

Exercise 5 - Algebra 4

- In the course we introduced $\text{Ext}^n(A, -)$ as the right n^{th} right derived functor of $\text{Hom}(A, -): \text{Mod}_R \rightarrow \text{Ab}$, but then mentioned that elements of $\text{Ext}^n(A, B)$ (for $n > 0$) are equiv. classes of exact sequences

$$0 \rightarrow B \rightarrow X_1 \rightarrow \dots \rightarrow X_n \rightarrow A \rightarrow 0$$

- Here we explore some of the algebra of such exact sequences.

① (Splicing of exact sequences)

Consider exact sequences

$$0 \rightarrow B \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \rightarrow X_n \xrightarrow{f_n} A \rightarrow 0 \quad \&$$

$$0 \rightarrow C \xrightarrow{g} Y \xrightarrow{h} B \rightarrow 0.$$

Show that

$$0 \rightarrow C \xrightarrow{g} Y \xrightarrow{f_0 \circ h} X_1 \xrightarrow{f_1} X_2 \rightarrow \dots \rightarrow X_n \xrightarrow{f_n} A \rightarrow 0$$

is exact.

- What is relationship to les of cohomology for Ext ?

- Can you generalise to longer sequences?

② (Decomposing exact sequences)

The first question shows how to "splice" exact sequences into longer ones.

Show that every exact sequence of length n can be obtained by splicing n short exact sequences.

③ (Pullback & pushout)

- Consider

$$\begin{array}{ccccccc}
 & & & & A' & & \\
 & & & & \downarrow \alpha & & \\
 0 & \rightarrow & B & \xrightarrow{f} & X & \xrightarrow{g} & A \rightarrow 0
 \end{array}$$

with exact row

Show that the top row is ex.

$$\begin{array}{ccccccc}
 0 & \rightarrow & B & \xrightarrow{f'} & P & \xrightarrow{g'} & A' \rightarrow 0 \\
 & & \downarrow & & \downarrow & \downarrow & \downarrow \alpha \\
 0 & \rightarrow & B & \xrightarrow{f} & X & \xrightarrow{g} & A \rightarrow 0
 \end{array}$$

where g' is pullback of g

& $f'x = (fx, 0)$.

- Explain the dual constr.

For pushouts,

④ Explain how the previous constructions give actions

$$\text{Ext}_1(A, B) \longrightarrow \text{Ext}_1(A', B)$$

$$E \longmapsto E \alpha \quad \&$$

&

$$\text{Ext}_1(A, B) \longrightarrow \text{Ext}_1(A, B')$$

$$E \longmapsto \beta E$$

where $\beta: B \rightarrow B'$.

making $\text{Ext}_1(-, -): \text{Mod}_R^{\oplus} \times \text{Mod}_R \rightarrow \text{Ab}$
a Functor.

5) (Baer sum)

Given $E: 0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0$

& $E': 0 \rightarrow B \rightarrow Y \rightarrow A \rightarrow 0$

in $\text{Ext}_1(A, B)$ we will define
the Baer sum

$$E + E' \in \text{Ext}_1(A, B).$$

Firstly, take

$$E \oplus E': 0 \rightarrow B \oplus B \rightarrow X \oplus Y \rightarrow A \oplus A \rightarrow 0$$

& then define

$$E + E' = \nabla_A (E \oplus E') \Delta_B$$

where $\nabla_A: A \oplus A \rightarrow A$ is codiagonal

$$(x, y) \mapsto x + y$$

$$\& \quad \Delta_B: B \longrightarrow B \oplus B$$

$$x \longmapsto (x, x)$$

is diagonal.

- Show that this makes $\text{Ext}_1(A, B)$

into an abelian group .

- What is the unit ?