Lecture 4

Today: projectives, proj. resolutions & derived functors.

Notation: In ab. cat, $\twoheadrightarrow$ for epi, $\rightarrowtail$ or $\hookrightarrow$ for mono.

Projectives (see Alg 3 - we will do it quickly here)

Def^n) An ob. $A \in \mathcal{C}$ in an abelian cat is projective if given any diagram

$$\begin{tikzcd} & A \arrow[d, two heads, "\alpha"] \\ B \arrow[r, two heads, "f"'] & C \end{tikzcd} \quad \exists\, A \xrightarrow{\alpha'} B \text{ st } \begin{tikzcd} & A \arrow[dl, "\alpha'"'] \arrow[d, two heads, "\alpha"] \\ B \arrow[r, two heads, "f"'] & C \end{tikzcd}$$

Remark: This says $\mathcal{C}(A,B) \xrightarrow{f_*} \mathcal{C}(A,C) \in Ab$
$g \longmapsto fg$
is surjective / epi, so
A is proj. $\Leftrightarrow$ $\mathcal{C}(A,-) : \mathcal{A} \longrightarrow Ab$ preserves epis.

Def^n) $\mathcal{C}$ has enough projectives if for each $A \in \mathcal{A}$ $\exists$ X projective & epi $X \twoheadrightarrow A$.

Prop^n) Mod_R has enough projectives. The projectives are the retracts of free modules.

$\boxed{\text{Proof}}$ See Alg. 3. Given A we take counit map $FUA \xrightarrow{\varepsilon_A} A$ relative to adj $Mod_R \underset{U}{\overset{F}{\rightleftarrows}} Set$, which takes formal sums to actual sums. $\square$

Proposition (Properties of projectives)

1. Direct sums / biproducts and retracts of projectives are projective.
2. $A \in \mathcal{C}$ is proj. $\Leftrightarrow$ $\mathcal{C}(A,-) : \mathcal{C} \to Ab$ is exact.

Proof

① is straightforward. Consider A, B proj & direct sum $A \oplus B$.

Given
$$\begin{tikzcd} & A\oplus B \arrow[d, "\alpha"] \\ C \arrow[r, "f"] & D \end{tikzcd}$$
we have
$$\begin{tikzcd} & A \arrow[d, "i_1"] \\ B \arrow[r, "i_2"] & A\oplus B \end{tikzcd}$$
& so

$$\begin{tikzcd} & A \arrow[dl, dashed, "\exists\theta_1"'] \arrow[d, "\alpha i_1"] \\ C \arrow[r, "f"] & D \end{tikzcd} \quad \& \quad \begin{tikzcd} & B \arrow[dl, dashed, "\exists\theta_2"'] \arrow[d, "\alpha i_2"] \\ C \arrow[r, "f"] & D \end{tikzcd}$$
& then by u.p. of coprod. $A \oplus B$

$\exists! \, \theta : A\oplus B \to C$ s.t. $\theta i_1 = \theta_1$, $\theta i_2 = \theta_2$.

Then
$$\begin{tikzcd} & A\oplus B \arrow[dl, "\theta"'] \arrow[d, "\alpha"] \\ C \arrow[r, "f"] & D \end{tikzcd}$$
commutes using u.p. of coproduct.

Hence $A \oplus B$ projective.

- For retracts - easy: did in proof of "Proj. Mods = retracts of frees" in Alg. 3.

② If $\mathcal{C}(A,-) : \mathcal{C} \to Ab$ is exact, let $B \xrightarrow{f} C$ be epi, ie. $B \xrightarrow{f} C \to 0$ is exact.

Then let $F = \mathcal{C}(A,-)$. Then as exact functors pres. exactness,

$FB \xrightarrow{Ff} FC \longrightarrow F0 = 0$ is exact, so Ff is epi.

