes, I)$ be a symm. mon. cat (smcat) eg. $(\text{Set}, \times, 1)$ or $(\text{Vect}, \otimes, k)$.
- Will write as if $\mathcal{C}$ is strict monoidal (strictly associative & unital) as is justified by Maclane's coherence theorem. And write AB for $A \otimes B$ etc, to save space.
- A monoid / algebra in $\mathcal{C}$ is an ob $A \in \mathcal{C}$ + $AA \xrightarrow{m} A$ & $I \xrightarrow{e} A$ st. the diagrams

$$\begin{tikzcd} AAA \arrow[r, "m1"] \arrow[d, "1m"'] & AA \arrow[d, "m"] \\ AA \arrow[r, "m"'] & A \end{tikzcd} \quad \& \quad \begin{tikzcd} A \arrow[r, "e1"] \arrow[dr, "1"'] & AA \arrow[d, "m"] & A \arrow[l, "1e"'] \arrow[dl, "1"] \\ & A & \end{tikzcd}$$

commute.

Ex: In $(\mathrm{Set}, \times, 1)$, an algebra $\equiv$ monoid
$(\mathrm{Vect}, \otimes, k)$, — — — — $\equiv$ k-algebra

- A comonoid / coalgebra is: ob $A \in \mathcal{C}$ + $A \xrightarrow{\Delta} AA$, $A \xrightarrow{q} I$ sat. dual coassoc., counit axioms.

(Equiv., algebra in $(\mathcal{C}^{op}, \otimes, I)$.)

- A bimonoid / bialgebra is
  - alg (A, m, e) +
  - coalg (A, Δ, q) s.t.

·
$$\begin{tikzcd} AA \arrow[r, "\Delta\Delta"] \arrow[d, "m"'] \arrow[dr, phantom, "="] & AAAA \arrow[d, "1s1"] \\ & AAAA \arrow[d, "mm"] \\ A \arrow[r, "\Delta"] & AA \end{tikzcd} \qquad \begin{tikzcd} I \arrow[d, "e"'] \arrow[dr, "ee"] & \\ A \arrow[r, "\Delta"] & AA \end{tikzcd}$$

&
$$\begin{tikzcd} I \arrow[r, "e"] \arrow[dr, "1"'] \arrow[r, phantom, shift right=3, "="] & A \arrow[d, "q"] \\ & I \end{tikzcd}, \qquad \begin{tikzcd} AA \arrow[r, "m"] \arrow[dr, "qq"'] & A \arrow[dl, "q"] \arrow[l, phantom, shift left=3, "="] \\ & I \end{tikzcd}.$$

Example

- In Set (or any cartesian mon. cat.) each ob. has ! coalgebra str.

$$\begin{array}{ccccc} 1 & \xleftarrow{!} & X & \xrightarrow{\Delta} & X \times X \\ & & x & \longmapsto & (x,x) \end{array}$$

- Then a bialgebra in Set $\equiv$ monoid.

A <u>Hopf algebra</u> is a bialgebra + a map $a : A \longrightarrow A$ (called <u>antipode</u>) st.

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[drr, "q"'] & AA \arrow[r, shift left, "a1"] \arrow[r, shift right, "1a"'] & AA \arrow[r, "m"] & A \\ & & I \arrow[ur, "e"'] & \end{tikzcd}$$

$=$

<u>Examples</u>

- In Set, this says
  $x \longmapsto (x,x) \longmapsto (a(x), x) \longmapsto a(x).x = e$
  & sim. $x.a(x) = e$, so
  <u>a Hopf algebra $\equiv$ group</u>
  where $a(x) = x^{-1}$.
- Hopf algebra in $(\mathrm{Vect}, \otimes, K)$ is what is trad. called a Hopf algebra.
  Will show that <u>group algebra</u> $K[G]$ is a <u>Hopf algebra</u>.

