Lecture 6

- Projective dimension & some calcs of Ext

Def^n) R a ring & $A \in \mathrm{Mod}_R$. A proj. resolution of form

$$\cdots 0 \to P_n \to \cdots P_1 \to P_0 \to A \to 0$$

with all $P_i \neq 0$ for $i \leq n$ is said to be a proj. resolution of length n.

Examples

(1) For $R = \mathbb{Z}$, so A an abelian group, consider

$$0 \longrightarrow \mathrm{Ker}(\varepsilon) \hookrightarrow \underbrace{\mathrm{Fu}A}_{\text{free ab. group on und. set}} \xrightarrow{\varepsilon} A \to 0$$

Then as each subgroup of a free abelian group is free, $\mathrm{Ker}(\varepsilon)$ is free, so the above is a projective resolution of length 1.

(2) In the exercise class, we saw for $R = \mathbb{Z}/4$, the $\mathbb{Z}/4$-mod. $\mathbb{Z}/2$ has a proj. resolution

$$\cdots \mathbb{Z}/4 \xrightarrow{2\cdot -} \mathbb{Z}/4 \xrightarrow{2\cdot -} \mathbb{Z}/4 \xrightarrow{\bmod 2} \mathbb{Z}/2 \to 0$$

of infinite length.

Def^n) The projective dimension of A, $\mathrm{pd}(A)$, is the least n such that A has a proj. resolution of length n.

$\boxed{\text{Notation}}$ Given proj. resolution P of A, $K_n = \mathrm{Ker}(P_n \to P_{n-1})$ is module of n-syzygies.

Theorem

For $A \in Mod_R$, tfae:

1. $pd A \leq n$,
2. $Ext^k(A,B)=0$ all B & $k \geq n+1$,
3. $Ext^k(A,B)=0$ all B & $k=n+1$,
4. For each proj. res. P of A, we have K_{n-1} is projective.

Proof

Assume 1 & let P be a proj. resolution of length $\leq n$.
Then $Ext^k(A,B)$ is k'th cohomology of

$$\cdots Hom(P_{k-1},B) \longrightarrow Hom(P_k,B) \longrightarrow Hom(P_{k+1},B) \cdots$$

but as $P_k=0$, $Hom(P_k,B)=0$, so $Ext^k(A,B)=0$ proving 2.

2 trivially implies 3.

For 3 $\Rightarrow$ 4, observe that

$$\cdots\; P_{n+1} \longrightarrow P_n \twoheadrightarrow K_{n-1} \hookrightarrow P_{n-1}$$

is a projective res. of K_{n-1}.

Hence $Ext^1(K_{n-1},B)$ is cohomology of

$$Hom(P_n,B) \to Hom(P_{n+1},B) \to Hom(P_{n+2},B)$$

so $Ext^1(K_{n-1},B) = Ext^{n+1}(A,B)=0$.

Hence, from result earlier in course, K_{n-1} is projective.

For 4 $\Rightarrow$ 3, take proj. resolution

$$0 \to K_n \hookrightarrow P_{n-1} \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A. \quad \square$$

Cor) $pd(A)=n$ if n is greatest number st $\exists$ non-zero $Ext^n(A,B)$ for some B.

Example

- In the resolution of $\mathbb{Z}/2$:

$$\cdots \mathbb{Z}/4 \xrightarrow{2\cdot} \mathbb{Z}/4 \xrightarrow{2\cdot -} \mathbb{Z}/4 \xrightarrow{\text{mod } 2} \mathbb{Z}/2 \to 0$$

as $\mathbb{Z}/4$-module,

$$K_n = \mathbb{Z}/2.$$

Exercise: show $\mathbb{Z}/2$ not projective as $\mathbb{Z}/4$-module

Hence $\mathbb{Z}/2$ has infinite proj. dim.

- Calculation of $Ext^n_{\mathbb{Z}}(A,B)$:

- $Ext^0(A,B) = Ab(A,B)$.
- By example earlier, $pd(A) \leq 1$, so by prev. prop $Ext^k(A,B) = 0$ for $k \geq 2$.
- What about $Ext^1(A,B)$?