Conversely, let F pres epis. We must show F preserves ses

Each ses is of form $0 \to \ker f \hookrightarrow A \xrightarrow{f} B \to 0$ & as F is lex it pres. kernels so above is sent to $0 \longrightarrow \ker Ff \hookrightarrow FA \xrightarrow{Ff} FB \to 0$

as F preserves epis

& this is a ses as Ff is epi.
Hence F preserves ses, and so is exact. $\square$

Injectives

- For $\mathcal{C}$ abelian, $A \in \mathcal{C}$ injective $\iff$ $A \in \mathcal{C}^{op}$ is projective.

So anything we can prove about projectives in a gen. ab. cat has dual for injectives.

- $A \in \mathcal{C}$ is injective if
$$\begin{tikzcd} B \arrow[r, tail, "\text{mono}"] \arrow[dr, "f"'] & C \\ & A \end{tikzcd} \quad \exists \quad \begin{tikzcd} B \arrow[r, tail, "\text{mono}"] \arrow[dr, "f"'] & C \arrow[d, "f'"] \\ & A \end{tikzcd}.$$

- In Ab, injective = divisible.
- $\mathcal{C}$ has enough injectives if $\forall A \in \mathcal{C}$ $\exists$ mono $A \rightarrowtail A'$ w/ A' inj.
- Mod_R has enough injectives - follows from Baer criterion in Alg 3 (relatively hard).

Projective resolutions

Notation: Ch. complex X st $X_n = 0$ for $n < 0$ can be id. w/ a positive chain complex

$$\cdots \to X_2 \xrightarrow{d} X_1 \xrightarrow{d} X_0 .$$ write $Ch(\mathcal{C})_{\geq 0}$ for cat of positive chain complexes.

Def) let $A \in \mathcal{C}$ abelian cat. A proj. resolution of A is a chain complex $C \in Ch(\mathcal{C})_{\geq 0}$ with a map $C_0 \xrightarrow{\varepsilon} A$ such that:

① $\cdots C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{\varepsilon} A \to 0$ is exact

② each C_i is projective.

Can give more conceptual definition, as below.

Def^n) - A chain map $f: A \to B \in Ch(\mathcal{C})$ quasi-isomorphism if $H_n f : H_n A \to H_n B$ is an iso $\forall n \in \mathbb{Z}$.

- For $A \in \mathcal{C}$, let $A[0]$ be chain complex

$$\cdots 0 \cdots 0 \to 0 \to \underset{\text{degree } 0}{A} \in Ch(\mathcal{C})_{\geq 0}$$

Proposition

Projective resolution C of A

$\equiv$

quasi-iso $C \to A[0]$ st. each C_i is projective.

~~Proof~~ - A chain map $C \longrightarrow A[0]$ is a diag.

$$\begin{tikzcd} \vdots \arrow[d, dotted] & \vdots \arrow[d, dotted] \\ C_2 \arrow[r] \arrow[d, "d"'] & 0 \arrow[d] \\ C_1 \arrow[r] \arrow[d, "d"'] & 0 \arrow[d] \\ C_0 \arrow[r, "\varepsilon"] & A \end{tikzcd}$$

is specified by a single morphism $\varepsilon : C_0 \longrightarrow A$ st. $\varepsilon d = 0$.

- To say it is a quasi-iso is to say

a) $H_n(C) \xrightarrow{\simeq} H_n(A[0]) = 0$ for $n \geq 1$ &

$$\begin{array}{ccc} H_0(C) & \xrightarrow{H_0(\varepsilon)} & H_0(A[0]) \\ \| & & \| \\ C_0/\operatorname{im} d & \xrightarrow{\bar{\varepsilon}} & A \\ x + \operatorname{im} d & \longmapsto & \varepsilon x . \end{array}$$

induced map is invertible

Now $\bar{\varepsilon}$ is invertible $\Longleftrightarrow$

b) $\bar{\varepsilon}$ is inj. (its kernel $\ker \varepsilon / \operatorname{im} d = 0$)

c) $\bar{\varepsilon}$ is surj. (equiv. ε is surj) so

(a), (b), (c) $\Longleftrightarrow$

$$\cdots C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{\varepsilon} A \longrightarrow 0 \text{ is exact}$$

where - (c) corr. to ex@A

- (b) - - - - ex@C_0

(c) ex @ other positions. $\square$

Proposition

If $\mathcal{C}$ has enough projectives, then each object has a projective resolution.

