Exercises 3

(0) Show that $0 \to A \to 0$ is exact $\iff A = 0$

& $0 \to A \to B \to 0$ is exact $\iff$ A is an iso,

(1) Recall that if $0 \to A \to B \to C \to 0$ is a ses of chain complexes, then we get the long exact sequence of homology

$$\dots H_{n+1}(A) \to H_{n+1}(B) \to H_{n+1}(C) \xrightarrow{\delta} H_n(A) \to H_n(B) \to \dots$$

Using this,
show that if any 2 of A, B & C are exact, then so is the third.

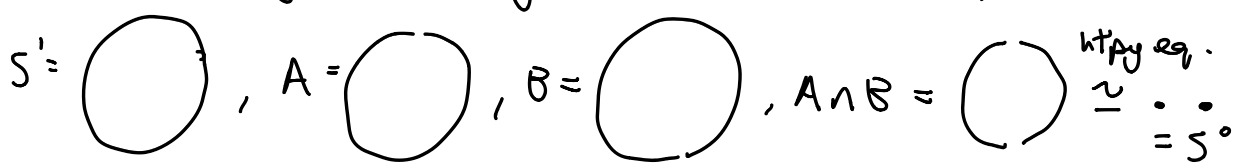
(2) Homology groups of spheres

- If X is a top. space, & A, B subspaces whose interiors jointly cover X, then by the les of homology one can get a les

$$\dots H_1 A \oplus H_1 B \to H_1(X) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(X) \to 0$$

called the Mayer-Vietorius sequence.

- For example, when $X = S^n$ the n-sphere, we can write it is as union of its subspaces A, B obtained by removing north & south pole:



Then $A \cap B \simeq S^{n-1}$ in general, &

$A, B \simeq \bullet$.

Using this, & that $H_n(S^1) = \mathbb{Z}$ if $n = 1$
else 0,

show $H_n(S^n) = \mathbb{Z}$ all $n \geq 1$.

(3) Show that there is a category whose objects are chain complexes & whose morphisms are homotopy classes of chain maps.

(4) Consider the categories $SES(\mathcal{C})$ $LES(\mathcal{C})$ whose objects are short exact & long exact sequences in an ab. cat $\mathcal{C}$. Show that the long exact sequence of homology gives a

Functor

$$SES(Ch(\mathcal{C})) \longrightarrow LES(\mathcal{C}).$$