Lecture 5

Today :- finish left derived functors
- dual cohomology & right derived functors
- examples : Ext, Tor

Recap : For $F: A \to B$ right exact, & A having enough projectives, defined

missing in video

$L_n F : A \to B : X \mapsto H_n(FX_\bullet)$ proj resolution of X

Proposition

For $F: A \to B$ right exact functor (between abelian categories) as above, we have a natural isomorphism $L_0 F \cong F$.

Proof At $X \in A$, we have $X_\bullet \xrightarrow{d} X[0]$ & so $L_0 FX = H_0 FX_\bullet \xrightarrow{H_0 F(d)} H_0 FX[0] = X$ gives the components of a natural transformation
$L_0 F \longrightarrow F$.

For invertibility, as F is rex,
$FX_1 \xrightarrow{Fd_0} FX_0 \xrightarrow{Fd} FX \longrightarrow 0$ is exact,
so $\ker Fd = \operatorname{im} Fd_0$ so

$$
\begin{aligned}
H_0 FX_\bullet &= FX_0 / \operatorname{im} Fd_0 \\
&\cong FX_0 / \ker Fd \\
&\cong FX \quad \text{as } Fd \text{ epi}. \ \square
\end{aligned}
$$

Further properties of left derived functors

① Horseshoe lemma (without proof)

- If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \in \mathcal{A}$ is ses we can lift it to a ses of chain complexes

$$\begin{tikzcd} 0 \arrow[r] & X_\bullet \arrow[r, "f_\bullet"] \arrow[d, phantom, "\perp"] & Y_\bullet \arrow[r, "g_\bullet"] \arrow[d, phantom, "\perp"] & Z_\bullet \arrow[r] \arrow[d, phantom, "\perp"] & 0 \\ 0 \arrow[r] & X[0] \arrow[r, "{f[0]}"'] & Y[0] \arrow[r, "{g[0]}"'] & Z[0] \arrow[r] & 0 \end{tikzcd}$$

as on top row, which is (moreover) componentwise

split exact: for each $n \geq 0$, of form

$$0 \longrightarrow A \xrightarrow{i_A} A \oplus B \xrightarrow{p_B} B \longrightarrow 0$$

for a biproduct diagram (such is always short exact).

② If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \in \mathcal{A}$ is ses

& $F : \mathcal{A} \longrightarrow \mathcal{B}$ we obtain a les

$$\cdots L_2FZ \to L_1FX \to L_1FY \to L_1FZ \to FX \to FY \to FZ \to 0$$

of left derived functors:

- Indeed, F takes split exact sequences to split exact sequences (as it preserves biproducts)
- Then apply les of homolology to ses

$$0 \longrightarrow FX_\bullet \xrightarrow{Ff_\bullet} FY_\bullet \xrightarrow{Fg_\bullet} FZ_\bullet \longrightarrow 0 .$$

(3) The right exact F is exact $\iff L_nF=0$ $\forall n\geq 1$
$\iff L_1F=0$:

- If F is exact, $FX_\bullet$ is exact sequence, so $H_nFX_\bullet \simeq 0$ all $n\geq 1$, so $L_nF=0$ all $n\geq 1$.
- Therefore $L_1F=0$.

Ass. $L_1F=0$, at ses $0\to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \in \mathcal{A}$ obtain les

$$\cdots \longrightarrow \underset{\underset{0}{\parallel}}{L_1FZ} \longrightarrow FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \to 0$$

so $0 \to FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \to 0$ is a ses.

Conclusion

The higher L_nF for $n\geq 1$ measure the failure of F is to be exact.

Cohomology & right derived functors

- So far - ch. complexes, homology, proj. res., in an abelian cat $\mathcal{A}$ & left derived functors $F : \mathcal{A} \to \mathcal{B}$
- Can also look at these concepts in opposite abelian cat $\mathcal{A}^{op}$, $\mathcal{A}^{op} \to \mathcal{B}^{op}$, which give dual notions of cochain complex, cohomology, inj. resolutions & right derived functors.
- A cochain complex in $\mathcal{A}$ is a diagram

$$\cdots X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \cdots \quad \text{s.t.}$$

$$d^n \circ d^{n-1} = 0 \quad (\text{ie. chain comp. in } \mathcal{A}^{op}).$$

- Cohomology $H^n X = \ker d^n / \operatorname{im} d^{n-1}$.

