Exercises 2 - Algebra 4

Reminder

- A pre-additive category $\mathcal{C}$ is one in which each hom $\mathcal{C}(a,b)$ is an abelian group & pre & post composition preserves the abelian group structure.
- In a pre-additive category $\mathcal{C}$, binary products & coproducts are the same thing - biproducts. These are diagrams

$$a \underset{p_1}{\overset{i_1}{\rightleftarrows}} c \underset{p_2}{\overset{i_2}{\leftrightarrows}} b$$

satisfying

- $p_1 i_1 = 1_A$, $p_2 i_1 = 0$ • $i_1 p_1 + i_2 p_2 = 1_C$
- $p_1 i_2 = 0$, $p_2 i_2 = 1_B$

- Sim. terminal & initial obs coincide, & are captured by the property that $c \xrightarrow{0 = id} c$ and we call them a zero object.
- A pre-additive category is additive if it has biproducts & a zero ob.
- Given $f : A \longrightarrow B \in \mathcal{C}$ its kernel $\ker f \xrightarrow{i} A \xrightarrow{f} B$ is the universal ob. such that $f \circ i = 0$. Cokernels are dual.
- It is abelian if $\operatorname{coker} \ker f \longrightarrow \ker \operatorname{coker} f$ is invertible, or equiv: each mono is kernel of its cokernel, each epi the cokernel of its kernel.

1. Show that a pre-additive $\mathcal{C}$ has kernels $\Leftrightarrow$ it has equalisers.

2. Suppose that A is abelian. Describe kernels and cokernels in the cat $Ch(A)$ in detail.

3. - An element $x \in A$ an abelian group has torsion if $\exists n \in \mathbb{Z}/0$ st $nx = 0$.
   - An abelian group is torsion-free if it has no non-trivial torsion elements.

   - Show that the category TFreeAb is additive with kernels & cokernels, but not abelian by finding a mono which is not the kernel of its cokernel.

4. A cat $\mathcal{C}$ is pointed if it has an object 0 which is both terminal & initial. Show that $\forall a, b$ $\exists$ a morphism $0_{a,b} : a \longrightarrow b$ preserved by pre & post composition.

(5) Show that in a pointed cat there always exists a canonical map $\lambda_{a,b} : a+b \longrightarrow a \times b$.

Using this, define biproducts in a pointed category.

(6) Show that in a pointed cat. $\mathcal{C}$ with biproducts, each homset $\mathcal{C}(a,b)$ has the str. of a commutative monoid.