Lecture 13 - A glance at Hopf algebras

(Not examinable!)

Part 1 - Tensor products & internal homs

- Familiar with tensor product $U \otimes W$ of k-vector spaces (eg. Algebra 3) which classifies bilinear maps.
- Elements of $U \otimes W$ are lin. combs of $v \otimes w$ where $v \in U$ & $w \in W$.
- We have isos of vector spaces
$$\alpha : (U \otimes W) \otimes X \cong U \otimes (W \otimes X),$$
$$s : U \otimes W \cong W \otimes U,$$
$$l : k \otimes U \cong U \quad \& \quad r : U \otimes k \cong U,$$
making $(\mathrm{Vect}, \otimes, k)$ into a symmetric monoidal cat.
- We have bijections
$$\mathrm{Vect}(U \otimes W, A) \cong \underbrace{\mathrm{Bilin}(U, W; A)}_{\text{set of bilin. maps}} \cong \mathrm{Vect}(U, [W, A])$$

where $[W, A] = \mathrm{Vect}(W, A)$ equipped w/ pointwise vector space structure.

Hence $- \otimes W \dashv [W, -]$ are adjoint, making $(\mathrm{Vect}, \otimes, k, [-,-])$ into a symmetric mon. closed category - & call $[W, A]$ the internal hom.

Example

$(\underset{}{\mathrm{Set}}, \underbrace{\times}_{\text{cart. prod.}}, \underbrace{1}_{\text{1 elt set}})$ is a symmetric monoidal closed category.

Here $[X, Y] = \mathrm{Set}(X, Y)$

- We have forgetful functor $U : G\text{-Mod} \to \mathrm{Vect}$ & (in fact) the symmetric monoidal closed structure lifts along U to G-Mod:

- if V, W are G-modules, then $V \otimes W$ becomes a G-module on defining

$$g(v \otimes w) = gv \otimes gw;$$

more abstractly, $g.- : V \otimes W \to V \otimes W$ is the unique linear map st

$$\begin{tikzcd} V \times W \arrow[r, "g.- \times g.-"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "="] & V \times W \arrow[d, "\otimes"] \\ V \otimes W \arrow[r, "\exists! g.-"'] & V \otimes W \end{tikzcd}$$

univ. bil. map

Follows from either description that $V \otimes W$ is a G-module.

In fact, $V \otimes W$ classifies <u>G-bilinear maps</u>: bilinear maps $V \times W \xrightarrow{F} A$ st.

$$g.F(v,w) = F(gv, gw).$$

- The unit is k equipped w/ trivial G-mod. structure.
- The isos α, s, l, r lift to isos of G-modules — so G-Mod has s.mon str. pres. by U.
- The internal hom $[V,W] = \mathrm{Vect}(V,W)$ becomes a G-module if we define at $f \in [V,W]$:

<u>$(g.f)v = g(f(g^{-1}v))$</u>:

equiv.

$$\begin{tikzcd} V \arrow[r, "g.f"] \arrow[d, "g^{-1}.-"'] \arrow[dr, phantom, "="] & W \\ V \arrow[r, "f"'] & W \arrow[u, "g.-"'] \end{tikzcd}$$

& so is linear, as a comb of 3 linear maps,

& is easy to see this makes $[U,W]$ a G-module, and that

$$G\text{-Mod}(U\otimes W, A) \cong G\text{-Bilin}(U,W;A) \cong G\text{-Mod}(U,[W,A])$$

so $-\otimes W \dashv [W,-]$ again making G-Mod s-mon closed.

- Can express the above story in much greater generality, using bialgebras & Hopf algebras.

Part 2 - Bialgebras & Hopf algebras

- Let $(\mathcal{C}, \otimes, I)$ be a symm. mon. cat (smcat) eg. $(\text{Set}, \times, 1)$ or $(\text{Vect}, \otimes, k)$.
- Will write as if $\mathcal{C}$ is strict monoidal (strictly associative & unital) as is justified by Maclane's coherence theorem. And write AB for $A\otimes B$ etc, to save space.
- A monoid/algebra in $\mathcal{C}$ is an ob $A\in\mathcal{C}$ + $AA \xrightarrow{m} A$ & $I \xrightarrow{e} A$ st. the diagrams

$$\begin{tikzcd} AAA \arrow[r, "m1"] \arrow[d, "1m"'] & AA \arrow[d, "m"] \\ AA \arrow[r, "m"'] & A \end{tikzcd} \quad \& \quad \begin{tikzcd} A \arrow[r, "e1"] \arrow[dr, "1"'] & AA \arrow[d, "m"] & A \arrow[l, "1e"'] \arrow[dl, "1"] \\ & A & \end{tikzcd}$$

commute.

Ex: In $(\mathrm{Set}, \times, 1)$, an algebra $\equiv$ monoid
$(\mathrm{Vect}, \otimes, k)$, — — — — $\equiv$ k-algebra

- A comonoid / coalgebra is: ob $A \in \mathcal{C}$ + $A \xrightarrow{\Delta} AA$, $A \xrightarrow{q} I$ sat. dual coassoc., counit axioms.

(Equiv., algebra in $(\mathcal{C}^{op}, \otimes, I)$.)