Examples

① If X is a chain complex in $\mathcal{A}$ & $A \in \mathcal{A}$, then

$$\cdots \mathcal{A}(X_n, A) \xrightarrow{\mathcal{A}(d_n, A)} \mathcal{A}(X_{n+1}, A) \cdots$$

$$X_n \xrightarrow{f} A \longmapsto X_{n+1} \xrightarrow{d_n} X_n \xrightarrow{f} A$$

is a cochain complex.

② If X is a topological space, recall that we can form chain comp.

$S_n X$ in Ab, whose homology is singular homology of X.

If $A \in Ab$, can form cochain

complex

$$\cdots \mathrm{Ab}(S_n X, A) \xrightarrow{\mathrm{Ab}(d_n, A)} \mathrm{Ab}(S_{n+1} X, A)$$

as in ① above, & by def^n its cohomology is singular cohomology of X with coefficients in A.

Right derived functors

- An inj resolution of $X \in \mathcal{A}$ is an exact sequence

  $0 \to X \to X^0 \to X^1 \to \cdots$ with each X^i injective.

  This gives a positive cochain comp. $X^\bullet$ & morphism $X[0] \to X^\bullet$ which induces an iso. on cohomology.
- If $\mathcal{A}$ has enough injectives, each ob X has an injective resolution.
- If $F : \mathcal{A} \to \mathcal{B}$ is lex functor & $\mathcal{A}$ has enough injectives, we can form its n'th right derived functor $R^n F : \mathcal{A} \to \mathcal{B}$, which has value $R^n F(X) = H^n(F X^\bullet)$.
- We have dual props to

these from before:

- $R^0 F \cong F$
- For each ses $0 \to X \to Y \to Z \to 0$ obtain les
  $0 \to FX \to FY \to FZ \to R^1FX \to R^2FY \to \dots$
- F is exact $\Leftrightarrow$
  $R^nF = 0$ all $n \geq 1$ $\Leftrightarrow$
  $R^1F = 0$.

Ext & Tor (no proofs!)

- Begin with Tor, but focus on Ext.
- For A right R-mod, B left R-module, can form their t.p. $A \otimes_R B \in Ab$ w/ universal property:

$A \otimes_R B \longrightarrow C \in Ab$ is in bij^n w/ function
$f : A \times B \longrightarrow C$ which is Ab-gp hom. in each var.

- In this way, we obtain a functor

$$\begin{aligned} - \otimes_R - : R\text{-Mod} \times \text{Mod}R &\longrightarrow Ab \\ (A, B) &\longmapsto A \otimes_R B, \end{aligned}$$ & so

functors $A \otimes_R - : \text{Mod}_R \to Ab$ & $- \otimes_R B : R\text{-Mod} \to Ab$, and these are rex.

Then $\mathrm{Tor}_n^A(A,B) := L_n(- \otimes_R B)(A)$, which is given by n'th homology of

$$\cdots \quad A_2 \otimes_R B \longrightarrow A_1 \otimes_R B \longrightarrow A_0 \otimes_R B$$

where $A_\bullet$ is proj. resolution of $A \in R\text{-Mod}$, so $H_n(A_\bullet \otimes_R B)$.

Equiv., it can be calc as

$L_n(A \otimes_R -)(B)$, so take proj. resolution of B, tensor by A & calc. homology,

then $H_n(A_\bullet \otimes_R B) \cong H_n(A \otimes_R B_\bullet)$.

Ext

- Given $A \in \mathrm{Mod}_R$, can form $\overset{\text{"}\mathrm{Hom}(A,-)\text{"}}{\mathrm{Mod}_R(A,-)} : \mathrm{Mod}_R \to \mathrm{Ab}$ which is rex & as Mod_R has enough injs, can form right der. functor
$\mathrm{Ext}_n(A,-) := R^n \mathrm{Hom}(A,-)$, so
$\mathrm{Ext}_n(A,B) := H^n \mathrm{Hom}(A, B^\bullet)$ for $B^\bullet$ a inj. resolution $B_0 \to B_1 \to B_2 \cdots$ of B.
- Note also have rex functor
$$\mathrm{Hom}(-,B) : \mathrm{Mod}_R^{op} \longrightarrow \mathrm{Ab}$$
& so can calculate
$$R^n \mathrm{Hom}(-,B) : \mathrm{Mod}_R^{op} \longrightarrow \mathrm{Ab} \text{ as}$$
$R^n \mathrm{Hom}(-,B)(A)$ as H_n app to
$$\mathrm{Hom}(A_0,B) \to \mathrm{Hom}(A_1,B) \to \mathrm{Hom}(A_2,B) \cdots$$
for $A_\bullet$ a proj resolution of A (ie. inj. resolution in Mod_R^{op}.)

