Lecture 3

Additive & exact functors

Def^n) Let $\mathcal{A}, \mathcal{B}$ be abelian cats. A functor $F: \mathcal{A} \longrightarrow \mathcal{B}$ is additive if each function $F_{X,Y}: \mathcal{A}(X,Y) \longrightarrow \mathcal{B}(FX, FY)$ is a homom. of abelian groups (ie. $F(f+g) = Ff + Fg$ & $F0_{X,Y} = 0_{FX,FY}$.

- From last week,
  - term/in. ob are char. by diags $a \xrightarrow{0 = id} a$ which we call zero object diags & a a zero ob., denoted by 0.
  - bin. products/coproducts are characterised by diagrams of form

$$a \underset{p_1}{\overset{i_1}{\rightleftarrows}} c \underset{p_2}{\overset{i_2}{\leftrightarrows}} b \quad \text{sat.}$$

- $p_1 i_1 = id$, $p_2 i_2 = id$ $\quad i_1 p_1 + i_2 p_2 = 1d_c$
- $p_1 i_2 = 0$, $p_2 i_1 = 0$,

which are called biproduct diagrams, and often denote biprod. by $a \oplus b$.

Proposition

Any additive functor F preserves finite prods & finite coproducts - ie. zero ob. & biproducts.

Proof

F preserves zero ob. diags / biproduct diagrams $\square$

Def^n) An additive functor $F : A \to B$ bet. abelian cats is

- left exact if it pres. kernels (lex)
- right exact if it pres. cokernels (rex)
- exact if it preserves both. (ex)

Remark : Lex functors are those preserving finite limits (fin. prods + equalisers $\equiv$ kernels).
Rex functors preserve finite colimits & ex functors preserve both.

Lemma

① F is lex $\Leftrightarrow$ it preserves exactness of sequences $0 \to A \xrightarrow{f} B \xrightarrow{g} C$

② F is rex $\Leftrightarrow$ it preserves exactness of sequences $A \xrightarrow{f} B \xrightarrow{g} C \to 0$

③ F is ex $\Leftrightarrow$ it preserves exactness everywhere $\Leftrightarrow$ it pres ses $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$.

Proof

First we prove ①. Observe $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is exact $\Leftrightarrow$ $\ker f = 0$ (ie f is mono) & $\ker g = f$ ($A \to \operatorname{im} f$ is an iso)

Hence if F pres. kernels, it pres ex. of such sequences.

Conversely, consider the exact sequence

$0 \to \ker f \xhookrightarrow{i} A \xrightarrow{f} B$ which F preserves

so $0 \to F\ker f \xrightarrow{Fi} FA \xrightarrow{Ff} FB$ is exact

$\Rightarrow F\ker f = \ker Ff$. $\square$

(2) is dual to (1).

For (3), if F is ex. it preserves kernels & cokers & so images. Therefore it preserves exactness everywhere & so in particular ses.

Suppose F pres ses. We will show F pres. monos & epis. If f is mono then $0 \to A \xrightarrow{f} B \to \operatorname{coker} f \to 0$ is ses which F preserves, so by ex @ FA, Ff is mono.

Dually F preserves epis.

Now show F preserves kernels (cokernels dual.)

At general $f: A \to B$ consider ses on hor. row.

$$\begin{tikzcd} & & & & B \\ 0 \arrow[r] & \ker f \arrow[r, hook, "q"] & A \arrow[r, two heads, "p"] \arrow[urr, "f"] & \operatorname{im} f \arrow[r] \arrow[ur, "i"] & 0 \end{tikzcd}$$

Then Fq, Fi mono, Fp epi & hor. sequence exact after applying F, so

$\ker(Ff) =$ as Fi mono

$\ker(Fp) =$ as F pres ses

$\operatorname{im}(Fq) =$ as Fq mono

$F(\ker f)$, so F pres. kernels. $\square$

Examples

① If $\mathcal{C}$ abelian cat & $A \in \mathcal{C}$ the functor

$$\begin{array}{cccc} \mathcal{C}(A,-) : \mathcal{C} & \longrightarrow & Ab & \\ X & & \mathcal{C}(A,\cancel{X}) & g \\ f\downarrow & & f_* \downarrow & \downarrow \\ Y & & \mathcal{C}(A,Y) & fg \end{array}$$

is additive & preserves all limits - therefore it is lex.