- A bimonoid / bialgebra is
  - alg (A, m, e) +
  - coalg (A, Δ, q) s.t.

$$\begin{tikzcd} AA \arrow[r, "\Delta\Delta"] \arrow[dd, "m"'] & AAAA \arrow[d, "1s1"] \\ & AAAA \arrow[d, "mm"] \\ A \arrow[r, "\Delta"] & AA \end{tikzcd} \qquad \begin{tikzcd} I \arrow[d, "e"'] \arrow[dr, "ee"] & \\ A \arrow[r, "\Delta"] & AA \end{tikzcd}$$

&
$$\begin{tikzcd} I \arrow[r, "e"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3, "="] & A \arrow[d, "q"] \\ & I \end{tikzcd}, \qquad \begin{tikzcd} AA \arrow[r, "m"] \arrow[dr, "qq"'] & A \arrow[d, "q"] \arrow[dl, phantom, "="] \\ & I \end{tikzcd}.$$

Example

- In Set (or any cartesian mon. cat.) each ob. has ! coalgebra str.

$$1 \xleftarrow{!} X \xrightarrow{\Delta} X \times X$$
$$x \longmapsto (x,x)$$

- Then a bialgebra in **Set** $\equiv$ monoid.

A Hopf algebra is a bialgebra + a map $a : A \longrightarrow A$ (called antipode) st.

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[drr, "q"'] & AA \arrow[r, shift left, "a1"] \arrow[r, shift right, "1a"'] & AA \arrow[r, "m"] \arrow[d, phantom, "="] & A \\
& & I \arrow[ur, "e"'] &
\end{tikzcd}$$

Examples

- In **Set**, this says
$$x \longmapsto (x,x) \longmapsto (a(x),x) \longmapsto a(x).x = e$$
& sim. $x.a(x) = e$, so
a Hopf algebra $\equiv$ group
where $a(x) = x^{-1}$.

- Hopf algebra in $(\mathrm{Vect}, \otimes, k)$ is what is trad. called a Hopf algebra.
Will show that group algebra $k[G]$ is a Hopf algebra.

Firstly, observe that free v. space functor $k[-] : \mathrm{Set} \longrightarrow \mathrm{Vect}$ sat.

$k[X \times Y] \cong k[X] \otimes k[Y]$, $k(1) \cong k$

in a way compatible with assoc., unit, symmetry isos in these smcats. Therefore $k[-]$ takes Hopf algebras in Set to Hopf algebras in Vect — so if G is a gp (ie. Hopf alg in Set) then $k[G]$ is a Hopf algebra in Vect.

Indeed $k[G]$ is an alg. (the group alg) w/ str.

$$k(G) \otimes k(G) \cong k[G \times G] \xrightarrow{k[m]} k(G) \text{ \&}$$

$$k \cong k(1) \xrightarrow{k(e)} k(G)$$

& coalg str

$$k[G] \xrightarrow{k[\Delta]} k[G \times G] \cong k[G] \otimes k[G]$$

& $k[G] \xrightarrow{k(!)} k[1] \cong k$ which act on basis elts as

$$g \longmapsto g \otimes g$$

$$g \longmapsto 1$$

& antipode

$$k[G] \xrightarrow{k[a]} k[G]$$

$$g \longmapsto g^{-1}.$$

So the group algebra $k[G]$ is a Hopf algebra!

Ex 3) Recall $Var^{op} \xrightarrow{k(-)}$ comm-k-Alg which sends products to tensor products.

Follows that sends groups in Var (algebraic groups) to Hopf algebras: i.e. co-ordinate ring of an algebraic group is Hopf algebra.

More examples – un. env. alg. of a Lie algebra, tensor algebras . . .

Modules over an algebra in symm. mon. cat

· If (A, m, e) is an alg a (left) A-module is ob $X \in \mathcal{C}$ + $AX \xrightarrow{x} X$ st.

$$\begin{tikzcd}
AAX \arrow[r, "1x"] \arrow[d, "m1"'] \arrow[dr, phantom, "="] & AX \arrow[d, "x"] & X \arrow[l, "e1"'] \arrow[dl, "1"] \arrow[dl, phantom, "="] \\
AX \arrow[r, "x"'] & X &
\end{tikzcd}$$

Morph. $f : (X, x) \to (Y, y)$ of A-modules is $f : X \to Y$ such that

$$\begin{array}{ccc} AX & \xrightarrow{1f} & AY \\ x\downarrow & = & \downarrow y \\ X & \xrightarrow{f} & Y \end{array}.$$

Obtain a cat Mod_A & forgetful functor $U : \mathrm{Mod}_A \longrightarrow \mathcal{C}$.

Example

If $A = kG$, $\mathrm{Mod}_{kG} \cong G\text{-Mod}$.

Theorem

Let A be an alg. in smcat $\mathcal{C}$.

① If A is a bialg, then smcat str on $\mathcal{C}$ lifts along U to a smcat on Mod_A pres by U strictly.

② When $\mathcal{C} = \mathrm{Vect}$, there is bij^n between bialg strs on A & smcat str on Mod_A pres strictly by $U : \mathrm{Mod}_A \to \mathcal{C}$.

③ If A is Hopf alg & $\mathcal{C}$ is closed, then internal hom in $\mathcal{C}$ lifts to Mod_A.

Sketch proof of ①

Let A be a bialg & X, Y A-modules. Then $X \otimes Y$ is A-module with action

$$AXY \xrightarrow{\Delta 11} AAXY \xrightarrow{1s1} AXAY \xrightarrow{xy} XY$$

& bialg. axioms imply this is in fact an A-module. $\square$

When $A = KG$, this coincides with the tensor prod. we saw earlier

$$g \otimes x \otimes y \longmapsto g \otimes g \otimes x \otimes y \mapsto g \otimes x \otimes g \otimes y \mapsto gx \otimes gy.$$

Summary: Representations of bialgebras & Hopf algebra (in part. group representations) admit well behaved tensor products & internal homs.