

lecture 13 - A glance at Hopf algebras

(Not examinable!)

Part 1 - Tensor products & internal homo

- Familiar with tensor product $U \otimes W$ of k -vector spaces (eg. Algebra 3) which classifies bilinear maps.
- Elements of $U \otimes W$ are lin. combos of $u \otimes w$ where $u \in U$ & $w \in W$.
- We have isos of vector spaces
$$\kappa: (U \otimes W) \otimes X \cong U \otimes (W \otimes X),$$
$$\sigma: U \otimes W \cong W \otimes U,$$
$$\lambda: k \otimes U \cong U \quad \& \quad \rho: U \otimes k \cong U,$$
 making $(\text{Vect}, \otimes, k)$ into a symmetric monoidal cat.
- We have bijections
$$\text{Vect}(U \otimes W, A) \cong \text{Bilin}(U, W; A) \cong \text{Vect}(U, [W, A])$$
set of bilin. maps

where $[W, A] = \text{Vect}(W, A)$ equipped w' pointwise vector space structure.

Hence $-\otimes W \dashv [W, -]$ are adjoint, making $(\text{Vect}, \otimes, k, [-, -])$ into a symmetric mon. closed category - & call $[W, A]$ the internal homo,

Example

$(\text{Set}, \times, 1)$ is a symmetric monoidal closed category.

Here $[X, Y] = \text{Set}(X, Y)$

- We have forgetful Functor $U: G\text{-Mod} \rightarrow \text{Vect}$
 & (in fact) the symmetric monoidal closed structure lifts along U to $G\text{-Mod}$:

• if U, W are G -modules, then $U \otimes W$ becomes a G -module on defining

$$g(u \otimes w) = gu \otimes gw;$$

more abstractly, $g_{-} : U \otimes W \rightarrow U \otimes W$ is the unique linear map st

$$\begin{array}{ccc} U \times W & \xrightarrow{g_{-} \times g_{-}} & U \times W \\ \text{univ. bil. map} \downarrow \cong & & \downarrow \cong \\ U \otimes W & \xrightarrow{\exists! g_{-}} & U \otimes W \end{array}$$

Follows from either description that $U \otimes W$ is a G -module.

In fact, $U \otimes W$ classifies G -bilinear maps:
 bilinear maps $U \times W \xrightarrow{F} A$ st.
 $g.F(u, w) = F(gu, gw)$.

• The unit is k equipped w' trivial G -mod. structure.

• The iso $\alpha, \sigma, \lambda, \nu$ lift to iso of G -modules - so $G\text{-Mod}$ has s. mon str. pres. by U .

• The internal hom $[U, W] = \text{Vect}(U, W)$

becomes a G -module if we define at $f \in [U, W]$:

$$\underline{(g.f)u = g(f(g^{-1}u))} :$$

$$\begin{array}{ccc} \text{equiv. } U & \xrightarrow{g.f} & W \\ g_{-}^{-1} \downarrow & \cong & \uparrow g_{-} \\ U & \xrightarrow{f} & W \end{array} \quad \begin{array}{l} \text{\& so is linear, as} \\ \text{a combo of 3 linear} \\ \text{maps,} \end{array}$$

& is easy to see this makes $[U, W]$ a G -module, and that

$$G\text{-Mod}(U \otimes W, A) \cong G\text{-Bilin}(U, W; A) \cong G\text{-Mod}(U, [W, A])$$

so $- \otimes W + [W, -]$ again making $G\text{-Mod}$ s -mon closed.

- Can express the above story in much greater generality, using bialgebras & Hopf algebras.

Part 2 - Bialgebras & Hopf algebras

- Let $(\mathcal{C}, \otimes, I)$ be a symm. mon. cat (smcat) eg. $(\text{Set}, \times, 1)$ or $(\text{Vect}, \otimes, k)$.

- Will write as if \mathcal{C} is strict monoidal (strictly associative & unital) as is justified by MacLane's coherence theorem.

And write AB for $A \otimes B$ etc, to save space.

- A monoid (algebra) in \mathcal{C} is an obj $A \in \mathcal{C}$

+ $AA \xrightarrow{m} A$ & $I \xrightarrow{e} A$ st. the

diagrams

$$\begin{array}{ccc}
 AAA & \xrightarrow{m_1} & AA \\
 \downarrow m & & \downarrow m \\
 AA & \xrightarrow{m} & A
 \end{array}
 \quad \& \quad
 \begin{array}{ccc}
 A & \xrightarrow{e_1} & AA & \xleftarrow{e_2} & A \\
 \searrow & & \downarrow m & & \swarrow \\
 & & A & &
 \end{array}$$

commute.

