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The multiplier approach to the projective Finsler metrizability problem

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ABSTRACT

This paper is concerned with the problem of determining whether a projective-equivalence class of sprays is the geodesic class of a Finsler function. We address both the local and the global aspects of this problem. We present our results entirely in terms of a multiplier, that is, a type $(0,2)$ tensor field along the tangent bundle projection. In the course of the analysis we consider several related issues of interest including the positivity and strong convexity of positively-homogeneous functions, the relation to the so-called Rapcsák conditions, some peculiarities of the two-dimensional case, and geodesic convexity for sprays.

1. Introduction

A Finsler function — a smooth function on the slit tangent bundle $\tau : T^\circ M \to M$ of a smooth paracompact manifold M, which is positive, positively (but not necessarily absolutely) homogeneous, and strongly convex — determines through its Euler–Lagrange equations an equivalence class of sprays whose base integral curves are its geodesics, where the curves with given initial point and direction defined by two different members of the class are the same up to an orientation-preserving reparametrization. The geodesics of the class, in other words, define oriented paths in M. Such sprays are projectively equivalent. That is to say, if Γ is one such spray — a vector field on $T^\circ M$ such that $\tau_*\Gamma_{(z,y)} = y$ for any $x \in M$ and $y \in T_xM$, $y \neq 0$, and such that $[\Delta, \Gamma] = \Gamma$ where Δ is the Liouville field — then any other takes the form $\Gamma - 2P\Delta$ for some function P on $T^\circ M$ which satisfies $\Delta(P) = P$ (the choice of numerical coefficient is made for later convenience). A set of sprays related in this way is called a projective-equivalence class, or simply projective class, of sprays.

The projective metrizability problem is the corresponding inverse problem: given a projective class of sprays, does it come from a Finsler function in the way just described? There have been several publications on this question in the last few years: see for example [1,3,6,8,17,18]. It is a curious fact that none of these uses what is, in the context of the inverse problem of the calculus of variations in general, probably the most studied, and certainly the historically most significant, approach, namely that of the multiplier. The necessary and sufficient conditions for the existence of a Lagrangian for a given system of second-order ordinary differential equations, when expressed as conditions on a multiplier matrix, are known as the Helmholtz, or sometimes Helmholtz–Sonin, conditions. Such conditions for systems with two degrees of freedom were

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originally formulated by Douglas in 1941 in [9]. The subject was resurrected by Henneaux, and a more modern version of the Helmholtz conditions was given by Sarlet, both in 1982 (see [10] and [14]). For a recent survey see [12].

Our first aim in this paper is to formulate the necessary and sufficient conditions on a multiplier, conceived as a symmetric type $(0,2)$ tensor field along the tangent bundle projection τ, for it to be the Hessian (with respect to the fibre coordinates) of a positively-homogeneous function on $T^\circ M$ for which a given projective class of sprays satisfies the Euler–Lagrange equations. These necessary and sufficient conditions play the role of Helmholtz conditions for the projective Finsler metrizability problem. They differ from the Helmholtz conditions for the general inverse problem of the calculus of variations in several respects, arising from the homogeneity of the sought-for Finsler function and the fact that we work with a projective class of sprays rather than a single differential equation field.

We must make a point of clarification here. As well as the projective metrizability problem, there is another, related but different and more straightforward, inverse problem in Finsler geometry, that of determining whether a single given spray is the canonical geodesic spray of a yet unknown Finsler function. Given a Finsler function F and its corresponding energy $E = \frac{1}{2}F^2$, there are actually two candidates for the role of a multiplier, namely the Hessians h and g of F and E respectively; they have significantly different properties in terms of homogeneity and regularity. The Helmholtz conditions for the canonical spray problem have been formulated previously (see [11,15]); the multiplier in this case is of the type of g. By constrast we use the Finsler function rather than the energy, and a multiplier of the type of h rather than g. We work with a projective class of sprays, and avoid reference to the canonical spray; moreover we express our results as far as possible in projectively-invariant terms.

We should also mention that when referring to a multiplier, and indeed more generally, we usually abbreviate the expression 'tensor field along the tangent bundle projection' to simply 'tensor field' or 'tensor'.

In addition to formulating the Helmholtz-like conditions for the projective metrizability problem, we analyse in detail the requirements on a Finsler function that it takes only positive values and is strictly convex.

Our first results on the metrizability problem, though global with respect to the fibres of τ, or y-global, are local with respect to M. As a third topic, we address the question of how these results may be made global in M. In the literature one will find mostly local results, but global questions are barely touched upon. We find that there is a cohomological obstruction on M to combining locally-defined Finsler functions into a global function; when this obstruction vanishes the resulting function satisfies all of the requirements of a Finsler function except positivity, and can be turned into a genuine Finsler function (by the addition of a total derivative) only locally. A well-known example of a spray which exhibits precisely this behaviour is Shen's circle example from [16].

Some authors, such as Álvarez Paiva (in [1]), deal only with reversible paths, that is, paths which have no preferred orientation; we, on the other hand, cover the more general case of oriented paths, or sprays in the fully general sense. At the level of Finsler geometry this distinction corresponds to that between positively-homogeneous Finsler functions (the general case) and absolutely-homogeneous functions (the case discussed by Álvarez Paiva). We do however specialise to the reversible case where appropriate, and we begin the discussion of this in Section 2.

One of the difficulties of working directly with the Finsler function, rather than with the energy, is that the conditions of positivity and strong convexity required of a Finsler function become somewhat tricky to deal with. We discuss this issue in Section 3. The argument is based on the theorem from [4] in which the triangle inequality and the fundamental inequality are established for Finsler functions: we prove a version which doesn't assume ab initio that the function in question is non-negative, and we use our result to show that any positively-homogeneous function whose Hessian is positive quasi-definite — as nearly positive definite as it can be — can be turned into a Finsler function locally by the addition of a total derivative.

Sections 2 and 3 deal with preliminary matters. In Section 4 we turn to the projective metrizability problem, the main business of the paper, and discuss the Rapcsák conditions, which are essentially ways of formulating or reformulating the Euler–Lagrange equations. In Sections 5 and 6 we give the necessary and sufficient conditions for the existence of a Finsler function in terms of a multiplier. We discuss them initially in local terms: local, that is, relative to M. In fact throughout the paper, with the exception of Section 7, we deal only with objects which are defined and smooth as functions of y^i for all $y \neq 0$, where $y = (y^i)$ are the fibre coordinates on TM. That is to say, we deal almost entirely with objects which are y-global. We discuss in Section 6 the problem of extending our local results to results which are global in M.

In the main body of the paper we make the assumption that $\dim M \geqslant 3$. The two-dimensional case has some special features which mean that our general results do not always apply. We make some remarks about the two-dimensional case in Section 7. The paper proper ends with the discussion of an illustrative example, and some concluding remarks.

Most of our local calculations are carried out in coordinates. We denote coordinates on M by (x^i), and the corresponding canonical coordinates on TM by (x^i, y^i). We denote a spray Γ by

$$\Gamma = y^i \frac{\partial}{\partial x^i} - 2\Gamma^i \frac{\partial}{\partial y^i}.$$

The corresponding horizontal local vector fields are denoted by

$$H_i = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial y^j}, \quad \Gamma_j^i = \frac{\partial \Gamma^i}{\partial y^j}.$$

We write V_i for $\partial/\partial y^i$ when it is convenient to do so.

The paper is written under the assumption that the reader is familiar with the differential geometry of sprays. However, we have given a number of basic results in Appendix A, for reference and to fix notations. As a general reference for Finsler geometry we use the well-known book by Bao, Chern and Shen, [4]. The standard modern reference for the geometry of sprays is Shen's book, [16]; for a survey using a coordinate-independent formalism see the paper by Bácsó and Z. Szilasi, [3].

Appendix B contains a discussion of the existence of geodesically convex sets for a spray, which can be used in the construction of so-called good open coverings, needed for the global results in Section 6. While this material on convex sets, which is of some interest in its own right, is not strictly speaking new, it seems to have been lost from view for some considerable time.

2. Reversible sprays

The geodesics of a spray are not in general reversible, even as paths. In this brief section we consider the special class of sprays for which the geodesics are reversible.

We denote by ρ the reflection map of $T^\circ M$, namely $(x,y)\mapsto(x,-y)$. For any spray Γ the vector field $\bar\Gamma=-\rho_*\Gamma$ is also a spray, which we call the reverse of Γ. We say that Γ is *reversible* if it is projectively equivalent to its reverse, and *strictly reversible* if the two are equal. Then the geodesic paths of Γ are reversible if and only if Γ is reversible. Moreover, the geodesics are reversible as parametrized curves, which is to say that if γ is a geodesic so is $t\mapsto\gamma(-t)$, if and only if Γ is strictly reversible.

If a spray is reversible so are all sprays projectively equivalent to it: that is to say, reversibility is a property of a projective class. We shall show that the projective class of a reversible spray contains a strictly-reversible spray.

Proposition 1. *If Γ is reversible then there is a projectively-equivalent spray which is strictly reversible.*

Proof. Let $\bar\Gamma$ be the reverse of Γ, and set

$$\tilde\Gamma=\frac{1}{2}(\Gamma+\bar\Gamma).$$

Then $\tilde\Gamma$ is a spray, and

$$\rho_*\tilde\Gamma=\frac{1}{2}(\rho_*\Gamma+\rho_*\bar\Gamma)=-\frac{1}{2}(\bar\Gamma+\Gamma)=-\tilde\Gamma.$$

Thus $\tilde\Gamma$ is strictly reversible. But $\bar\Gamma$ is projectively equivalent to Γ, from which it follows immediately that $\tilde\Gamma$ is projectively equivalent to Γ also. $\square$

3. Positivity and strong convexity

For a positively-homogeneous smooth function F on $T^\circ M$ to be a Finsler function it must be positive and strongly convex: that is to say,

$$g_{ij}=F\frac{\partial^2F}{\partial y^i\partial y^j}+\frac{\partial F}{\partial y^i}\frac{\partial F}{\partial y^j}$$

must be positive definite. The conditions for a projective class of sprays to be metrizable to be discussed below consist in the first instance of conditions for the existence of a positively-homogeneous function F whose Euler–Lagrange equations are satisfied by the given sprays, which we may describe as differential conditions. These differential conditions must then be supplemented by algebraic conditions which will ensure that there is such a function F which is positive and strongly convex. Formulating such algebraic conditions is somewhat tricky, for reasons we now explain.

It follows from the fact that F is homogeneous of degree 1 in the fibre variables that its Hessian satisfies

$$y^j\frac{\partial^2F}{\partial y^i\partial y^j}(y)=0.$$

At any point $(x,y)\in T^\circ M$ such that $F(x,y)\neq0$, any vector u may be written uniquely as the sum of a scalar multiple of y and a vector v which is g-orthogonal to y, namely

$$v=u-\lambda y\quad\text{where }\lambda=\frac{1}{F}u^k\frac{\partial F}{\partial y^k}.$$

Then

$$g_{ij}u^iu^j=g_{ij}v^iv^j+\lambda^2g_{ij}y^iy^j=F\frac{\partial^2F}{\partial y^i\partial y^j}v^iv^j+\left(u^k\frac{\partial F}{\partial y^k}\right)^2.$$

Thus provided that $F(x, y)$ is positive, g is positive definite if and only if the Hessian is *positive quasi-definite*, in the sense that

$$\frac{\partial^2 F}{\partial y^i \partial y^j}(y) v^i v^j \geqslant 0$$

with equality only if v is a scalar multiple of y.

So for F to be a Finsler function it is necessary that its Hessian be positive quasi-definite; but this property is not sufficient. The problem is that we require F to be positive, but there is no way of ensuring this by imposing a condition on the Hessian. Note that if F is positively homogeneous so is any function obtained by adding a total derivative to it, that is, any function of the form

$$F + y^i \frac{\partial \phi}{\partial x^i}$$

where ϕ is a function on M; the new function has the same Hessian, and its Euler–Lagrange equations are satisfied by the given sprays if those of F are. But adding a total derivative to a Finsler function may destroy the Finsler property.

Conversely, however, given a positively-homogeneous function F whose Hessian is positive quasi-definite in the sense specified above, it is always possible to modify F by the addition of a total derivative so as to obtain a local Finsler function, that is, a function on $T^\circ U$ for some open neighbourhood U of any point in M, which is positive, positively homogeneous and strongly convex. We shall prove this below.

It turns out, as will become apparent below, that there is no difficulty about positivity if F is absolutely rather than just positively homogeneous, that is if $F(ky) = |k| F(y)$ for all $k \in \mathbb{R}$.

The results in this section are not to be confused with that of Lovas [13], who showed in effect that assuming positivity one can prove strong convexity: he showed, that is, that for a positive, positively-homogeneous function F, if

$$\frac{\partial^2 F}{\partial y^i \partial y^j}(y) v^j = 0$$

only when v is a scalar multiple of y then the Hessian is positive quasi-definite (this formulation is to be found in [7]). Lovas's result is converse to ours.

For the first part of the argument we fix $x \in M$, that is, we essentially work in $\mathbb{R}^n$. The following result is Theorem 1.2.2 of [4], but without the assumption that F is non-negative.

Proposition 2. *Let F be C^∞ on $\mathbb{R}^n - \{0\}$, continuous on $\mathbb{R}^n$, positively homogeneous, and suppose that its Hessian is non-negative in the sense that for any $w \in \mathbb{R}^n$*

$$w^i w^j \frac{\partial^2 F}{\partial y^i \partial y^j}(y) \geqslant 0.$$

Then for all $y_1, y_2 \in \mathbb{R}^n$

$$F(y_1) + F(y_2) \geqslant F(y_1 + y_2) \quad (\textit{the triangle inequality});$$

and for all $y, z \in \mathbb{R}^n$ with $y \neq 0$

$$F(z) \geqslant z^i \frac{\partial F}{\partial y^i}(y) \quad (\textit{the fundamental inequality}).$$

Proof. Provided that 0 does not lie in the line segment $[y, y + w]$ we can apply the second mean value theorem to the function $t \mapsto F(y + tw)$ to obtain

$$F(y + w) = F(y) + w^i \frac{\partial F}{\partial y^i}(y) + w^i w^j \frac{\partial^2 F}{\partial y^i \partial y^j}(y + \epsilon w)$$

for some ϵ with $0 < \epsilon < 1$. The final term is non-negative, so

$$F(y + w) \geqslant F(y) + w^i \frac{\partial F}{\partial y^i}(y),$$

and similarly

$$F(y - w) \geqslant F(y) - w^i \frac{\partial F}{\partial y^i}(y),$$

where we must now assume that 0 does not lie in the segment $[y - w, y + w]$. Summing,

$$F(y+w)+F(y-w)\geqslant 2F(y).$$

On setting $y_1=\frac{1}{2}(y+w)$ and $y_2=\frac{1}{2}(y-w)$ we obtain the triangle inequality, for all pairs of points y_1, y_2 such that the segment $[y_1, y_2]$ does not contain the origin. Now by continuity $F(0)=0$, so the triangle inequality holds self-evidently if either argument is 0. Suppose that 0 is an interior point of the segment $[y_1, y_2]$. Consider first the case $y_1+y_2=0$. For any $y\neq 0$ and any positive s, $F(sy+y_1)+F(sy+y_2)\geqslant 2sF(y)$, and on letting s tend to 0 we obtain $F(y_1)+F(y_2)\geqslant 0$. Now suppose that $y_2=-ky_1$ with $k>0$, and furthermore suppose that $k<1$ (otherwise reverse the roles of y_1 and y_2). Then $y_1+y_2=(1-k)y_1$, and

$$F(y_1)+F(y_2)=F(y_1)+kF(-y_1)\geqslant(1-k)F(y_1)=F(y_1+y_2).$$

Thus the triangle inequality holds for all y_1, y_2.

For the fundamental inequality replace $y+w$ by z to obtain

$$F(z)\geqslant F(y)+\left(z^i-y^i\right)\frac{\partial F}{\partial y^i}(y)=z^i\frac{\partial F}{\partial y^i}(y).$$

We have to exclude $y=0$ since F won't be differentiable there. We also have to assume that the segment $[y, z]$ does not include the origin, that is, that z is not a negative multiple of y. But if $z=-ky$ with $k>0$ then

$$F(z)=kF(-y)\geqslant -kF(y)=z^i\frac{\partial F}{\partial y^i}(y). \qquad \square$$

Note that by positive homogeneity F can only be constant if it is identically zero.

Corollary 1. *If F is not constant then it must take positive values.*

Proof. If F is not constant there must be a point y at which the gradient of F is non-zero. Then for all z in an open half-space, $z^i\partial F/\partial y^i(y)>0$; and $F(z)>0$ for all such z. $\square$

We assume for the rest of this section that the Hessian of F is positive quasi-definite.

Lemma 1. *It cannot be the case that for fixed non-zero z,*

$$z^i\frac{\partial F}{\partial y^i}(y)$$

is constant (i.e. takes the same value for all $y\neq 0$).

Proof. If it were constant it would follow that

$$z^j\frac{\partial^2 F}{\partial y^i\partial y^j}(y)=0$$

for all y, which cannot be the case when the Hessian is positive quasi-definite if $z\neq 0$. $\square$

Lemma 2. *Except when z is a positive scalar multiple of y, the fundamental inequality is strict*:

$$F(z)>z^i\frac{\partial F}{\partial y^i}(y).$$

Proof. Evidently equality holds when z is a positive scalar multiple of y. Moreover, it follows from the second mean value theorem that the inequality is strict otherwise, except possibly when z is a negative scalar multiple of y. From the proof of Proposition 2 we know that if $z=-ky$ with $k>0$ then

$$F(z)=kF(-y)\geqslant -kF(y)=z^i\frac{\partial F}{\partial y^i}(y).$$

So the question is whether we can have $F(-y)=-F(y)$ with $y\neq 0$. Now for every $w\neq 0$,

$$F(-y)\geqslant -y^i\frac{\partial F}{\partial y^i}(w), \qquad -F(y)\leqslant -y^i\frac{\partial F}{\partial y^i}(w);$$

so if $F(-y)=-F(y)=-c$ say then

$$y^i\frac{\partial F}{\partial y^i}(w)=c \quad \text{for all } w\neq 0.$$

But by Lemma 1 this cannot hold. Thus $F(-y)>-F(y)$ and the inequality is strict in this case too. $\square$

Proposition 3. *If F is absolutely homogeneous and its Hessian is positive quasi-definite then it is everywhere positive on $\mathbb{R}^n - \{0\}$.*

Proof. It follows from the previous proof that if $F(y) = 0$ (where $y \neq 0$) then $F(-y) > 0$. But if F is absolutely homogeneous and $F(y) = 0$ then it would have to be the case that $F(-y) = 0$ also. So F can never vanish. We know that F must take positive values. So it must be positive everywhere on $\mathbb{R}^n - \{0\}$. □

This result has appeared previously in [7].

Proposition 4. *If F is positively homogeneous and its Hessian is positive quasi-definite then there is a linear function $y \mapsto \alpha_i y^i$ such that if $\tilde{F}(y) = F(y) + \alpha_i y^i$ then $\tilde{F}$ is everywhere positive on $\mathbb{R}^n - \{0\}$.*

Proof. Fix $z \in \mathbb{R}^n - \{0\}$ and define $\bar{F}$ by

$$\bar{F}(y) = F(y) - y^i \frac{\partial F}{\partial y^i}(z).$$

Then by Lemma 2, $\bar{F} > 0$ on $\mathbb{R}^n - \{0\}$, except along the ray through z; $\bar{F}(z) = 0$. Now consider the restriction of $\bar{F}$ to the unit coordinate Euclidean sphere S. Denote by H_z the closed hemisphere opposite the point where the ray through z intersects S, that is, the southern hemisphere where $\hat{z} = z/|z|$ is the north pole. Then $\bar{F}$ is positive on H_z, which is compact, so there is some $k > 0$ such that $\bar{F}(y) \geqslant k$ for $y \in H_z$. Now set $\varphi(y) = \frac{1}{2}k\hat{z}_i y^i$ where $\hat{z}_i = \delta_{ij}\hat{z}^j$. Note that $-\frac{1}{2}k \leqslant \varphi(y) \leqslant \frac{1}{2}k$ for $y \in S$: in fact φ takes its minimum value $-\frac{1}{2}k$ at the south pole, is zero on the equator, and is positive on the open northern hemisphere $S - H_z$. Set $\tilde{F} = \bar{F} + \varphi$. Then $\tilde{F}$ is positive on $S - H_z$, since $\bar{F}$ is non-negative and φ is positive; and also on H_z, since $\bar{F} \geqslant k$ and $\varphi \geqslant -\frac{1}{2}k$. Thus $\tilde{F}$ is positive everywhere on S, and so everywhere on $\mathbb{R}^n - \{0\}$; and $\tilde{F}(y) = F(y) + \alpha_i y^i$ where

$$\alpha_i = \frac{1}{2}k\hat{z}_i - \frac{\partial F}{\partial y^i}(z)$$

for the chosen non-zero z. □

Theorem 1. *Let F be a positively homogeneous function on $T^\circ M$ such that for every $x \in M$ the Hessian of F on $T^\circ_x M$ is positive quasi-definite. Then for any $x_0 \in M$ there is a neighbourhood U of x_0 in M and a function $\tilde{F}$ defined on $T^\circ U$ such that $\tilde{F}$ is a Finsler function which differs from F by a total derivative. If F is absolutely, rather than just positively, homogeneous then it itself is a Finsler function.*

Proof. Take any point $x_0 \in M$ and choose coordinates at x_0. By the previous proposition there is a (constant) covector α such that if $\tilde{F}(x, y) = F(x, y) + \alpha_i y^i$ then $\tilde{F}(x_0, y) > 0$ for $y \neq 0$. It follows that there is an open neighbourhood U of x_0 within the coordinate patch such that $\tilde{F}(x, y) > 0$ for all $x \in U$ and $y \in T^\circ_x M$. (To see this, suppose the contrary. Consider $\tilde{F}$ as function on $P \times S$ where P is the coordinate patch and S, as before, is the unit Euclidean sphere. Then there is a sequence of points $\{(x_r, y_r)\}$ in $P \times S$ with $x_r \to x_0$ such that $\tilde{F}(x_r, y_r) \leqslant 0$ for all r. But $\{y_r\}$, being a sequence in a compact set, has a convergent subsequence, which converges to y_0 say. Then $\tilde{F}(x_0, y_0) \leqslant 0$, which is a contradiction.) So $\tilde{F}$ is a Finsler function on $T^\circ U$, and it differs from F by $\alpha_i y^i$ which is the total derivative of the local function $\alpha_i x^i$.

If F is absolutely homogeneous then it is everywhere positive by Proposition 3, and therefore a Finsler function. □

4. The Euler–Lagrange equations and the Rapcsák conditions

For a function F on $T^\circ M$ which is positively homogeneous of degree 1, the Euler–Lagrange equations

$$\Gamma\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = 0$$

have the property that if there is a solution Γ which is a spray, then every spray projectively equivalent to Γ is also a solution.

At a basic level therefore solving the projective metrizability problem for a given projective class of sprays involves finding a positively-homogeneous function F such that the Euler–Lagrange equations are satisfied for any spray Γ in the class. These equations, considered as conditions on F, constitute (in the terminology of [17,18]) the first set of *Rapcsák conditions*. There is a second equivalent set which is given in the next proposition. (As a matter of fact there are many versions of the Rapcsák conditions to be found in the literature: [3] contains seven. The two we give are the ones most useful for our purposes.)

Proposition 5. *The positively-homogeneous function F satisfies the first Rapcsák conditions if and only if*

$$\frac{\partial^2 F}{\partial x^j \partial y^i} - \Gamma^k_j \frac{\partial^2 F}{\partial y^i \partial y^k} = \frac{\partial^2 F}{\partial x^i \partial y^j} - \Gamma^k_i \frac{\partial^2 F}{\partial y^j \partial y^k}.$$

Proof. The first Rapcsák conditions are

$$y^k\frac{\partial^2 F}{\partial x^k\partial y^i}-2\Gamma^k\frac{\partial^2 F}{\partial y^i\partial y^k}-\frac{\partial F}{\partial x^i}=0.$$

On differentiating with respect to y^j one obtains

$$\Gamma\left(\frac{\partial^2 F}{\partial y^i\partial y^j}\right)+\frac{\partial^2 F}{\partial x^j\partial y^i}-2\Gamma^k_j\frac{\partial F}{\partial y^i}y^k-\frac{\partial^2 F}{\partial x^i\partial y^j}=0.$$

The part of this which is skew in i and j gives the required conditions. Conversely, if the given conditions hold and F is homogeneous then contraction with y^j produces the first Rapcsák conditions. □

It is of interest, though not of any significance here, that the second Rapcsák conditions follow from the first regardless of whether or not F is homogeneous; homogeneity is required for the converse step only.

The second Rapcsák conditions may be written

$$H_j(\theta_i)=H_i(\theta_j)\quad\text{where } \theta_i=\frac{\partial F}{\partial y^i}.$$

The symmetric part of the equation whose skew part gives the second Rapcsák conditions may be written

$$(\nabla h)_{ij}=0\quad\text{where } h_{ij}=\frac{\partial^2 F}{\partial y^i\partial y^j},$$

where ∇ is the dynamical covariant derivative operator associated with Γ. These conditions have important roles to play in the next section.

5. The Helmholtz-like conditions for a multiplier

This section is concerned with conditions on a multiplier necessary and sufficient for it to be the Hessian of a Finsler function for a given projective class of sprays.

Initially we work in a coordinate neighbourhood U of M. Without essential loss of generality we may and shall assume that the Poincaré Lemma holds on U: in fact we shall assume that U is contractible. We further assume, for reasons that will shortly become clear, that $\dim M\geqslant 3$; and indeed we make this assumption for most of the rest of this paper, until the discussion of the two-dimensional case in Section 7 in fact.

We denote a putative Hessian of a Finsler function by h, and its components by h_{ij}, to distinguish it from the putative Hessian of an energy function, which is written g or g_{ij} in the conventional way. We shall say that h is *quasi-regular* if $h_{ij}v^j=0$ if and only if $v^i=ky^i$ for some constant k. We shall call a positively-homogeneous function F whose Hessian is quasi-regular a *pseudo-Finsler function*.

Lemma 3. *If h_{ij} is defined and smooth on $T^\circ U$, is symmetric and satisfies $h_{ij}y^j=0$ and*

$$\frac{\partial h_{ij}}{\partial y^k}=\frac{\partial h_{ik}}{\partial y^j}$$

then

1. *h_{ij} is positively homogeneous of degree -1;*
2. *there is a smooth function F on $T^\circ U$, positively homogeneous of degree 1, such that*

$$h_{ij}=\frac{\partial^2 F}{\partial y^i\partial y^j}.$$

Proof. 1. We have

$$0=\frac{\partial}{\partial y^j}\left(h_{ik}y^k\right)=\frac{\partial h_{ik}}{\partial y^j}y^k+h_{ij}=\frac{\partial h_{ij}}{\partial y^k}y^k+h_{ij}$$

as required.

2. For each i we may consider $h_{ij}\,dy^j$ as a 1-form on $\mathbb{R}^n-\{0\}$, $n=\dim M$, with coordinates y^i, where we regard the x^i as parameters. The condition

$$\frac{\partial h_{ij}}{\partial y^k}=\frac{\partial h_{ik}}{\partial y^j}$$

says that this 1-form is closed. For $\dim M \geqslant 3$ it follows that $h_{ij}\,dy^j$ is exact, that is, there are functions $\bar{F}_i$ depending smoothly on y^i for $y \neq 0$, and smoothly also on the parameters x^i, such that

$$h_{ij} = \frac{\partial \bar{F}_i}{\partial y^j}.$$

But by the symmetry of h_{ij}, $\bar{F}_i dy^i$ is also closed, and hence exact. By a similar argument there is a smooth function $\bar{F}$ on $T^\circ U$ such that

$$h_{ij} = \frac{\partial^2 \bar{F}}{\partial y^i \partial y^j}.$$

Then

$$\frac{\partial}{\partial y^i}\left(y^j \frac{\partial \bar{F}}{\partial y^j} - \bar{F}\right) = h_{ij} y^j = 0,$$

so that $y^i \partial \bar{F}/\partial y^i - \bar{F}$ is a function of x^i alone, say f. Then $F = \bar{F} + f$ has the required properties; and so indeed has any function differing from F by a function linear in the y^i. □

The first result also follows from the second, of course, but the direct proof is available if one does not want to invoke the existence of F.

Note that we require that $\dim M = n \geqslant 3$ to be able to conclude that a closed 1-form on $\mathbb{R}^n - \{0\}$ is exact, and hence that $F(x, y)$ is defined for all $y \neq 0$. It is for this reason that we make this assumption for most of the rest of the paper.

Lemma 4. *Let Γ, $\tilde{\Gamma}$ be projectively-equivalent sprays, ∇, $\tilde{\nabla}$ be the corresponding dynamical covariant derivative operators. If h_{ij} is symmetric and homogeneous of degree -1 then $(\tilde{\nabla} h)_{ij} = (\nabla h)_{ij}$.*

Proof. With $\tilde{\Gamma} = \Gamma - 2P\Delta$ and $P_i = V_i(P)$ we have

$$\begin{aligned}
(\tilde{\nabla} h)_{ij} &= \tilde{\Gamma}(h_{ij}) - \tilde{\Gamma}_i^k h_{kj} - \tilde{\Gamma}_j^k h_{ik} \\
&= \Gamma(h_{ij}) - 2P\Delta(h_{ij}) \\
&\quad - \left(\Gamma_i^k + P\delta_i^k + P_i y^k\right)h_{kj} - \left(\Gamma_j^k + P\delta_j^k + P_j y^k\right)h_{ik} \\
&= (\nabla h)_{ij} + 2Ph_{ij} - Ph_{ij} - Ph_{ij} \\
&= (\nabla h)_{ij}. \qquad \square
\end{aligned}$$

Thus ∇h is projectively invariant for a given projective class.

The following lemma involves the curvature tensors R^i_j and R^i_{jk} of a spray Γ.

Lemma 5. *If h_{ij} satisfies the conditions of Lemma 3 then the following conditions are equivalent to each other:*

$$\begin{aligned}
&h_{ik} R^k_j = h_{jk} R^k_i, \\
&h_{il} R^l_{jk} + h_{jl} R^l_{ki} + h_{kl} R^l_{ij} = 0.
\end{aligned}$$

Proof. To obtain the second expression, differentiate $h_{il} R^l_j - h_{jl} R^l_i$ with respect to y^k to obtain

$$\frac{\partial h_{il}}{\partial y^k} R^l_j + h_{il} \frac{\partial R^l_j}{\partial y^k} - \frac{\partial h_{jl}}{\partial y^k} R^l_i + h_{jl} \frac{\partial R^l_i}{\partial y^k} = 0,$$

and add the two similar expressions obtained by cyclically permuting i, j and k. Terms involving derivatives of the h_{ij} cancel in pairs due to the symmetry condition they satisfy, while pairs of terms involving derivatives of the R^l_i give the R^l_{ij}.

To obtain the first from the second, simply contract with y^k. □

This lemma is also to be found in [3].

The Weyl curvature W^i_j of a spray Γ is defined by

$$W^i_j = R^i_j - R\delta^i_j - \rho_j y^i \quad \text{where } R = \frac{1}{n-1} R^k_k \text{ and } \rho_j = \frac{1}{n+1}\left(\frac{\partial R^k_j}{\partial y^k} - \frac{\partial R}{\partial y^j}\right).$$

The Weyl curvature is projectively invariant. See [16] for details.

Lemma 6. *If h_{ij} satisfies the conditions of Lemma 3 then the condition $h_{ik}R^k_j = h_{jk}R^k_i$ is equivalent to $h_{ik}W^k_j = h_{jk}W^k_i$, and in particular is projectively invariant.*

Proof. We have

$$\begin{aligned} h_{ik}W^k_j - h_{jk}W^k_i &= h_{ik}\big(R^k_j - R\delta^k_j - \rho_j y^k\big) - h_{jk}\big(R^k_i - R\delta^k_i - \rho_i y^k\big) \\ &= h_{ik}R^k_j - Rh_{ij} - h_{jk}R^k_i + Rh_{ij} \\ &= h_{ik}R^k_j - h_{jk}R^k_i \end{aligned}$$

as claimed. □

Lemma 7. *Let $\bar F$ be a smooth function on $T^\circ U$ which is positively homogeneous and satisfies*

$$(\nabla h)_{ij} = 0 \quad \textit{and} \quad h_{ik}W^k_j = h_{jk}W^k_i$$

where

$$h_{ij} = \frac{\partial^2 \bar F}{\partial y^i \partial y^j}.$$

Then there is a smooth positively-homogeneous function F on $T^\circ U$, with the same Hessian as $\bar F$, which satisfies the second Rapcsák conditions.

Proof. We show first that $H_i(\bar\theta_j) - H_j(\bar\theta_i)$ is independent of the y^k, where $\bar\theta_i = \partial \bar F/\partial y^i$. Now

$$\frac{\partial}{\partial y^k}\big(H_i(\bar\theta_j)\big) = H_i(h_{jk}) - \Gamma^l_{ik}h_{jl}, \qquad \Gamma^k_{ij} = \frac{\partial \Gamma^k_i}{\partial y^j} = \Gamma^k_{ji}.$$

It is a simple and well-known consequence of the first assumption, together with the evident fact that

$$\frac{\partial h_{ij}}{\partial y^k} = \frac{\partial h_{ik}}{\partial y^j},$$

that

$$H_i(h_{jk}) - \Gamma^l_{ik}h_{jl} = H_j(h_{ik}) - \Gamma^l_{jk}h_{il},$$

whence $H_i(\bar\theta_j) - H_j(\bar\theta_i)$ is independent of the y^k. Thus

$$\big(H_i(\bar\theta_j) - H_j(\bar\theta_i)\big)\, dx^i \wedge dx^j$$

is a basic 2-form, say χ. We show next that χ is closed. In computing $d\chi$ we may replace partial derivatives with respect to x^k with H_k. We have

$$\bigoplus H_k\big(H_i(\bar\theta_j) - H_j(\bar\theta_i)\big) = \bigoplus [H_j, H_k](\bar\theta_i) = -\bigoplus R^l_{jk}h_{il}$$

where $\bigoplus$ indicates the cyclic sum over i, j and k. But this vanishes if $h_{ik}W^k_j = h_{jk}W^k_i$ by Lemmas 5 and 6. So χ is closed, and hence exact. Choose $\psi = \psi_i\, dx^i$ such that $\chi = d\psi$, and set $F = \bar F - \psi_i y^i$, so that $\theta_i = \bar\theta_i - \psi_i$. Then

$$\big(H_i(\theta_j) - H_j(\theta_i)\big)\, dx^i \wedge dx^j = \chi - d\psi = 0,$$

and F satisfies the second Rapcsák conditions. □

After all these preliminaries we can now give the main result of this section.

Theorem 2. *Let $U \subset M$ be any contractible open subset of a coordinate patch. Given a projective class of sprays, the following conditions are necessary and sufficient for the existence of a positively-homogeneous function F on $T^\circ U$, such that every spray in the class satisfies the Euler–Lagrange equations for F: there is a tensor h defined on $T^\circ U$ whose components satisfy*

$$h_{ji} = h_{ij},$$
$$h_{ij} y^j = 0,$$
$$\frac{\partial h_{ij}}{\partial y^k} = \frac{\partial h_{ik}}{\partial y^j},$$
$$(\nabla h)_{ij} = 0,$$
$$h_{ik} W^k_j = h_{jk} W^k_i,$$

where ∇ is the dynamical covariant derivative operator of any spray in the class.

Proof. If such a function F exists then its Hessian satisfies these conditions. Conversely, from the first three conditions and Lemma 3 there is a smooth positively-homogeneous function $\bar{F}$ on $T^\circ U$ whose Hessian is h. Take any Γ in the projective class. Then by Lemma 7 and the remaining conditions there is a smooth positively homogeneous function F on $T^\circ U$, whose Hessian is also h, which satisfies the second Rapcsák conditions. By the equivalence of the first and second Rapcsák conditions Γ, and hence every spray in the class, satisfies the Euler–Lagrange equations for F. □

Corollary 2. *Let $U \subset M$ be any contractible open subset of a coordinate patch. If there is a tensor h which satisfies the conditions of Theorem 2 and F is a corresponding positively-homogeneous function on $T^\circ U$ such that every spray in the class satisfies the Euler–Lagrange equations for F then another function $\tilde{F}$ has the same properties if and only if differs from F by the total derivative of a function on U.*

Proof. The function $\tilde{F}$ has the same Hessian as F and is positively homogeneous, so $\tilde{F} = F + \alpha_i y^i$ for some functions α_i on U. If Γ satisfies the Euler–Lagrange equations for both F and $\tilde{F}$ we must have

$$y^j \frac{\partial \alpha_i}{\partial x^j} - y^j \frac{\partial \alpha_j}{\partial x^i} = 0;$$

equivalently $\alpha_i\, dx^i$ is closed and therefore exact; if $\alpha_i\, dx^i = d\phi$ then $\tilde{F} - F$ is the total derivative of ϕ. □

6. Some global results

We now consider the problem of extending these results from coordinate neighbourhoods in M to the whole of M. We shall work with an open covering $\mathfrak{U} = \{U_\lambda\colon\ \lambda \in \Lambda\}$ of M by coordinate patches. We shall assume that $\mathfrak{U}$ has the property that every U_λ, and every non-empty intersection of finitely many of the U_λ, is contractible. A covering with this property is known as a good covering; it can be shown (see Appendix B) that every manifold over which is defined a spray admits good open coverings by coordinate patches.

Let us assume that there is a tensor field h which satisfies the conditions of Theorem 2, and in addition is everywhere quasi-regular. Then for each U_λ there is a pseudo-Finsler function F_λ defined on $T^\circ U_\lambda$; and on $U_\lambda \cap U_\mu$, if it is non-empty, there is a function $\phi_{\lambda\mu}$, determined up to the addition of a constant, such that

$$F_\lambda - F_\mu = y^i \frac{\partial \phi_{\lambda\mu}}{\partial x^i}.$$

On $U_\lambda \cap U_\mu \cap U_\nu$, if it is non-empty, $\phi_{\mu\nu} - \phi_{\lambda\nu} + \phi_{\lambda\mu}$ is a constant, say $k_{\lambda\mu\nu}$ (add the expressions for $F_\lambda - F_\mu$ etc., and note that $U_\lambda \cap U_\mu \cap U_\nu$, being contractible, is connected). Furthermore, for any four members $U_\kappa, U_\lambda, U_\mu, U_\nu$ of $\mathfrak{U}$ whose intersections in threes are non-empty

$$k_{\lambda\mu\nu} - k_{\kappa\mu\nu} + k_{\kappa\lambda\nu} - k_{\kappa\lambda\mu} = 0.$$

That is to say, k satisfies a cocycle condition. If k is in fact a coboundary we can modify each $\phi_{\lambda\mu}$ by the addition of a constant, so that (after modification) $\phi_{\mu\nu} - \phi_{\lambda\nu} + \phi_{\lambda\mu} = 0$.

We now show that if, for an open covering $\mathfrak{U}$ of M by coordinate patches we can find functions $\phi_{\lambda\nu}$ which satisfy this cocycle condition then there is a globally-defined pseudo-Finsler function F on $T^\circ M$ for the given projective class of sprays.

Since by assumption M is paracompact the covering $\mathfrak{U}$ of M admits a locally finite refinement $\mathfrak{V} = \{V_\alpha\colon\ \alpha \in A\}$. (The covering $\mathfrak{V}$ may not be good, but this will not matter for what follows.) There is a partition of unity subordinate to $\mathfrak{V}$, that is, for every α there is a smooth function f_α such that $\operatorname{supp}(f_\alpha) \subset V_\alpha$, $0 \leqslant f_\alpha \leqslant 1$ and $\sum_\alpha f_\alpha = 1$. By local finiteness, for every $x \in M$, $\sum_\alpha f_\alpha(x)$ is a finite sum. Indeed, since $\operatorname{supp}(f_\alpha) \subset V_\alpha$

$$\sum_\alpha f_\alpha(x) = \sum_{\alpha\colon x \in V_\alpha} f_\alpha(x).$$

Since $\mathfrak{V}$ is a refinement of $\mathfrak{U}$ we can define a map $\sigma : A \to \Lambda$ such that $V_\alpha \subseteq U_{\sigma(\alpha)}$. For every $\alpha, \beta \in A$ such that $V_\alpha \cap V_\beta$ is non-empty we define a real-valued function $\phi_{\alpha\beta}$ by

$$\phi_{\alpha\beta} = \phi_{\sigma(\alpha)\sigma(\beta)}|_{V_\alpha \cap V_\beta}.$$

Then the $\phi_{\alpha\beta}$ satisfy the cocycle condition, namely

$$\phi_{\beta\gamma} - \phi_{\alpha\gamma} + \phi_{\alpha\beta} = 0,$$

since the $\phi_{\lambda\mu}$ do. Moreover, if we set $F_\alpha = F_{\sigma(\alpha)}|_{\tau^{-1}V_\alpha}$ then

$$F_\alpha - F_\beta = y^i \frac{\partial \phi_{\alpha\beta}}{\partial x^i}.$$

Proposition 6. *Under these assumptions there is a globally-defined function F on $T^\circ M$ such that F_α differs from $F|_{\tau^{-1}V_\alpha}$ by a total derivative.*

Proof. Let V_α be any member of the covering $\mathfrak{V}$. For $x \in V_\alpha$ set

$$\psi_\alpha(x) = \sum_{\gamma:\, x \in V_\gamma} f_\gamma(x)\phi_{\alpha\gamma}(x).$$

Consider $\psi_\alpha(x) - \psi_\beta(x)$ where $x \in V_\alpha \cap V_\beta$. We have

$$\begin{aligned}\psi_\alpha(x) - \psi_\beta(x) &= \sum_{\gamma:\, x \in V_\gamma} f_\gamma(x)\big(\phi_{\alpha\gamma}(x) - \phi_{\beta\gamma}(x)\big)\\ &= \sum_{\gamma:\, x \in V_\gamma} f_\gamma(x)\phi_{\alpha\beta}(x)\\ &= \left(\sum_{\gamma:\, x \in V_\gamma} f_\gamma(x)\right)\phi_{\alpha\beta}(x)\\ &= \phi_{\alpha\beta}(x).\end{aligned}$$

Then for any two members of the covering $\mathfrak{V}$ with non-empty intersection

$$F_\alpha - y^i \frac{\partial \psi_\alpha}{\partial x^i} = F_\beta - y^i \frac{\partial \psi_\beta}{\partial x^i}$$

on $V_\alpha \cap V_\beta$; and we can define F consistently by

$$F(x, y) = F_\alpha(x, y) - y^i \frac{\partial \psi_\alpha}{\partial x^i}(x)$$

where V_α is any member of the covering containing x. □

The first part of this proof is a simple special case of the argument used to show that Čech cohomology is a sheaf cohomology theory in [20].

So for $\dim M \geqslant 3$ the multiplier argument gives a global pseudo-Finsler function provided the cocycle k is a coboundary. Now k is in fact an element of the Čech cochain complex for the covering $\mathfrak{U}$ with values in the constant sheaf $M \times \mathbb{R}$. This cocycle will certainly be a coboundary if the corresponding cohomology group on M, namely $\check{H}^2(\mathfrak{U}, \mathcal{R})$, is zero (we use the notation of [20]). It follows from the fact that $\mathfrak{U}$ is a good open covering that this cohomology group is isomorphic to the de Rham cohomology group $H^2(M)$. We have the following result.

Theorem 3. *If F is a (global) Finsler function on $T^\circ M$ then its Hessian h satisfies the conditions of Theorem 2 for the sprays of its geodesic class, and is in addition positive quasi-definite. Conversely, suppose given a projective class of sprays on $T^\circ M$. If there is a tensor field h which everywhere satisfies the conditions of Theorem 2 and is in addition positive quasi-definite, and if $H^2(M) = 0$, then the projective class is the geodesic class of a global pseudo-Finsler function, and of a local Finsler function over a neighbourhood of any point of M.*

We give examples of sprays which admit a global pseudo-Finsler function but only local Finsler functions in Section 8, so the result above would appear to be the best possible.

The situation in the case of a reversible spray, or without loss of generality in the light of Proposition 1 a strictly reversible spray, is much more clear cut. We observe first that if a spray Γ satisfies the Euler–Lagrange equations of a positively-homogeneous function F, and we set $\bar{F} = \rho^* F$ (where ρ is the reflection map), then $\bar{\Gamma}$, the reverse of Γ, satisfies the Euler–Lagrange equations of the (positively-homogeneous) function $\bar{F}$. So if Γ is strictly reversible, it satisfies the Euler–Lagrange equations for both F and $\bar{F}$, and therefore for their sum. But if F and $\bar{F}$ are positively homogeneous, $F + \bar{F}$ is absolutely homogeneous: for if $k < 0$

$$\begin{aligned} F(ky)+\bar F(ky) &= F\bigl(-|k|y\bigr)+\bar F\bigl(-|k|y\bigr) \\ &= |k|\bigl(F(-y)+\bar F(-y)\bigr) = |k|\bigl(\bar F(y)+F(y)\bigr). \end{aligned}$$

So for a strictly reversible spray, on U_λ there is a pseudo-Finsler function F_λ which is absolutely homogeneous. Now two functions (such as F_λ, F_μ) which are known to differ by a total derivative and are both absolutely homogeneous must be equal. In the light of these remarks and the final assertion of Theorem 1 we have the following result.

Theorem 4. *The projective class of a reversible spray on $T^\circ M$ is the geodesic class of a globally-defined absolutely-homogeneous Finsler function if and only if there is a tensor h which satisfies the conditions of Theorem 2 and is in addition positive quasi-definite.*

7. The two-dimensional case

It is well known that in two dimensions every spray is locally projectively metrizable (see for example [2,6]). 'Locally' here, however, refers to the fibre as well as to the base: the claim is only that a y-local Finsler function exists. We may seek to establish the existence of a Finsler, or at least pseudo-Finsler, function as follows.

Lemma 8. *Let h be a symmetric smooth tensor field on $\mathbb{R}^2-\{0\}$ which is positively homogeneous of degree -1 and satisfies $h_{ij}y^j=0$. Then*

$$\frac{\partial h_{ij}}{\partial y^k}=\frac{\partial h_{ik}}{\partial y^j}.$$

Proof. We have

$$h_{11}y^1+h_{12}y^2=0,$$
$$h_{21}y^1+h_{22}y^2=0.$$

Thus by symmetry h has just one independent component, which we may take to be h_{12}; this must be non-zero except where $y^1=0$ or $y^2=0$ (separately) for h to be non-trivial. Moreover h_{12} must satisfy the homogeneity condition

$$y^1\frac{\partial h_{12}}{\partial y^1}+y^2\frac{\partial h_{12}}{\partial y^2}+h_{12}=0.$$

On differentiating the equation $h_{11}y^1+h_{12}y^2=0$ with respect to y^2 we obtain

$$y^1\frac{\partial h_{11}}{\partial y^2}+y^2\frac{\partial h_{12}}{\partial y^2}+h_{12}=0,$$

whence by homogeneity of h_{12}

$$y^1\left(\frac{\partial h_{11}}{\partial y^2}-\frac{\partial h_{12}}{\partial y^1}\right)=0.$$

Thus

$$\frac{\partial h_{11}}{\partial y^2}=\frac{\partial h_{12}}{\partial y^1}$$

except possible where $y^1=0$. But by continuity this must hold where $y^1=0$ (but $y^2\neq 0$) also. This is the first of the two non-trivial cases of the equations it is required to prove; the other is proved similarly. □

Lemma 9. *Let U be a coordinate neighbourhood in a two-dimensional manifold on which is defined a spray Γ. Suppose that a symmetric tensor h_{ij} satisfies $(\nabla h)_{ij}=0$. If $\eta_i=h_{ij}y^j$ then $\nabla\eta=0$, and if $\lambda_{ij}=\Delta(h_{ij})+h_{ij}$ then $\nabla\lambda=0$.*

Proof. The first follows immediately from the fact that $\nabla y=0$ (that is to say, the dynamical covariant derivative of the total derivative $y^i\partial/\partial x^i$ vanishes). For the second we use the facts that $[\Delta,\Gamma]=\Gamma$ and that $\Delta(\Gamma^i_j)=\Gamma^i_j$ to show that $[\Delta,\nabla]=\nabla$. □

The Weyl tensor W^i_j vanishes when $\dim M=2$. In order for the conditions of Theorem 2 to be satisfied it is therefore enough that $\nabla h=0$ everywhere, and that $\eta=0$ and $\lambda=0$ on some cross-section of the flow of Γ, which we must assume to contain, for every y in it, the ray through y, for the latter condition to make sense. Indeed, we can always find a multiplier h locally in $T^\circ M$ by specifying its values on such a cross-section of the flow of Γ, arbitrarily subject to the conditions that $\eta=0$ and $\lambda=0$, and requiring that $\nabla h=0$: this is a first-order differential equation along each integral curve of Γ, and so the value of h along the curve is determined by the specified initial value. Moreover, η and λ satisfy similar first-order differential equations along the integral curves of Γ; and now we can use the fact that the equations are linear to show that since $\eta = 0$ initially, $\eta = 0$ everywhere, and likewise for λ. Since Γ never vanishes on $T^\circ M$ we can always find a local cross-section to its flow. Indeed, we can take a local two-dimensional submanifold of $T^\circ M$ which is transverse to Γ, and extend it to a local cross-section by including, for every y in it, the ray through y. This, together with the evident y-local version of Theorem 2, establishes the result stated above.

Theorem 5. *When* $\dim M = 2$*, every spray is locally projectively metrizable.*

It is moreover easy to specify the freedom in the choice of multiplier.

Proposition 7. *Suppose that in two dimensions h satisfies the conditions of Theorem 2. Let f be a function on $T^\circ M$ such that $Z(f) = 0$ for every vector field $Z \in \langle \Delta, \Gamma \rangle$ (so that f is homogeneous of degree 0, and constant along the integral curves of any spray in the projective class). Then $\hat{h} = fh$ also satisfies the conditions of Theorem 2. