Lecture 2

Abelian categories

- What is correct categorical context for homological algebra from a categorical perspective?
- Must talk about zero map, add and subtract morphisms & need to form kernels, images & quotients & these should behave as in Mod_R.
- Resulting notion: abelian category

Def^n) A pre-additive category (or Ab-enriched cat) $\mathcal{C}$ is a cat. in which hom-set $\mathcal{C}(a,b)$ has the structure of an abelian group -

(ie. we have $a \overset{f}{\underset{g}{\rightrightarrows}} b \mapsto a \xrightarrow{f+g} b, a \xrightarrow{-f} b, a \xrightarrow{0} b$)

and moreover pre & postcomposition preserve the abelian group structure:

(given $x \xrightarrow{r} a \overset{f}{\underset{g}{\rightrightarrows}} b \xrightarrow{s} y$ we have

$(f+g)r = fr+gr$ & $s(f+g) = sf+sg$.

$0.r = 0$ & $s.0 = 0$)

Example

- Mod_R is pre-additive:

Given $M \overset{f}{\underset{g}{\rightrightarrows}} N$, $f+g : M \longrightarrow N$ is def by $(f+g)x = fx+gx$ which is an abelian gp. hom. by commut. of $+$ & $(f+g)(rx) = f(rx)+g(rx)$

$= rfx + rgx$

$= r.(fx+gx)$

$= r(f+g(x))$.

- $M \xrightarrow{0} N$ is constant at 0
- $M \xrightarrow{-f} N : x \mapsto -fx$.
- $s(f+g) = sf + sg$ holds as s a homo. whilst $(f+g)r = fr + gr$ is trivial.

Remark

$\mathcal{C}$ is preadditive $\iff$ $\mathcal{C}^{op}$ is -

hence we can apply duality to pre-additive cats.

Proposition

Let $\mathcal{C}$ be a pre-additive category.

(1) Then $\mathcal{C}$ has a term. ob $\iff$ it has an init. ob

(2) $\mathcal{C}$ has binary prods $\iff$ $\mathcal{C}$ has bin coprods.

Proof

(1) Let t be terminal.

- Then $t \xrightarrow{0} t = id$.
- Now given x, we have $0 : t \longrightarrow x$ & must show it is unique, so consider $t \overset{f}{\underset{0}{\rightrightarrows}} x$. Then $f = f \circ id_t = f \circ 0 = 0$.
- Converse is dual.

(2) let $\begin{tikzcd}[row sep=small] & a \\ c \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & b \end{tikzcd}$ be a product diagram.

- We have

$$\begin{tikzcd}[row sep=small] & a \\ a \arrow[ur, "id"] \arrow[dr, "0"'] & \\ & b \end{tikzcd} \text{ ind } a \xrightarrow{i_1 = \langle id, 0\rangle} c \ \ \& \text{ sim } \begin{tikzcd}[row sep=small] & a \\ b \arrow[ur, "0"] \arrow[dr, "id"'] & \\ & b \end{tikzcd} \text{ ind } b \xrightarrow{i_2 = \langle 0, id\rangle} c$$

& so $\begin{tikzcd}[row sep=small] a \arrow[dr, "i_1"] & \\ & c \\ b \arrow[ur, "i_2"'] & \end{tikzcd}$, which we must show is a coprod. diagram.

- The key point is to observe that the diagram $a \underset{p_1}{\overset{i_1}{\rightleftarrows}} c \underset{p_2}{\overset{i_2}{\leftrightarrows}} b$ satisfies

- $p_1 i_1 = 1_A$, $p_2 i_1 = 0$
- $p_2 i_2 = 1_B$, $p_1 i_2 = 0$
- $i_1 p_1 + i_2 p_2 = 1_c$

Such a diagram is called a biproduct diagram.

To see the last use the u.p. of the prod:

$$p_1(i_1p_1 + i_2p_2) = p_1i_1p_1 + p_1i_2p_2$$
$$= 1p_1 + 0 = p_1 \quad \&$$

sim for p_2.

- Any biproduct diagram is both a prod. & coproduct. let's show the latter.
- Indeed, given $\begin{tikzcd}[row sep=small] a \arrow[dr, "f"] & \\ & d \\ b \arrow[ur, "g"'] & \end{tikzcd}$ must find k st

$$\begin{tikzcd}[row sep=small] a \arrow[dr, "i_1"] \arrow[drr, bend left, "f"] & & \\ & c \arrow[r, dashed, "t"] & d \\ b \arrow[ur, "i_2"'] \arrow[urr, bend right, "g"'] & & \end{tikzcd}.$$

But then $t = t.1 = t(i_1p_1 + i_2p_2) = fp_1 + gp_2$.
Let's check $fp_1 + gp_2$ has required props:
$(fp_1+gp_2)i_1 = fp_1i_1 + gp_2i_1 = f.1 + g.0 = f$
& sim. $(fp_1+gp_2)i_2 = g$, as required.
The converse is dual. □

$\boxed{\text{Def}^n}$ A preadditive cat is additive if it has finite products (equivalently by above, finite coproducts).

$\boxed{\text{Example}}$ Mod_R is additive. Prop 3 generalises fact from Algebra 3 that in Mod_R finite products & coproducts coincide.

Kernels & quotients

- In an additive category $\mathcal{C}$ consider $f: a \to b$.
- The kernel of f is an ob & arrow $\ker f \xrightarrow{i} a$ such that $\ker f \xrightarrow{i} a \xrightarrow{f} b = 0$ & it is universal with this property.
- The cokernel of f is dual: we have $a \xrightarrow{f} b \xrightarrow{p} \operatorname{coker} f$ & $pf = 0$ & universal with this prop.

Remark:- Equivalently, $\ker f$ & $\operatorname{coker} f$ are the equaliser & coequaliser of $a \overset{f}{\underset{0}{\rightrightarrows}} b$.

- Note this implies that $i : \ker f \to a$ is mono & $p : \operatorname{coker} f \longrightarrow b$ is epi.

Example

- Mod_R has kernels & cokernels
- Given $f : A \longrightarrow B$,

$$\ker f = \{x : fx = 0\}$$
$$\operatorname{coker} f = B/\operatorname{im} f.$$

- If $a \overset{i}{\hookrightarrow} b$ is mono in $\mathcal{C}$, one often writes $b/a := \operatorname{coker}(i)$

Lemma

In an additive cat, $A \xrightarrow{f} B$ is mono $\iff$ $\ker f = 0$

$A \xrightarrow{f} B$ is epi $\iff$ $\operatorname{coker} f = 0$,

where 0 is the term/init obj.

Proof

By duality, it suffices to prove the first.

- Let $A \xrightarrow{f} B$ be mono & consider $0 \xrightarrow{0} A \xrightarrow{f} B$.
- Given $C \xrightarrow{g} A$ if $f \circ g = 0 = f \circ 0$ so $g = 0_{C,A}$

but

$$\begin{tikzcd} C \arrow[rr, "0"] \arrow[dr, "0"'] & & A \\ & 0 \arrow[ur, "0"'] & \end{tikzcd}$$

, giving the factorisation, which is clearly unique.

Conversely let $0 \xrightarrow{0} A \xrightarrow{f} B$ be kernel.

$$C \overset{g}{\underset{h}{\rightrightarrows}} A$$

If $fg = fh$, then $f(g-h) = 0$ so $\exists!\, C \xrightarrow{t} 0$ st

$g - h = 0t = 0$ so $g = h$. $\square$

- In an additive category $\mathcal{C}$ with kernels & cokernels, we can factor each arrow in 2 ways:

①
$$\begin{tikzcd} \ker f \arrow[r, "i"] & a \arrow[rr, "f"] \arrow[dr, "p"'] & & b \\ & & \mathrm{ck}f \arrow[ur, "t"'] & \end{tikzcd}$$

where t is unique map ind. by $fi = 0$

②
$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "\ell"'] & & b \arrow[r, "q"] & cf \\ & \mathrm{kc}f \arrow[ur, "m"'] & & \end{tikzcd}$$

where ℓ is ! map ind by $qf = 0$. As $0 = qf = qtp$ & p is epi, $qt = 0$ so we get ! map $\mathrm{ck}f \xrightarrow{\alpha} \mathrm{kc}f$ such that

$$\begin{tikzcd} a \arrow[rrrr, "f"] \arrow[dr, "p"'] & & & & b \\ & \mathrm{ck}f \arrow[r, "\alpha"] & \mathrm{kc}f \arrow[urr, "m"'] & & \end{tikzcd}$$

$\boxed{\text{Def}^n}$ An add. cat w' kernels & cokernels is abelian if $\alpha : \mathrm{ck}f \to \mathrm{kc}f$ is an iso.