② If $\mathcal{C} = Mod_R$, we have $Mod_R(A,-) : Mod_R \longrightarrow Ab$ & this has a left adjoint $A \otimes_{\mathbb{Z}} - : Ab \to Mod_R$.

- Here $A \otimes_{\mathbb{Z}} B$ classifies functions

$$K : A \times B \longrightarrow C \sim R\text{-module}$$

& $K(a,-) : B \to C$ is hom. of ab. groups

& $K(-,b) : A \to C$ is hom. of R-modules,

& $A \otimes_{\mathbb{Z}} B$ is constructed as a quotient sim. to the tensor prod. of R-modules in Alg. 3.

In particular, as a left adjoint it preserves colimits & so is right exact.

More gen, if right adjoint is lex then left adjoint is rex & conversely.

③ The forgetful functor $U : \mathrm{Mod}_R \longrightarrow \mathrm{Ab}$ is left exact, since $U \cong \mathrm{Mod}_R(R,-)$ which is lex by ① (or directly).

④ If $\mathcal{C}$ is abelian, so is $\mathrm{Ch}(\mathcal{C})$ & then

$$\begin{aligned} \mathrm{Ch}(\mathcal{C}) &\xrightarrow{(-)_n} \mathcal{C} \\ X &\longmapsto X_n \end{aligned}$$

is exact since kernels & cokernels are componentwise in $\mathrm{Ch}(\mathcal{C})$.

Theorem (Freyd - Mitchell)

If $\mathcal{C}$ is a small abelian cat., $\exists$ a ring R & an exact fully faithful embedding

$$F : \mathcal{C} \longrightarrow \mathrm{Mod}_R .$$

- We will not prove it.

Consequence :- When proving things about diagrams in an abelian cat it suffices to supp. we are working in a cat. of R-modules, since the theorem lets us view our category as a full subcat. of Mod_R such that the inclusion pres all structure - kernels, cokernels etc.

- For instance, suffices to prove snake lemma in Mod_R.

The snake lemma & long exact sequence of homology

Snake lemma

Given a diagram
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \end{tikzcd}$$

with exact rows in an abelian cat., there is an exact sequence

$$\ker(\alpha) \xrightarrow{f} \ker(\beta) \xrightarrow{g} \ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha) \xrightarrow{f'} \operatorname{coker}(\beta) \xrightarrow{g'} \operatorname{coker}(\gamma).$$

Proof - Maps except for δ are ind. by u.p.'s.

- Exactness @ $\ker(\beta)$ & $\operatorname{coker}(\beta)$ are dual. So prove ex @ $\ker(\beta)$. Enough to prove it in Mod_R by FM-thm which assume for rest of proof.
- let $b \in \ker(\beta)$ & suppose $gb = 0$. By ex. $\exists a$ st $fa = b$. Then $f'\alpha a = \beta f a = \beta b = 0$ & as f' is mono (since bottom row exact) therefore $\alpha a = 0$ so $a \in \ker(\alpha)$ w/ $fa = b$. Hence ex @ $\ker(\beta)$.
- Now we construct $\delta : \ker\gamma \to \operatorname{coker}\alpha$. Consider $x \in \ker\gamma$. As g epi $\exists x' \in B$ st $gx' = x$. Consider $\beta x'$. Then $g'\beta x' = \gamma g x' = \gamma x = 0$ so $\beta x' \in \ker(g') = \operatorname{im}(f')$ so $\exists! x'' \in A'$ as f' is mono s.t. $f'(x'') = x'$. Define $\delta(x) = x'' + \operatorname{im}(\alpha) \in \operatorname{coker}(\alpha)$.
- To show δ well defined, let $gy' = x$.
- Then $g(y' - x') = 0$ so by ex. of top row, $\exists a \in A$ st $f(a) = y' - x'$, so $\alpha a \in A'$. Then $f'\alpha a = \beta f a = \beta y' - \beta x' = f'y'' - f'x''$ so as f' mono $\alpha a = y'' - x''$. Hence $y'' + \operatorname{im}\alpha = x'' + \operatorname{im}\alpha$.

Therefore δ is well defined.
Easy to see it is a homomorphism
& exactness @ $\ker(\gamma)$, $\operatorname{coker}(\alpha)$
are left an exercise. $\square$

<u>Theorem</u>
Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a ses of chain complexes in an ab. cat. Then we obtain a long exact sequence of homology

$$\cdots H_{n+1}(A) \xrightarrow[H_{n+1}(f)]{} H_{n+1}(B) \xrightarrow[H_{n+1}(g)]{} H_{n+1}(C) \xrightarrow{\delta} H_n(A) \xrightarrow[H_n(f)]{} H_n(B) \xrightarrow[H_n(g)]{} H_n(C) \cdots$$

$\boxed{\text{Proof}}$ For a chain complex A

$$\cdots A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots\cdots$$

we have an ind. homomorphism

$$A_n/\mathrm{im}(d_{n+1}) \xrightarrow{d} \ker(d_{n-1}) = Z_{n-1}(A)$$
$$x + \mathrm{im}(d_{n+1}) \longmapsto d_n x$$

- whose kernel contains elements $x+\mathrm{im}(d_{n+1})$ with $d_n(x) = 0$ & so is precisely $\ker(d_n)/\mathrm{im}(d_{n+1}) = H_n(A)$.
- whose cokernel is $\ker(d_{n-1})/\mathrm{im}(d_n) = H_{n-1}(A)$

Then we obtain a comm. diag.

$$\begin{tikzcd}
 & A_n/\mathrm{im}(d_{n+1}) \arrow[r, "f_n"] \arrow[d, "d"] & B_n/\mathrm{im}(d_{n+1}) \arrow[r, "g_n"] \arrow[d, "d"] & C_n/\mathrm{im}(d_{n+1}) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r, "f_{n-1}"'] & Z_{n-1}(B) \arrow[r, "g_{n-1}"'] & Z_{n-1}(C) &
\end{tikzcd}$$

& we will show these rows are

exact, & then by snake lemma we obtain les of homology. It remains to check rows are exact.

- Ex @ $C_n/\mathrm{im}(d_n)$, $Z_{n-1}(A)$ is easy as f_{n-1} is mono & g_n is epi.
- At $x \in Z_{n-1}(B)$, suppose $gx = 0$. Then $\exists y \in A_{n-1}$ st $fa = x$. Must show $d_{n-1}a = 0$, but as F is inj, $fd_{n-1}a = d_{n-1}fa = d_{n-1}x = 0$ implies $d_{n-1}a = 0$ so $a \in Z_{n-1}(A)$, as required.
- At $B_n/\mathrm{im}(d_{n+1})$ let $g(x + \mathrm{im}\, d_{n+1}) = 0$. Then $gx \in \mathrm{im}\, d_{n+1}$ so $gx = dc$ some $c \in C_{n+1}$. Then $\exists y \in B_{n+1}$ st $gy = c$. Now consider $x - dy \in B_n$. Then

$$g(x - dy) = gx - gdy = gx - dgy = gx - dc = 0,$$

so $\exists a \in A_n$ st $fa = x - dy$. Then

$$f(a + \mathrm{im}\, d_{n+1}) = x - dy + \mathrm{im}\, d_{n+1} = x + \mathrm{im}\, d_{n+1},$$

as required. $\square$

Stop here for today.