Lecture 11 - Group representations ctd.

Last time: decomposition into irred.

Today: finer decomposition results,
in partic. when $k = \mathbb{C}$.

$\boxed{\text{Prop}}$ $k[G]$ is the free G-module on 1.

$\boxed{\text{Proof}}$ Follows from fact (last time) that

G-modules $\equiv$ $k[G]$-modules for $k[G]$ ring,

& free $k[G]$-module on 1 is of course $k[G]$.

Explicitly,

$$\begin{tikzcd} k[G] \arrow[dr, dashed, "\exists ! f"] & \\ 1 \arrow[u, "e"] \arrow[r, "a"'] & U \end{tikzcd}$$

G-mod map

Must have $f(e) = a$.

Then need $f(g) = f(g.e) = g.f(e) = g.a$

& for linearity

$$f\Big(\sum_{i \in I} \lambda_i g_i\Big) = \sum_{i \in I} \lambda_i g_i . a . \qquad \square$$

Schur's lemma

Let $\theta : V \longrightarrow W$ be a morph. of irred. G-modules.

① Then $\theta = 0$ or θ is an isomorph.

② If K is algebraically closed, & $V = W$ of finite dimension then $\exists \lambda \in K$ st

$$\theta(v) = \lambda v \text{ all } v \in V.$$

① As V irr., $\ker\theta = 0$ or V

As W irr, $\operatorname{im}\theta = 0$ or W.

- If $\theta \neq 0$, then $\ker\theta \neq V$ & $\operatorname{im}\theta \neq 0$

so $\ker\theta = 0$ & $\operatorname{im}\theta = W$.

Hence θ is inj & surj $\Rightarrow$ an iso.

② Since K is alg. closed,

the poly. in λ, $\det(\theta - \lambda.\mathrm{Id}) = 0$,

has a solution (eigenvalue).

But then $\theta - \lambda.\mathrm{Id} : V \longrightarrow V$ has non-zero kernel (the eigenvector)

& so kernel $= V$ - hence $\theta = \lambda.\mathrm{Id}$. $\square$

Prop

Let G be finite gp st. $\mathrm{char}(K)$ does not divide $|G|$.
Let $K[G] = \mathcal{U}_1 \oplus \dots \oplus \mathcal{U}_n$ be a decomp. into irreducibles (last time).
Then every irreducible G-module $\mathcal{U}$ is iso. to one of the $\mathcal{U}_i$.

Proof

- Let $v \in \mathcal{U}$ be non-zero.
- By freeness of $K[G]$, $\exists!\, \theta : K[G] \to \mathcal{U}$ st $\theta(e) = v$.
- Then $\mathrm{im}\,\theta \leq \mathcal{U}$; since non-zero & $\mathcal{U}$ irr. $\mathrm{im}\,\theta = \mathcal{U}$.
- Consider $\ker\theta \leq K[G]$. By Maschke's Theorem, $K[G] = \ker\theta \oplus U$; consider composite

$$\begin{tikzcd} U \arrow[r, hook, "i"] \arrow[rr, bend right, "\bar{\theta}"'] & \ker\theta \oplus U \arrow[r, "\theta"] & \mathcal{U} \end{tikzcd}.$$

will show $\bar{\theta}$ is an iso; then

$$K[G] = \ker\theta \oplus U \cong \ker\theta \oplus \mathcal{U}$$

so that $\mathcal{U}$ is iso. to a submodule of $K[G]$.

Now $\bar{\theta}(v) = \theta(v)$.

- Let's show $\ker\bar{\theta} = 0$.

If $\bar{\theta}(v) = 0$ then $\theta(v) = 0$ so $v \in \ker\theta \cap V = 0$; hence $v = 0$.

- Since θ is surj., given $u \in U$ $\exists\ a+b \in \ker\theta \oplus V$ st. $\theta(a+b) = u$.

But $\theta(a+b) = \theta a + \theta b = \theta b$ as $a \in \ker\theta$; hence $\bar{\theta}(b) = u$, as required, so $\bar{\theta}$ surj.

Hence $\bar{\theta}$ an iso.

Therefore suffices to prove Theorem when U is a submodule of $K[G]$.

- For $K[G] = U_1 \oplus \cdots \oplus U_n$, consider

$$K[G] \xrightarrow{p_i} U_i$$
$$u_1 + \cdots + u_n \longmapsto u_i.$$

- Since U is non-zero, one of the composites

$$U \hookrightarrow K[G] \longrightarrow U_i$$

must be non-zero, & so invertible by Schur's lemma Part i. $\square$

Cor) Let G be finite gp st. char(K) does not divide $|G|$.
Then there are only finitely many irreducible G-modules up to iso.

Defn) $\{U_1, \ldots, U_n\}$ is a complete set of irred. G-modules if no two are iso. & every irreduc. G-module is iso to one of them.

Defn) Let U, W be G-modules. Write $\mathrm{Hom}_{K[G]}(U, W)$ for vector space of G-module maps from U to W with pointwise operations.

Remark: $\mathrm{Hom}_{K[G]}(U, W)$ need not be a G-module unless G is commutative!

Finer results when $K = \mathbb{C}$

In this subsection, assume G is finite & $K = \mathbb{C}$.