Ex: In $(\text{Set}, \times, 1)$, an algebra \equiv monoid
 $(\text{Vect}, \otimes, k)$, - - - \equiv k -algebra

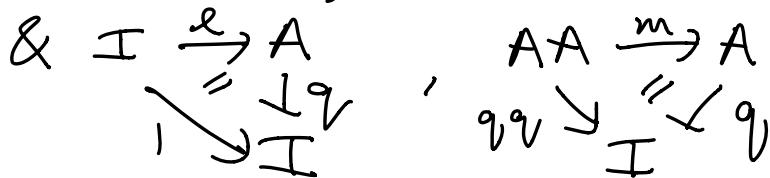
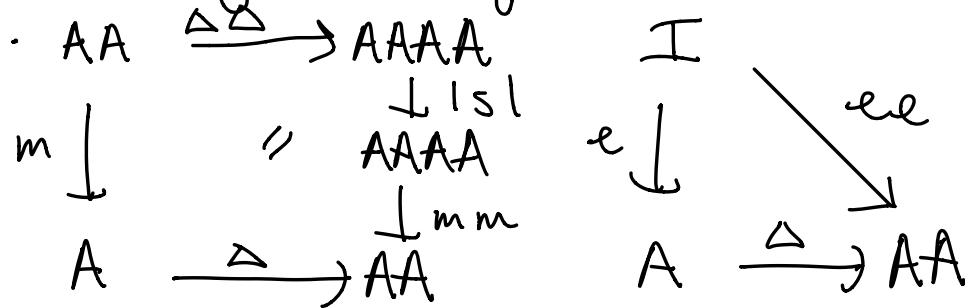
- A comonoid / coalgebra is: ob $A \in \mathcal{C}$
 $+ A \xrightarrow{\Delta} AA$, $A \xrightarrow{q} I$ sat. dual
 coassoc., counit axioms.

(Equiv., algebra in $(\mathcal{C}^{\text{op}}, \otimes, I)$.)

- A bimonoid / bialgebra is

- alg (A, m, e) +

- coalg (A, Δ, q) s.t.



Example

- In Set (or any cartesian mon. cat.)
 each ob. has ! coalgebra str.

Firstly, observe that free v. space
 functor $K[-] : \text{Set} \longrightarrow \text{Vect}$ sat
 $K[X \times Y] \cong K[X] \otimes K[Y]$, $K(1) \cong K$
 in a way compatible with assoc.,
 unit, symmetry iso in these sm. lats.
 Therefore $K[-]$ takes Hopf algebras
 in Set to Hopf algebras in Vect -
 so if G is a gp (ie. Hopf alg in Set)
 then $K[G]$ is a Hopf algebra in Vect .

Indeed $K[G]$ is an alg. (the group
alg) w' str.
 $K(G) \otimes K(G) \cong K[G \times G] \xrightarrow{K[m]} K(G)$ &
 $K \cong K(1) \xrightarrow{K[e]} K(G)$

& coalg str
 $K[G] \xrightarrow{K[\Delta]} K[G \times G] \cong K[G] \otimes K[G]$
 & $K[G] \xrightarrow{K[!]} K(1) \cong K$ which
 act on basis elts as

$$\begin{array}{ccc}
 g & \longmapsto & g \otimes g \\
 g & \longmapsto & 1 \quad \& \quad \text{antipode} \\
 K[G] & \xrightarrow{K[a]} & K[G] \\
 g & \longmapsto & g^{-1}.
 \end{array}$$

So the group algebra $K[G]$ is a Hopf algebra.

Ex 3) Recall $\text{Var} \xrightarrow{\varphi} \text{comm-k-Alg}$ which sends products to tensor products.

Follows that sends groups in Var (algebraic groups) to Hopf algebras: i.e. co-ordinate ring of an algebraic group is Hopf algebra.

More examples - un. env. alg. of a Lie algebra, tensor algebras ...

Modules over an algebra in symm. mon. cat

If (A, m, e) is an alg a (left) A -module is obj $X \in \mathcal{C} + AX \xrightarrow{x} X$ st.

$$\begin{array}{ccccc} AAX & \xrightarrow{1_X} & AX & \xleftarrow{e_1} & X \\ m_1 \downarrow & & \downarrow x & \swarrow & \\ AX & \xrightarrow{x} & X & & \end{array}$$

Morph. $f: (X, x) \rightarrow (Y, y)$ of A -modules is $f: X \rightarrow Y$ such that

$$\begin{array}{ccc}
 AX & \xrightarrow{IF} & AY \\
 \times \downarrow & & \downarrow Y \\
 X & \xrightarrow{F} & Y
 \end{array}$$

Obtain a cat Mod_A & forgetful functor $U: \text{Mod}_A \rightarrow \mathcal{C}$.

Example

IF $A = KG$, $\text{Mod}_A \cong G\text{-Mod}$.

Theorem

Let A be an alg. in smcat \mathcal{C} .

- ① IF A is a bialg, then smcat str on \mathcal{C} lifts along U to a smcat on Mod_A pres by U strictly.
- ② When $\mathcal{C} = \text{Vect}$, there is bijⁿ between bialg str on A & smcat str on Mod_A pres strictly by $U: \text{Mod}_A \rightarrow \mathcal{C}$.
- ③ IF A is Hopf alg & \mathcal{C} is closed, then internal hom in \mathcal{C} lifts to Mod_A .

Sketch proof of ①

Let A be a bialg & X, Y A -modules.

Then $X \otimes Y$ is A -module with action

$$AX \otimes Y \xrightarrow{\Delta} A \otimes AX \otimes Y \xrightarrow{1st} A \otimes X \otimes AY \xrightarrow{\times} X \otimes Y$$

& bialg. axioms imply This is in fact an A -module. \square

When $A = KG$, this coincides with the tensor prod. we saw earlier
 $g \otimes x \otimes y \mapsto g \otimes g \otimes x \otimes y \mapsto g \otimes x \otimes g \otimes y \mapsto g \otimes g \otimes y$.

Summary: Representations of
bialgebras & Hopf algebras (in
part. group representations)
admit well behaved tensor
products & internal homs.