Lecture 10 - Representation theory of groups

Next 4 weeks - 2 weeks : basics, theory
- 1 week : symmetric group
- 1 week : Hopf algebras

Basic def^{n}s

- let k be a field & Vect denote the category of k-vector spaces & linear transf.
- Given V a vect. space, $End(V) = Vect(V, V)$ is a monoid with composition given by comp. of lin. transformations.
- When V is n-dim vect. space, $End(V) \cong Mat(n, k)$, the monoid of $n \times n$-matrices w/ values in k.

~~Def^n~~ (monoid would suffice)

Let G be a group. A G-module / G-repres. is a monoid homomorphism

$$\rho : G \longrightarrow End(V) :$$

that is, for each $g \in G$ an invertible lin. transformation $\rho(g) : V \to V$ such that

$$\rho(gh) = \rho(g)\rho(h) \text{ \& } \rho(e) = Id.$$

Remark

Equivalently,

- for each $g \in G$, $v \in V$ an elt $\rho(g)(v)$, which we write as gv:

such that

- $g(v+w) = gv + gw$ & $g(\lambda v) = \lambda(gv)$
- $gh(v) = g(hv)$ & $ev = v$ where $e \in G$ is id.

Remark

- When U is n-dim vect. space, $End(U) \cong Mat(n, K)$, the monoid of $n \times n$-matrices w/ values in K.

Hence a G-module str. on U is specified by a homomorphism

$$\rho : G \longrightarrow Mat(n, K),$$

which is often called an n-dimensional matrix representation.

Examples

① (Trivial representation)

$$G \longrightarrow End(K, K) : g \longmapsto 1 : K \to K.$$

② (Some more 1-d representations)

$C_n = \langle g \mid g^n = 1 \rangle$ cyclic group of order n

$K = \mathbb{C}$. Then a 1-d rep. of C_n is a homomorph $C_n \longrightarrow End(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$: such is specified by an $\sigma(g) \in \mathbb{C}$ st $\sigma(g)^n = 1$ - hence $\sigma(g)$ is a n'th root of unity.

There are n such roots:

$$\sigma(g) = \cos(2k\pi/n) + i\sin(2k\pi/n) \text{ for}$$

$k = 0, \ldots, n-1$. Eg. for C_4 these are $\{1, i, -1, -i\}$ & so n 1-d representations.

③ $D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ the dihedral group. This captures symmetries of the square, generated

by a rotation of $90°$ & a reflection.
Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be the matrices for such a rot. & refl.
Defining $a, b \longmapsto A, B$ gives a 2-d representation since $A^4 = B^2 = 1$ &
$B^{-1}AB = A^{-1}$.

Def^n) Let V, W be G-modules. A homomorph. of G-modules $\theta : V \longrightarrow W$ is a linear transformation such that $\theta(gv) = g\,\theta(v)$ all $g \in G$, $v \in V$.

G-modules & homomorphisms form a category G-Mod.

Examples from group actions

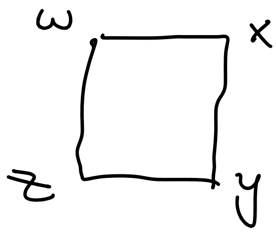
- Let G act on a set X: we have bijections
  $g.- : X \longrightarrow X$ st $g(hx) = (gh)x$ & $ex = x$.
- Let FX be the free vector space on X, with basis elements of X – we obtain

$$g.- := F(g.-) : FX \longrightarrow FX$$
$$\lambda_1 x_1 + \dots + \lambda_n x_n \longmapsto \lambda_1 g x_1 + \dots + \lambda_n g x_n.$$

by linear extension.

- This is clearly a G-module, & such G-modules arising from actions are called permutation representation.

Ex ① : $D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ acts on corners $\{w, x, y, z\}$ of square



where a rotates by 90° & b swaps w & y.

Then we obtain permutation D_8-module w/ basis w, x, y, z. The matrix for a is then

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

② The regular G-module KG is the permutation module associated to the (left) action of G on itself: i.e. KG has basis elts of G,

$g(\lambda_1 g_1 + \cdots + \lambda_n g_n) = \lambda_1 g g_1 + \cdots + \lambda_n g g_n$.