If A is fin. generated,

$$A \cong \mathbb{Z}^m \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_k}.$$

- Not hard to see $Ext^1(-,B)$ is additive so

$$Ext^1(A,B) = Ext^1(\mathbb{Z}^m,B) \oplus Ext(\mathbb{Z}_{p_1},B) \oplus \cdots$$
$$= Ext(\mathbb{Z}_{p_1},B) \oplus \cdots \oplus Ext(\mathbb{Z}_{p_k},B)$$

as $\mathbb{Z}^m$ is free, so $Ext^1(\mathbb{Z}^m,B) = 0$.

- What about $Ext^1(\mathbb{Z}_p,B)$?
- Then

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p\cdot -} \mathbb{Z} \xrightarrow{\text{mod} p} \mathbb{Z}_p \longrightarrow 0$$

is a ses & so a proj. resolution of $\mathbb{Z}_p$, so
$Ext^1(\mathbb{Z}_p, B)$ is cohomology of

$$\begin{array}{ccccc}
hom(\mathbb{Z}, B) & \xrightarrow{(p\cdot -)^*} & hom(\mathbb{Z}, B) & \longrightarrow & 0 \\
\wr\| & '' & \wr\| & & \\
B & \xrightarrow{p\cdot -} & B & \longrightarrow & 0
\end{array}$$

so that $\underline{Ext^1(\mathbb{Z}_p, B) = B/pB}$.

Putting together,

$$\underline{Ext^1(A, B) = B/p_1 B \oplus \cdots \oplus B/p_n B}.$$

- If A not finitely gen, more complex.

Def^n) The <u>left projective dimension</u> of a ring R is defined by
$$lpd(R) = \sup\{pd(A) : A \text{ a left } R\text{-mod}\}.$$

$\boxed{\text{Remark}}$ • Dually one can consider injective dimension $id(A)$, the shortest length of an inj. resolution of A.

• Since $Ext^k(A, B)$ can be calc. using a proj. res. of A or an inj. res. of B, the preceding proposition about Ext.. has dual version for inj. dimension.

• Again, can consider left injective dimension of a ring $\mathrm{lid}(R)$. In fact,

Prop) $\mathrm{lpd}(R) = \mathrm{lid}(R)$

$\boxed{\text{Proof}}$ $\mathrm{lpd}(R) \leq n \iff$

$\forall A \in \mathrm{Mod}_R$, $\mathrm{pd}(A) \leq n \iff$ (by Prop.)

$\forall A, B \quad \mathrm{Ext}^k(A,B) = 0$ for $k \geq n+1 \iff$ (by dual prop.)

$\forall B \quad \mathrm{id}(B) \leq n \iff$

$\mathrm{lid}(R) \leq n$. Hence,

$\mathrm{lpd}(R) = \mathrm{lid}(R)$ since this is true $\forall n$. $\square$

The common value
$$\mathrm{lgd}(R) := \mathrm{lpd}(R) = \mathrm{lid}(R)$$
is called the <u>left global dimension</u> of R,

• Of course, there is also <u>right</u> proj, inj, global dimension but these coincide with left versions when R is <u>commutative</u>, in which case we speak of <u>global dimension $\mathrm{gd}(R)$</u>.

<u>Examples</u>

① $\mathrm{gd}(\mathbb{Z}) = 1$.

② $\mathrm{gd}(\mathbb{Z}/4) = \infty$.

Hilbert's Syzygy Theorem

If R is a field, then the polynomial ring $R[x_1, ---, x_n]$ has global dimension n.

- We will not prove this.

Exercise

Consider poly ring $R = \mathbb{C}(x, y)$ & let $M \subseteq \mathbb{C}(x, y)$ be ideal of polys with no zero term & consider quotient ring R/M as an R-module.

Consider surj. $R \longrightarrow R/M \longrightarrow 0$ of R-modules & extend this to a proj. resolution of length 2.