Example

- Mod_R has kernels & cokernels
- Given $f : A \longrightarrow B$.

$$\ker f = \{x : fx = 0\}$$
$$\operatorname{coker} f = B / \operatorname{im} f.$$

Thus $\mathrm{ck}f = A/\ker f$ &

$\mathrm{kc}f = \mathrm{im} f$ & the induced map $A/\ker f \longrightarrow \mathrm{im} f$

$[x] \longmapsto fx$ is indeed an iso (first iso theorem).

Hence Mod_R is abelian.

- In an abelian category, the two ways of factorising ① & ② above coincide (up to unique iso) & we write

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "e"'] & & B \\ & \mathrm{im} f \arrow[ur, "m"'] & \end{tikzcd}$$

$$\mathrm{im} f = \ker\mathrm{coker} f = \mathrm{coker}\ker f.$$

for this common fact

In particular e is epi & m is mono.

There is another characterisation of abelian cats. Note that ② shows that epi $\Longleftrightarrow$ regular epi.

In particular, in Mod_R.

Prop^n Let $\mathcal{C}$ be additive w/ kernels & cokernels. Then $\mathcal{C}$ is abelian

$\Leftrightarrow$ (1) Each mono is the kernel of its cokernel.

& (2) Each epi is the cokernel of its kernel.

Proof) We will only show $\Rightarrow$).

- Let $A \xrightarrow{f} B$ be mono, so $kf = 0$

- Then

$$\begin{tikzcd}
kf = 0 \arrow[r] & A \arrow[rr, "f"] \arrow[d, "1"'] & & B \arrow[r] & \operatorname{coker} f \\
& ckf = A \arrow[urr, "f"] \arrow[r, "\cong"', "\alpha"] & kcf \arrow[ur, "m"'] & &
\end{tikzcd}$$

- This shows that f is iso to the kernel of its cokernel, as required.
- To show each epi is cokernel of kernel is dual.

I leave $\Leftarrow$) to interested reader. $\square$

Examples

① Mod_R.

② - If A is an abelian cat then so is A^{op}! This allows us to apply duality to abelian cats.

③ If A is abelian, we can consider chain complexes in A:

$$\begin{tikzcd} \cdots A_{n+1} \arrow[r, "d_{n+1}"] \arrow[rr, bend right, "0"'] & A_n \arrow[r, "d_n"] & A_{n-1} \arrow[r] & \cdots \end{tikzcd},$$

We have
$$\begin{tikzcd} \mathrm{im}(d_{n+1}) \arrow[r, hook] \arrow[dr] & A_n \\ & \ker(d_n) \arrow[u, hook] \end{tikzcd}$$
& so can form

$H_n(A) := \ker(d_n) / \mathrm{im}(d_{n+1})$. Thus can speak of homology in an ab. cat.

We can also speak of exact sequences, chain maps, homotopies in

any abelian category &
the results of last week all
hold in this setting:

- We can form the cat $Ch(A)$
of chain complexes in A &
it is again abelian:
indeed $(f+g)_n = f_n + g_n$ for chain
maps & kernels & cokernels
are also constructed
componentwise.

- Homology is a functor
$H_n : Ch(A) \longrightarrow A$ as before.

④ If C is a small cat & A abelian,
the functor cat $[C, A]$ is abelian
with componentwise structure.

⑤ If X is a top. space we can
look at the poset $O(X)$ of open

sets of X. A presheaf of R-modules is a functor $O(X)^{op} \longrightarrow Mod_R$.

By the above the cat

$$[O(X)^{op}, Mod_R] \text{ is abelian.}$$

A sheaf is a presheaf

$$F : O(X)^{op} \longrightarrow Mod_R$$