Lecture 9 - Varieties & commutative algebras

last time : varieties vs ideals of poly. rings

varieties & maps of varieties vs commutative k-algs

Def^n) Let k be a field & $A \subseteq k^n$ & $B \subseteq k^\ell$ be varieties. A <u>polynomial map</u> $f : A \to B$ is a function such that $\exists$ polys $f_1, \dots, f_\ell \in k[x_1, \dots, x_n]$ with $\forall a \in A$ $f(a) = (f_1(a), \dots, f_\ell(a))$.

$\boxed{\text{Prop}^n}$ Varieties & polynomial maps form a category <u>Var</u>.

Proof) Consider

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \cap & & \cap & & \cap \\ k^\ell & & k^n & & k^m \end{array}$$

rep. by polynomials $(f_1, \dots, f_n)$ & $(g_1, \dots, g_m)$: then gf is represented by polys $h_1, \dots, h_m$ where $h_i(x_1, \dots, x_\ell)$
$= g_i(f_1(x_1, \dots, x_\ell), \dots, f_n(x_1, \dots, x_\ell))$.

The identity $\begin{array}{ccc} A & \xrightarrow{id} & A \\ \cap & & \cap \\ k^n & & k^n \end{array}$ is polynomial since rep. by $(x_1, \dots, x_n)$.

Clearly associative & unital since just function composition. $\square$

Def) For A a variety, the <u>co-ordinate ring</u> <u>$k(A)$</u> $=$ Var(A, k) is a commutative K-algebra whose elements are polynomial maps $A \to k$ with operations pointwise as in k:

- $f+g\,(a) = f(a)+g(a)$
- $f.g\,(a) = f(a).g(a)$
- $\lambda f\,(a) = \lambda.f(a)$.

• $K(A)$ can also be described more algebraically.

Proposition

There is an iso of K-algebras

$$k[x_1,\dots,x_n]/I(A) \cong K(A)$$

where $I(A)$ is ideal of polys vanishing at A.

~~Proof~~ The function $k[x_1,\dots,x_n] \twoheadrightarrow K(A)$

$$f \longmapsto A \hookrightarrow k^n \xrightarrow{f} k$$

is a surjective K-algebra homomorphism & its kernel consists exactly of $I(A)$.

Hence we obtain iso, by first iso thm, $k[x_1,\dots,x_n]/I(A) \cong K[A]$. □

Properties: ① As k is a field, $k[x_1,\dots,x_n]$

is Noetherian; hence so is quotient $k[A]$.

② As $I(A)$ is radical, the quotient $k[A]$ is reduced: (ie. $f^n = 0 \Rightarrow f = 0$).

③ The comm. k-alg $k[x_1, \ldots, x_n]$ is freely generated by $x_1, \ldots, x_n$; therefore $k(A)$ is generated by the image of thes under

$$k[x_1, \ldots, x_n] \twoheadrightarrow k(A) :$$
$$x_i \longmapsto p_i : A \hookrightarrow k^n \xrightarrow{x_i} k$$
$$a = (a_1, \ldots, a_n) \longmapsto a_i \;;$$

ie. $k(A)$ is finitely generated by the n projections $p_i : A \longrightarrow k$.

- Given a morphism $f : A \longrightarrow B \in Var$ we obtain

$$k(B) = Var(B, k) \xrightarrow{f^*} Var(A, k) = k(A)$$
$$B \xrightarrow{g} k \longmapsto A \xrightarrow{f} B \xrightarrow{g} k$$

which is a k-algebra homomorphism since operations are component-wise as in k.

- This makes $k(-) : Var^{op} \longrightarrow Comm\text{-}k\text{-}Alg$ a functor.

Theorem

$K(-) : Var^{op} \longrightarrow$ Comm-k-Alg is fully faithful.

(ie. given $K(A) \xrightarrow{\alpha} K(B) \in$ Comm.k-Alg $\exists !$ $B \xrightarrow{f} A \in Var$ st $\alpha = f^*$.)

