Algebra 4 - 2021

1) Homological algebra
2) Commutative algebra (related to Hilbert.. algebraic geometry)
3) Representation theory of finite groups.

Course details

- Weekly lecture uploaded to IS
- Exercise class on Tuesday on Zoom
- 3 marked assignments (50%) + oral exams.

Homological algebra

- R a ring, Mod_R cat of left R-modules.

Exact sequences

- Consider $A \xrightarrow{f} B \xrightarrow{g} C \in \mathrm{Mod}_R$ in which $g \circ f = 0$.
- Then $g(fx) = 0$ all $x \in A$ so $\mathrm{im}(f) \leq \ker(g)$.

Def^n) The sequence is exact at B if $\mathrm{im}(f) = \ker(g)$.

Examples

- $0 = \{0\} \longrightarrow A \xrightarrow{f} B$ is exact $\iff \ker f = 0$ $\iff$ f is injective (or mono).
- $A \xrightarrow{f} B \longrightarrow 0$ is exact $\iff \mathrm{im} f = B$ $\iff$ f is surjective (equiv. epi, as see later.)
- A short exact sequence is a sequence
$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \quad \text{st } g \circ f = 0$$
which is exact at each position:
- ex @ A : f is injective,
- ex @ C : g is surjective,
- ex @ B : $\mathrm{im} f = \ker g$, but as g is surj
$$C \cong B/\ker g = B/\mathrm{im} f.$$

$\boxed{\text{Exercise}}$ Using this, show that each ses is of the form
$$0 \longrightarrow A \hookrightarrow B \longrightarrow B/A \longrightarrow 0$$
up to isomorphism of sequences.

Def) A chain complex A is a sequence
$$\cdots A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to \cdots$$ of R-modules
where $n \in \mathbb{Z}$ & $d_n \circ d_{n+1} = 0 \quad \forall n$.

- $Z_n := \ker(d_n) \leqslant A_n$ has elements called n-cycles.
- $B_n := \operatorname{im}(d_{n+1}) \leqslant \ker(d_n)$ are called n-boundaries.

Def^n) The n'th homology $H_n(A)$ is the R-module $Z_n / B_n = \dfrac{\ker(d_n)}{\operatorname{im}(d_{n+1})}$

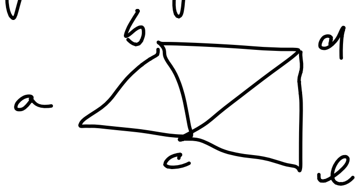
Remark) - Observe that
$$A \text{ is exact } @ A_n \iff H_n(A) = 0.$$
Thus the n-th homology measures the failure of A to be exact at A_n.

- If A is exact at all n, it is called a long exact sequence.

Examples

① Simplicial homology

K a geometric simp. complex - eg triang. space



- K_n set of n-simplices, each of which has n+1 faces.
- If set of vertices is ordered, so are the face maps $d_i : K_n \longrightarrow K_{n-1}$ for $i = 0, 1, \ldots, n$, where d_i omits i'th vertex.
- Forming free R-modules, we obtain $C_n \xrightarrow{d_i} C_{n-1}$ applying d_i to basis vectors & then

$$\therefore C_n \xrightarrow{\sum_{i=0}^{n} (-1)^i d_i} C_{n-1} \ldots$$ is a chain complex

called C_K, with

$H_n(C_K)$ = n-th simplicial homology of K.

② Singular homology

X a top. space,

$S_n(X)$ = free R-module on set of all cts maps

$\Delta_n \longrightarrow X$ where
Δ_n is standard n-simplex
$\Delta_n \subseteq \mathbb{R}^{n+1}$
$= \{ (x_0, \ldots, x_n) : \Sigma x_i = 1 \}$.

- Sim. to before, obtain face maps & chain comp. $\ldots S_n(X) \xrightarrow{d = \Sigma(-1)^i d_i} S_{n-1}(X) \ldots$ whose homology is called <u>singular homology</u> of X.

If X is a geom. simplicial complex, same as simplicial homology.

- A chain map $f: A \to B$ of ch. complexes consists of maps $f_n : A_n \to B_n$ such that

$$\begin{tikzcd} A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] \arrow[dr, phantom, "\circlearrowleft"] & A_n \arrow[d, "f_n"] \\ B_{n+1} \arrow[r, "d_{n+1}"'] & B_n \end{tikzcd} \quad \forall n.$$

Notation: often one writes $d: A_{n+1} \to A_n$ when context is clear.

- Chain complexes and chain maps form a category $Ch(Mod_R)$.

Proposition

The n'th homology determines a functor

$$H_n : Ch(Mod_R) \longrightarrow Mod_R$$

$\boxed{\text{Proof}}$ - It sends $A \longmapsto H_n(A)$

- At $f: A \to B$, then given $x \in Z_n(A) = \ker(d: A_n \to A_{n-1})$ we have $dfx = fdx = 0$ so $fx \in Z_n(B)$; similarly if $x \in B_n(A)$ then $fx \in B_n(B)$; hence we obtain

$$H_n(A) = \frac{Z_n(A)}{B_n(A)} \longrightarrow \frac{Z_n(B)}{B_n(B)} = H_n(B)$$

$$x + B_n(A) \longmapsto fx + B_n(B)$$

or just i.e. $[x] \longmapsto [fx]$

This is clearly functorial. $\square$

Homotopy

Def^n) let $f, g : A \rightrightarrows B$ be chain maps. A chain htpy s from f to g (written $s : f \sim g$) is a sequence of maps $s_n : A_n \longrightarrow B_{n+1}$ (as in the picture below)

$$\begin{tikzcd}
\cdots A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}-g_{n+1}"'] & A_n \arrow[r, "d_n"] \arrow[dl, "s_n"'] \arrow[d, "f_n-g_n"] & A_{n-1} \cdots \arrow[dl, "s_{n-1}"'] \arrow[d, "f_{n-1}-g_{n-1}"] \\
\cdots B_{n+1} \arrow[r, "d_{n+1}"] & B_n \arrow[r, "d_n"] & B_{n-1} \cdots
\end{tikzcd}$$

such that

$$d_{n+1}s_n + s_{n-1}d_n = f_n - g_n \quad \forall n \in \mathbb{Z}.$$

- A chain map $f : A \longrightarrow B$ is null homotopic if $f \sim 0$.

- It is a <u>htpy equivalence</u> if $\exists g : B \to A$ such that $fg \sim 1_B$ & $gf \sim 1_A$.

Lemma

Homotopy is an equivalence relation, compatible with composition.

Proof Consider $f : A \to B$.

- Taking $s_n = 0$ shows $f \sim f$.
- If $s : f \sim g$ & $t : g \sim h$ take $s+t : f \sim h$; indeed $h - f = (h-g) + (g-f) = (ds+sd) + (dt+td)$
$= d(s+t) + (s+t)d$.
- If $s : f \sim g$ then $-s : g \sim f$.
- If

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "s"] & B \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "s'"] & C \end{tikzcd}$$

must show $f'f \sim g'g$. By transitivity of $\sim$ suff. to show $f'f \sim f'g \sim g'g$.

Consider $f's$ with $(f's)_n = A_n \xrightarrow{s_n} B_{n+1} \xrightarrow{f'_{n+1}} C_{n+1}$.

Then $d(f's) + (f's)d = f'ds + f'sd$
$= f'(ds + sd)$
$= f'(g-f) = f'g - f'f$.

Sim. $d(s'g) + (s'g)d$
$= ds'g + s'dg = (ds' + s'd)g = f'g - g'g$. $\square$

Lemma

If $f \sim g$, they induce the same map $H_n f = H_n g$ on homology.

$\boxed{\text{Proof}}$

$$\begin{array}{ccc} H_n A & \overset{H_n f}{\underset{H_n g}{\rightrightarrows}} & H_n B \\ \| & & \| \\ \dfrac{Z_n A}{B_n A} & \rightrightarrows & Z_n B / B_n B \end{array}.$$

So must show given $x \in Z_n A$ that
$$f(x) + B_n B = g(x) + B_n(B)$$

But $g(x) - f(x) = sdx + dsx = sdx \in \operatorname{Im}(d_{n+1}) = B_n B$

as $sdx = s0 = 0$

Hence $H_n(f) = H_n(g)$

Corollary

If $f : A \longrightarrow B$ is htpy equivalence then $H_n f : H_n A \longrightarrow H_n B$ is invertible.

$\boxed{\text{Proof}}$ If $gf \sim 1_A$ & $fg \sim 1_B$ then

$H_n(g) H_n(f) =$ by funct.

$H_n(gf) =$ by prev. prop

$H_n(1_A) =$ by funct

$1_{H_n(A)}$. Sim. $H_n(f) H_n(g) = 1_{H_n(B)}$. $\square$