G-modules as representations over a ring/algebra

- Observe that KG is K-algebra (non-comm.):
- cert. KG is a K-vector space;
- Multiplication

$$\left(\sum_{i=1}^{n}\lambda_i g_i\right)\left(\sum_{j=1}^{m}\mu_j g_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m}(\lambda_i\mu_j)(g_i g_j)$$

& multiplicative unit e.

- If V is a K-vector space, then $\mathrm{End}(V)$ is a K-algebra where K-vector space str. is componentwise as in V.

By defn, a representation of a K-algebra A is a K-alg. hom. $A \longrightarrow \mathrm{End}(V)$.

Propn There is a bijn between
1. Representations of G;
2. Representations of K-algebra KG;
3. Modules over the ring KG.

Proof
- A rep of G is equally a monoid hom $G \xrightarrow{\sigma} \mathrm{End}(V)$.

This admits a ! extension along $G \to KG : g \mapsto g$ to a k-alg hom. $kG \xrightarrow{\sigma} \mathrm{End}(U)$ given by

$$\lambda_1 g_1 + \dots + \lambda_n g_n \longmapsto \lambda_1 \sigma(g_1) + \dots + \lambda_n \sigma(g_n),$$

& this gives the bij^{n} between ① & ②.

(Exercise: show $G \mapsto kG$ provides left adjoint to forg. functor $k\text{-Alg} \longrightarrow \mathrm{Mon}$.)

– Given ②, we have a <u>kG-module</u> str. on underlying abelian group of U:

$kG \longrightarrow \mathrm{Vect}(U,U) \longrightarrow \mathrm{Ab}(U,U)$.

Conversely, given a ring hom

$\sigma : kG \longrightarrow \mathrm{Ab}(A,A)$ (i.e. kG-module), then restriction along

$k \longrightarrow kG : \lambda \longmapsto \lambda.e$ gives a ring hom $k \longrightarrow kG \longrightarrow \mathrm{Ab}(A,A)$ making A a k-vector space, in such a way that σ lifts to an alg map

$kG \longrightarrow \mathrm{Vect}(A,A)$. These constructions are inverse. $\square$

· In particular, $G\text{-Mod} \cong \mathrm{Mod}_{kG}$, so everything we know about module cats (kernels, quotients, direct sums etc) holds for $G\text{-Mod}$.

A few Facts & Terminology for G-modules

- Given $\theta : V \to W \in G\text{-Mod}$, we can form $\ker\theta \leq V$ & $\operatorname{im}\theta \leq W$ which are G-submodules.

Direct sums (of submodules)

- If W a vector space, $U, V \leq W$ are subspaces then $U+V = \{u+v : u \in U, v \in V\} \leq W$ is a subspace.
- If given $w \in W$ $\exists! (u,v) \in U \times V$ st $w = u+v$, we say W is (internal) direct sum of U & V and write $W = U \oplus V$.
  This is equiv. to saying $W = U+V$ & $U \cap V = 0$.
- Of course, then $U \times V \cong U \oplus V$, so this is direct sum in usual sense.
- If W is a G-module, & $U, V \leq W$ submodule s.t. $W = U \oplus V$ as above, say W is direct sum $U \oplus V$ of G-submodules.

Projections

- A projection $p : V \to V$ of G-modules is a homom. s.t. $p^2 = p$.

Prop^n (Note: holds for modules over any ring.)

If p is a proj^n, then $V = \operatorname{im} p \oplus \ker p$ & each direct sum arises from a proj^n in this way.

Proof ~ Write $v = \overset{\in \operatorname{im} p}{pv} + \overset{\in \ker p}{(v - pv)}$

This is unique since if $v \in \operatorname{im} p \cap \ker p$, then $pv = 0$ but as $v = px$, $0 = pv = ppx = px = v$.
- If $W = U \oplus V$, define $p : W \to W : u + v \mapsto u$. This is proj^n with $\operatorname{im} p = U$ & $\ker p = V$. $\square$

Decomposing G-modules

Def^n) A G-module V is reducible if it contains a non-trivial submodule. A non-triv. G-module is irreducible if it is not reducible.