In fact, $\mathrm{Ext}^n(A,B) =$
$$H^n \mathrm{Hom}(A, B^\bullet) \cong H^n \mathrm{Hom}(A_\bullet, B)$$
so 2 ways of calc. $\mathrm{Ext}^n(A,B)$.

- Ext can be understood more explicitly using extensions:
an extension of A by B is a ses
$0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0$.
- Two extensions of A by B are equiv ($\simeq$) if $\exists$ iso of ses of form

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "1"'] & X \arrow[r] \arrow[d, "S"] & A \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & B \arrow[r] & Y \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

- Then $\mathrm{Ext}^1(A,B) \equiv \{\text{extensions of } A \text{ by } B\}/\simeq$ modulo equivalence

- From earlier work,
A is proj. $\Longleftrightarrow$ $\hom(A,-)$ is exact
$\Longleftrightarrow$ $\mathrm{Ext}^1(A,B) = 0$ all B:
this says that each extension as above of A by B is iso to trivial ext.

$$0 \longrightarrow B \longrightarrow A \oplus B \longrightarrow A \longrightarrow 0,$$

or split exact.
- For higher n, elts of $\mathrm{Ext}^n(A,B)$ are exact sequences of length n
$0 \to B \to \cdots \to A \to 0$ modulo equivalence.

Group cohomology

- G a group, can form group (co)homology using above:
- consider category Mod_G of G-modules: such consists of an ab. group A with an assoc. action $G \times A \xrightarrow{\cdot} A : (g,a) \longmapsto g.a$, equiv. a hom. $G \longrightarrow Ab(A,A)$.

(will look at these more closely in group representation theory!!)

- These are $\mathbb{Z}G$-modules where $\mathbb{Z}G$ the group ring, whose elements are sums

$$n_1 g_1 + \dots + n_k g_k \quad , n_i \in \mathbb{Z}, g_i \in G.$$

- The inclusion

$$Ab \xrightarrow{\quad i \quad} G\text{-Mod}$$

sends A to same ab. group A w/ trivial action $g.a = a$.

- We have adjunctions $\mathrm{Coinv} \dashv i \dashv \mathrm{Invar}$.
- Here $\mathrm{Invar}(M) = \{x \in M : gx = x \text{ all } g \in G\}$ or equally the limit of the diagram

$$\Sigma G \xrightarrow{\quad M \quad} Ab$$

1 ob. cat.

$$\cdot \circlearrowleft g \longmapsto M \circlearrowleft g\cdot -$$

ie.

$$\begin{tikzcd} \mathrm{Invar}(M) \arrow[r, hook] \arrow[dr, hook] \arrow[dr, phantom, shift left=3, "="] & M \arrow[d, "g\cdot -"] \\ & M \end{tikzcd} \quad \forall g.$$

- It's colimit is $\mathrm{Coinvar}(M) = M / \langle gx - x : x \in M, g \in G \rangle$.
- In partic.,
  $\mathrm{Invar} : \mathbb{Z}G\text{-Mod} \longrightarrow \mathrm{Ab}$ is lex,
  & the group cohomology of G w' coefficients in M is defined as $R^n(\mathrm{Invar}(-))(M)$. Note, however that $\mathrm{Invar}(A) \cong \mathbb{Z}G\text{-Mod}(i\mathbb{Z}, A)$, so
  $R^n(\mathrm{Invar}(-))(M) \cong \mathrm{Ext}_n(i\mathbb{Z}, M)$, which can be calc. using proj. resolution of $i\mathbb{Z}$ called Bar resolution.
  Group cohomology described in this way captures extensions, connections with crossed homomorphisms, Factor systems ... arising in group Theory.
- When M is an ab. group, viewed as triv. G-mod. iM, then this coincides w' cohomology of Top. space BG w' coefficients in M described earlier.