Exercise 8

① The Zariski Topology

For k a field, the set k^n has a topology with closed sets the varieties $V(I)$ where I is a set of polynomials.

Recall that the axioms for a topology on X (expressed using closed sets) say:

- ϕ, X are closed
- closed sets are closed under inf. intersection & finite union.

Verify these for k^n by showing

(a) $V(\phi) = k^n$, $V(k[x_1, \ldots, x_n]) = \phi$,

(b) $\bigcap_{i \in X} V(A_i) = V(\bigcup_{i \in I} A_i)$

(c) $V(A) \cup V(B) = V(AB)$ where

$AB = \{ fg : f \in A, g \in B \}$.

② Hilbert's Nullstellensatz fails over $\mathbb{R}$

- The Nullstellensatz says that if k is algebraically closed, & if $S \subseteq k[x_1, \ldots, x_n]$ a set of polynomials, then

$$I\,V(S) = \mathrm{Rad}\langle S \rangle.$$

- Show this is false over $\mathbb{R}$, by considering the polynomial $x^2 + 1 \in \mathbb{R}[x]$.

(3) Algebras versus quotient rings

- Show that A is a finitely generated commutative k-algebra $\Leftrightarrow$ it is iso to a quotient ring $k[x_1, \dots, x_n]/I$ where I is an ideal.
- Which commutative k-algebras correspond to radical ideals?

(4) Show that each linear subspace $V \subseteq k^2$ is a variety.