Theorem (Maschke)

Let G be a finite group & suppose $\operatorname{char}(K)$ does not divide order of G, $|G|$. (eg. if $k = \mathbb{R}$ or $\mathbb{C}$)

If V is a G-module & $U \leq V$ a proper submodule, then G-submodule W st $V = U \oplus W$.

~~Proof~~

~~- Firstly~~, as U subspace of V, can find linearly independent vectors giving subspace W_0 s.t. $V = U \oplus W_0$ as a vector space.
- This gives a projection of vector spaces $p : V \to V : u + w \mapsto u$ with image U & kernel W_0, but p need not be a G-module map.
- We will modify p to a G-module map

$q : V \to V$ st $q^2 = q$ & $\operatorname{im} q = U$;
then $V = \operatorname{im} q \oplus \ker q = U \oplus \ker q$, a dir. sum of G-submodules, as required.

- For $g \in G$, let $p_g : U \to U : v \mapsto g^{-1}(p(gv))$. As a composite of 3 linear maps, p_g is linear map.
- Define $q = \frac{1}{|G|} \sum_{g \in G} p_g$ as the "average" of these maps, which is again linear as it is a linear comb. of linear maps.
- To check q a G-module map:

$$q(hv) = \frac{1}{|G|} \sum_{g \in G} g^{-1} p(g.hv).$$

Since each elt of G is uniquely of form gh^{-1} for some $g \in G$,

$$\frac{1}{|G|} \sum_{g \in G} g^{-1} p(g.hv) =$$

$$\frac{1}{|G|} \sum_{g \in G} (gh^{-1})^{-1} p(gh^{-1}.hv) \quad \text{using } U \text{ a } G\text{-mod.}$$

$$= \frac{1}{|G|} \sum_{g \in G} hg^{-1} p(gv)$$

$$= h.\frac{1}{|G|} \sum_{g \in G} g^{-1} p(gv) = h.q(v).$$

- Remains to show $\operatorname{im} q = U$.

Let $u \in U$. Then

$$q(u) = \frac{1}{|G|} \sum_g g^{-1} p(gu) \quad \left(\begin{array}{c}\text{as } gu \in U \text{ so} \\ p(gu) = gu\end{array}\right)$$

$= \frac{1}{|G|} \sum_g g^{-1} g u$

$= \frac{1}{|G|} \sum_{g \in G} u = \frac{|G|}{|G|} u = u.$

- Hence $U \leq \operatorname{im} q$. To see $\operatorname{im} q \leq U$, observe that since p takes it's image in U, each $g^{-1} p(gv) \in U$; hence $q(v) \in U$ all $v \in V$.

Therefore $\operatorname{im} q = U$.

- Since $qv \in U$ & $qu = u$ all $u \in U$, we get $q\,qv = qv$ all $v \in V$, as required. □

Theorem

Let G be a finite group & suppose $\operatorname{char}(K)$ does not divide order of G, $|G|$. (eg. if $K = \mathbb{R}$ or $\mathbb{C}$).

Then each non-zero finite dim. G-module V admits a decomposition

$V = V_1 \oplus \cdots \oplus V_n$ as direct sum of irreducible G-submodules.

Proof) - By ind. on dimension of V.

- If $\dim V = 1$, trivial as each 1-d G-module is irreducible.
- Else, suppose it is true for all W st. $\dim(W) < \dim(V)$.
- Suppose $U \leq V$ is a proper G-submodule. Then by Mashke's Theorem,

$V = U \oplus W$ for U, W proper submodules

Then $\dim(U), \dim(W) < \dim(V)$ so

$$V = U \oplus W = (U_1 \oplus \cdots \oplus U_m) \oplus (V_1 \oplus \cdots \oplus V_\ell)$$

where all the U_i & V_j are irreducible.

$\square$