# Exercise 5 - Algebra 4

- In the course we introduced $\mathrm{Ext}^n(A,-)$ as the right $n$th right derived functor of $\mathrm{Hom}(A,-) : \mathrm{Mod}_R \to \mathrm{Ab}$, but then mentioned that elements of $\mathrm{Ext}^n(A,B)$ (for $n>0$) are equiv. classes of exact sequences

$$0 \to B \to X_1 \to \dots \to X_n \to A \to 0 .$$

- Here we explore some of the algebra of such exact sequences.

## ① (Splicing of exact sequences)

Consider exact sequences

$$0 \to B \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \to X_n \xrightarrow{f_n} A \to 0 \quad \&$$

$$0 \to C \xrightarrow{g} Y \xrightarrow{h} B \to 0 .$$

Show that

$$0 \to C \xrightarrow{g} Y \xrightarrow{f_0 h} X_1 \xrightarrow{f_1} X_2 \to \dots \to X_n \xrightarrow{f_n} A \to 0$$

is exact.

- What is relationship to les of cohomology for Ext?
- Can you generalise to longer sequences?

## ② (Decomposing exact sequences)

The first question shows how to "splice" exact sequences into longer ones. Show that every exact sequence of length $n$ can be obtained by splicing $n$ short exact sequences.

## ③ (Pullback & pushout)

- Consider

$$
\begin{tikzcd}
 & & & A' \arrow[d, "\alpha"] & \\
0 \arrow[r] & B \arrow[r, "f"'] & X \arrow[r, "g"'] & A \arrow[r] & 0
\end{tikzcd}
$$

with exact row

Show that the top row is ex.

$$
\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "f'"] \arrow[d, "1"'] & P \arrow[r, "g'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & B \arrow[r, "f"'] & X \arrow[r, "g"'] & A \arrow[r] & 0
\end{tikzcd}
$$

where $g'$ is pullback of $g$

& $f'x = (fx, 0)$.

- Explain the dual constr. for pushouts.

4) Explain how the previous constructions give actions

$$\mathrm{Ext}_1(A,B) \longrightarrow \mathrm{Ext}_1(A',B)$$
$$E \longmapsto E\alpha \quad \&$$

&

$$\mathrm{Ext}_1(A,B) \longrightarrow \mathrm{Ext}_1(A,B')$$
$$E \longmapsto \beta E$$

where $\beta : B \to B'$.

making $\mathrm{Ext}_1(-,-) : \mathrm{Mod}_R^{op} \times \mathrm{Mod}_R \to \mathrm{Ab}$ a functor.

⑤ (Baer sum)

Given $E : 0 \to B \to X \to A \to 0$

& $E' : 0 \to B \to Y \to A \to 0$

in $\mathrm{ExT}_1(A,B)$ we will define the Baer sum

$$E + E' \in \mathrm{ExT}_1(A,B).$$

Fistly, take

$$E \oplus E' : 0 \to B \oplus B \to X \oplus Y \to A \oplus A \to 0$$

& then define

$$E + E = \nabla_A (E \oplus E') \Delta_B$$

where $\nabla_A : A \oplus A \to A$ is codiagonal

$$(x,y) \longmapsto x+y$$

& $\Delta_B : B \longrightarrow B \oplus B$

$$x \longmapsto (x,x)$$

is diagonal.

- Show that this makes $\mathrm{ExT}_1(A,B)$

into an abelian group.

- What is the unit?