Prop^n) Let V, W be irreducible finite-dimensional G-modules. Then

$$\dim \mathrm{Hom}_{\mathbb{C}(G)}(U, W) = \begin{cases} 1 & \text{if } U \cong W \\ 0 & \text{otherwise} \end{cases}$$

<u>Proof</u>

- $\dim \mathrm{Hom}_{\mathbb{C}(G)}(U,W) = 0 \iff$ only homomorphism $U \to W$ is zero $\iff$ (by Schur) $U \not\cong W$.
- If $\dim \mathrm{Hom}_{\mathbb{C}(G)}(U,W) = 1$, $\exists$ non-zero hom. $U \to W$, which is an iso (by Schur)
- If $U \cong W$, we obtain iso of vector spaces

$$\mathrm{Hom}_{\mathbb{C}(G)}(U, U) \cong \mathrm{Hom}_{\mathbb{C}(G)}(U, W)$$

so it suffices to show lhs has dim 1.

But by Schur's lemma Part 2, each $f : U \to U$ equals $\lambda . \mathrm{Id}$ - thus lhs has basis $\{\mathrm{Id} : U \to U\}$, & so has dim. 1 $\square$

Theorem

Let $V \neq 0$ be a f.d. G-module.

Then

1. $V = V_1 \oplus \cdots \oplus V_n$ where the V_i are irreducible.
2. Each irreducible G-module W appears in decomp., up to iso, $\dim(\mathrm{Hom}_{\mathbb{C}G}(V, W))$ times.
3. In particular, let $U_1, \ldots, U_m$ is a complete set of irreducible G-modules. Then $V \cong U_1^{d_1} \oplus \cdots \oplus U_m^{d_m}$ where $d_i = \dim(\mathrm{Hom}_{\mathbb{C}G}(V, U_i))$.

Proof

- Proved 1 last week.
- For 2, we have

$$\mathrm{Hom}_{\mathbb{C}G}(V, W) = \mathrm{Hom}_{\mathbb{C}G}(V_1 \oplus \cdots \oplus V_n, W)$$
$$\cong \mathrm{Hom}_{\mathbb{C}G}(V_1, W) \oplus \cdots \oplus \mathrm{Hom}(V_n, W)$$

since direct sum is a coproduct & restriction along each $V_i \hookrightarrow V$ is linear.

Taking dimensions,

$$\dim(\mathrm{Hom}_{\mathbb{C}G}(V, W)) = \sum_{i=1}^{n} \dim(\mathrm{Hom}_{\mathbb{C}G}(V_i, W)$$
$$= \sum_{i : V_i \cong W} 1 \text{ by prev. proposition, i.e.}$$

the number of i st. $V_i \cong W$.

For ③, by ②,

$$U_1 \oplus \cdots \oplus U_n =$$

$$(\underbrace{U_{11} \oplus \cdots \oplus U_{1n_1}}_{\text{those } U_i \text{ iso to } U_1 \text{ - by ② there are } d_1 \text{ of these}}) \oplus \cdots \oplus (\underbrace{U_{m1} \oplus \cdots \oplus U_{mn_m}}_{\text{those } U_i \text{ iso to } U_m\text{, of which there are } d_m})$$

$$\cong$$

$$U_1^{d_1} \oplus \cdots \oplus U_m^{d_m}. \quad \square$$

Corollary

Let $U_1, \ldots, U_m$ be complete set of irreducibles. Then $\mathbb{C}[G] \cong U_1^{\dim(U_1)} \oplus \cdots \oplus U_m^{\dim(U_m)}$.

Proof

By Part 3 of previous result,
we must show

$$d_i := \dim(\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U_i)) = \dim(U_i)$$

In fact, since $\mathbb{C}[G]$ is free G-module on 1,
we have bij^n

$$\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U_i) \cong U_i$$
$$F \longmapsto F(e)$$

& this evaluation map is clearly linear,
hence an iso. of vector spaces.
Therefore lhs & rhs have same dimension. $\square$

Cor

$$|G| = \sum_{U_1,\dots,U_m} \dim(U_i)^2 .$$

Proof

Since

$$\mathbb{C}[G] \cong U_1^{\dim(U_1)} \oplus \cdots \oplus U_m^{\dim(U_m)}$$

Taking dimensions of lhs & rhs proves claim as $\dim(\mathbb{C}[G]) = |G|$. $\square$

Remark: Above formula relating order of G with number of its irreducible reps is very useful in calculating all irreps. of a finite group.

Example

- Consider $D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$.

By prev. Thm,

$$|D_8| = 8 = \sum_{U_1, \dots, U_m} \dim(U_i)^2$$

$$= 2^2 + 4 . 1^2$$

$$= 8 . 1^2$$

so 1 2-d irrep & 4 1 d irreps or 8 1 d irreps.

- Recall 2-d real rep:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \&$$

view as a complex 2-d rep — ie. rep. on $\mathbb{C}^2$.

- As a 2-d rep, a non-Triv. submodule must be 1-d subspace $\langle v \rangle$ st. $gv = \lambda v \in \langle v \rangle$ for each $g \in V$:

ie. v should be eigenvector for

both A & B.

- Can calc. eigenvectors of A which are $(1, i)$ & $(1, -i)$ & of B $(1,0)$ & $(0,1)$ but they have none in common. Hence this is irreducible 2-d rep.

Therefore D_8 has one 2-d irrep & 4 1-d irreps: ie 4 1-d reps.

A 1-d rep is simply a homomorphism

$$D_8 \longrightarrow (\mathbb{C}, \cdot, 1)$$

& it is easy to see these are given by

$$a, b \longrightarrow (\pm 1, \pm 1)$$

so these are 1-d irreps.

Hence we have calc.

all complex irreps of D_8.