Proof

- Let $A \in \mathcal{C}$ & consider $C_0 \xrightarrow{\varepsilon} A$ w/ C_0 proj. Then $C_0 \xrightarrow{\varepsilon} A \to 0$ exact.
- Now form

$$
\begin{tikzcd}
 & & \ker \varepsilon \arrow[dr, hook, "i"] & & & \\
C_1 \arrow[urr, two heads, "p_0"] \arrow[rrr, "d"] & & & C_0 \arrow[r, two heads, "\varepsilon"] & A \arrow[r] & 0
\end{tikzcd}
$$

with C_1 proj, p_0 epi.

- Then $\ker \varepsilon = \operatorname{im} p_0 \leq \operatorname{im} d \leq \ker \varepsilon$ so $\operatorname{im} d = \ker \varepsilon$.
- Now we continue in this way

$$
\begin{tikzcd}
 & & & & \ker \varepsilon \arrow[dr, hook, "i"] & & & \\
\cdots & C_2 \arrow[rr, "d"] \arrow[dr, two heads] & & C_1 \arrow[ur, two heads, "p_0"] \arrow[rr, "d"] & & C_0 \arrow[r, two heads, "\varepsilon"] & A \arrow[r] & 0 \\
 & & \ker p_0 \arrow[ur, hook] & & & & &
\end{tikzcd}
$$

obtaining a projective resolution. $\square$

Derived Functors

Let $F: A \to B$ be a right exact functor
be abelian categories.
The n'th left derived functor $L_nF: A \to B$
is defined as follows:
@ $x \in A$, let
stand. terminology $X_\bullet \xrightarrow{d'} X[0]$ be a proj. resolution
of X ser
$X_\bullet = \dots X_2 \xrightarrow{d} X_1 \xrightarrow{d} X_0$, so can form
$FX_\bullet = \dots \to FX_2 \xrightarrow{Fd} FX_1 \xrightarrow{Fd} FX_0$.
Then we set

$$L_nF(X) = H_n(FX_\bullet).$$

Still have to define L_nF on morphisms.

Remark: Strange definition, since pr. resol.
not functorial nor unique up to iso.
However they are so, up to homotopy, &
we will use this to define L_nF on morphisms.

Lemma
Consider $f: A \to B \in \mathcal{C}$ & proj. resolutions
$A_\bullet \xrightarrow{d} A[0]$ & $B_\bullet \xrightarrow{d} B[0]$ of A & B.
Then $\exists$ a chain map $f_\bullet$ s.t. square

$$\begin{array}{ccc} A_\bullet & \xrightarrow{f_\bullet} & B_\bullet \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ A[0] & \xrightarrow{f[0]} & B[0] \end{array}$$

* commutes, & it is unique w/ this prop. up to chain homotopy.
f in deg-0, else 0.

Proof) – Will constr. $f_\bullet$ inductively.

$$\begin{tikzcd}
\cdots \arrow[r] & A_1 \arrow[r, "d"] \arrow[d, "(3) f_1"'] \arrow[dr, "(2)\,\exists"] & A_0 \arrow[rr, "d"] \arrow[dd, "(1) f_0"] & & A \arrow[r] \arrow[dd, "f"] & 0 \\
 & & \ker d \arrow[d] & & & \\
\cdots \arrow[r] & B_1 \arrow[ur] \arrow[r, "d"'] & B_0 \arrow[rr, "d"'] & & B \arrow[r] & 0
\end{tikzcd}$$

Since rows ex., $d : B_0 \to B$ surj. As A_0 proj., $\exists f_0$ as in (1). Next, $d f_0 d = f d d = 0$ so $f_0 d$ factors through $\ker d \to B_0$. As row is ex., $\mathrm{im}(B_1) = \ker(d)$ so $B_1 \to \ker d$ surj., so as A_1 is proj., (2) factors through B_1 as a map f_1, as in (3). Then continue in same way.