Firstly, observe that free v. space functor $k[-] : \mathrm{Set} \longrightarrow \mathrm{Vect}$ sat

$$k[X \times Y] \cong k[X] \otimes k[Y], \quad k(1) \cong k$$

in a way compatible with assoc., unit, symmetry isos in these smcats.

Therefore $k[-]$ takes Hopf algebras in Set to Hopf algebras in Vect — so if G is a gp (ie. Hopf alg in Set) then $k[G]$ is a Hopf algebra in Vect.

Indeed $k[G]$ is an alg. (the group alg.) w/ str.

$$k(G) \otimes k(G) \cong k[G \times G] \xrightarrow{k[m]} k(G) \;\&$$

$$k \cong k(1) \xrightarrow{k(e)} k(G)$$

& coalg str

$$k[G] \xrightarrow{k[\Delta]} k[G \times G] \cong k[G] \otimes k[G]$$

& $\quad k[G] \xrightarrow{k(!)} k[1] \cong k$ which act on basis elts as

$$g \longmapsto g \otimes g$$

$$g \longmapsto 1$$

& antipode

$$k[G] \xrightarrow{k[a]} k[G]$$

$$g \longmapsto g^{-1}.$$

So the group algebra $k[G]$ is a Hopf algebra!

Ex 3) Recall $Var^{op} \xrightarrow{k(-)} \text{comm-}k\text{-Alg}$ which sends products to tensor products.

Follows that sends groups in Var (algebraic groups) to Hopf algebras: ie. co-ordinate ring of an algebraic group is Hopf algebra.

More examples – un. env. alg. of a Lie algebra, tensor algebras ...

Modules over an algebra in symm. mon. cat

- If (A, m, e) is an alg a (left) A-module is ob $X \in \mathcal{C}$ + $AX \xrightarrow{x} X$ st.

$$\begin{tikzcd}
AAX \arrow[r, "1x"] \arrow[d, "m1"'] \arrow[dr, phantom, "="] & AX \arrow[d, "x"] & X \arrow[l, "e1"'] \arrow[dl, "1"] \arrow[l, phantom, "=", shift right=3] \\
AX \arrow[r, "x"'] & X &
\end{tikzcd}$$

Morph. $f: (X, x) \longrightarrow (Y, y)$ of A-modules is $f: X \longrightarrow Y$ such that

$$\begin{tikzcd} AX \arrow[r, "1f"] \arrow[d, "x"'] & AY \arrow[d, "y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Obtain a cat Mod_A & forgetful functor $U : \mathrm{Mod}_A \longrightarrow \mathcal{C}$.

Example

If $A = kG$, $\mathrm{Mod}_{kG} \cong G\text{-Mod}$.

Theorem

Let A be an alg. in smcat $\mathcal{C}$.

① If A is a bialg, then smcat str on $\mathcal{C}$ lifts along U to a smcat on Mod_A pres by U strictly.

② When $\mathcal{C} = \mathrm{Vect}$, there is bij^n between bialg strs on A & smcat str on Mod_A pres strictly by $U : \mathrm{Mod}_A \to \mathcal{C}$.

③ If A is Hopf alg & $\mathcal{C}$ is closed, then internal hom in $\mathcal{C}$ lifts to Mod_A.

Sketch proof of ①

Let A be a bialg & X, Y A-modules. Then $X \otimes Y$ is A-module with action

$$AXY \xrightarrow{\Delta 11} AAXY \xrightarrow{1s1} AXAY \xrightarrow{xy} XY$$

& bialg. axioms imply this is in fact an A-module. □

When $A = KG$, this coincides with the tensor prod. we saw earlier

$$g \otimes x \otimes y \longmapsto g \otimes g \otimes x \otimes y \mapsto g \otimes x \otimes g \otimes y \mapsto gx \otimes gy.$$

Summary: Representations of bialgebras & Hopf algebra (in part. group representations) admit well behaved tensor products & internal homs.