Proof

- Firstly, we show faithfulness: consider

$$\begin{array}{ccc} A & \overset{f}{\underset{g}{\rightrightarrows}} & B \\ \cap & & \cap \\ k^n & & k^m \end{array} \in Var$$

& suppose $K(B) \overset{f^*}{\underset{g^*}{\rightrightarrows}} K(A)$, $f^* = g^*$. Must show $f = g$.

- So given $B \xrightarrow{h} k$ we have $f^* h = g^* h$, ie. $hf = hg$.
- In partic, consider $p_i : B \longrightarrow k$, $a \longmapsto a_i$ for $i \in \{1, \ldots, m\}$
- Then $p_i f(a) = f_i(a)$ where $f(a) = (f_1(a), \ldots, f_n(a))$.
- Therefore $p_i f = p_i g$ all i says $f_i(a) = g_i(a)$ all i so $f(a) = g(a)$ all $a \in B$; hence $f = g$.

- For fullness, consider $\alpha : K(B) \longrightarrow K(A)$. We must find f such that $\alpha = f^*$, but then $\alpha(p_i) = f^*(p_i) = p_i \circ f$.

Therefore, we must define f by

$F(a) = (\alpha(p_1)a, -----, \kappa(p_m)a)$.

Certainly this is polynomial since each $\alpha(p_i) \in K(A)$ is polynomial map.

It remains to show that if $a \in A$ then $fa \in B$: indeed, suppose $B = V(g_1, ----, g_r) = \{b \in K^m : g_i b = 0 \text{ all } i \in \{1, \dots, r\}\}$.

- We must show $g_i f(a) = 0$ all $a \in A$:

ie. $g_i F(a) = g_i(\alpha(p_1)a, -----, \kappa(p_m)a)$
$$= (g_i \kappa(p_1)a, \dots\dots, g_i \kappa(p_m)a) = 0 \text{ all } a \in A.$$

- This is equally to say
$$g_i(\kappa(p_1), \dots, \kappa(p_m)) = 0 \text{ in } K(A)$$

- But as $\alpha : K(B) \longrightarrow K(A)$ is a homomorphism of K-algebras, we have this equals $\alpha(g_i(p_1, -------, p_m))$ so it suff. to show $g_i(p_1, ---, p_m) = 0 \in K(B)$.

- But this is precisely to show
$$g_i(p_1(b), \dots, p_m(b)) = 0 \quad \text{all } b \in B$$
$$\parallel$$
$$g_i(b_1, ----, b_m) \quad \text{where } b = (b_1, ---, b_m)$$

& this is zero by assumption : ie. $B = V\{g_1, ---, g_r\}$.

$\square$

Corollary

Two varieties A & B are iso $\Longleftrightarrow$ $K(A)$ & $K(B)$ are iso as comm. K-algebras.

Proof : (Exercise : Fully faithful functor reflects isos.)

Corollary

- If k is algebraically closed, then a comm. k-algebra S is iso to some $K(A)$ $\iff$ S is a finitely gen. reduced k-alg.

Proof

- Certainly $K[A]$ is reduced, as we have seen & f.g.
- Conversely suppose S is f.g. reduced. As f.g., have surj. alg. hom.

$$k[x_1, \dots, x_n] \xrightarrow{p} S$$

whose kernel $\ker(p)$ is radical since

$$S \cong k[x_1, \dots, x_n]/\ker(p) \text{ is reduced},$$

Hence by the Nullstellensatz,

$$\ker(p) = IV\ker(p);$$

so

$$S \cong k[x_1, \dots, x_n]/IV\ker(p) = K(V\ker p).$$

$\square$

Remark : Therefore we have a fully faithful

Functor $Var^{op} \xrightarrow{k(-)}$ Red-Comm-k-Alg
which if k is algebraically closed
is essentially surjective (surj. up to iso):
in other words, for k alg. closed
we have an equivalence of categories

$$Var^{op} \xrightarrow{k(-)} \text{Red-Comm-}k\text{-Alg}.$$