– For uniqueness, suppose we have $g_\bullet : A_\bullet \to B_\bullet$ making * commute. Then we have

$$\begin{tikzcd}
\cdots \arrow[r] & A_2 \arrow[r, "d"] & A_1 \arrow[r, "d"] \arrow[dll, bend right, "h_1"'] \arrow[dl, "(4)\; f_1 - g_1"] & A_0 \arrow[r, "d"] \arrow[dl, bend right, "(3)\; h_0"'] \arrow[d, "(2)"'] \arrow[dd, "(1)\; f_0 - g_0"] & A \arrow[r] \arrow[dd, "f"] & 0 \\
 & \ker d \arrow[d] & & \ker d \arrow[d] & & \\
B_2 \arrow[ur] \arrow[r, "d"'] & B_1 \arrow[urr] \arrow[rr, "d"'] & & B_0 \arrow[r, "d"'] & B \arrow[r] & 0
\end{tikzcd}$$

Then $d(f_0 - g_0) = fd - fd = 0$, so (1) factors as (2) through $\ker d$ & as $B_1 \to \ker d$ is surj & A_0 proj, we obtain (3) h_0 satisfying
$$d h_0 = f_0 - g_0 .$$
Next we need a map $h_1 : A_1 \to B_2$ s.t.
$d h_1 + h_0 d = f_1 - g_1$ or equiv.
$d h_1 = f_1 - g_1 - h_0 d := k$. Now $dk = 0$ as
$d f_1 - d g_1 - d h_0 d = f_0 d - g_0 d - (f_0 - g_0) d = 0$, so
k factors through $\ker d$ as in (4), & now as

A_1 proj, $B_2 \twoheadrightarrow \ker d$ epi, obtain ⑤ h_1 sat
$dh_1 = K$, as required.
Then continue inductively to constr $h_2, \dots$.
$\square$

With this in place, we can define
L_nF on morphisms:
@ $f : A \to B$ obtain $f_\bullet : A_\bullet \to B_\bullet$ sat. *
so $Ff_\bullet : FA_\bullet \to FB_\bullet$ inducing

$$\begin{array}{ccc} L_nF(A) & \xrightarrow{L_nF(f)} & L_nF(B) \\ \| & & \| \\ H_n(FA_\bullet) & \xrightarrow{H_n(Ff_\bullet)} & H_n(FB_\bullet) \end{array}$$

__Proposition__
L_nF is a functor.

__Proof__
Firstly observe that if

$$\begin{array}{ccc} A_\bullet & \overset{g_\bullet}{\underset{f_\bullet}{\rightrightarrows}} & B_\bullet \\ d\downarrow & & \downarrow d \\ A[0] & \xrightarrow{f[0]} & B[0] \end{array}$$

as in $\cancel{*}$, then by lemma $f_\bullet \overset{h}{\simeq} g_\bullet$.

Then $Ff_\bullet \overset{Fh}{\simeq} Fg_\bullet$ so
$H_n(Ff_\bullet) = H_n(Fg_\bullet)$ as
homology identifies homotopic maps.
Hence L_nF is

well defined on morphisms.

Consider $A \xrightarrow{f} B \xrightarrow{g} C$. Then

$$\begin{tikzcd}
A_\bullet \arrow[r, "f_\bullet"] \arrow[d, "d"'] & B_\bullet \arrow[r, "g_\bullet"] \arrow[d, "d"] & C_\bullet \arrow[d, "d"] \\
A[0] \arrow[r, "f[0]"] \arrow[rr, bend right, "gf[0]"'] & B[0] \arrow[r, "g[0]"] & C[0]
\end{tikzcd}$$

so

by Lemma, have $g_\bullet f_\bullet \simeq (gf)_\bullet$,

so $F(g_\bullet)F(f_\bullet) \simeq F(gf_\bullet)$ so

$$H_nF(g_\bullet)\,H_nF(f_\bullet) = H_nF(gf_\bullet)$$

$$L_nF(g)\,L_nF(f) \qquad\qquad L_nF(gf).$$

Similarly, L_nF pres. identities & so is a functor. $\square$

Next time, more theory of der. functors & examples – Ext, Tor.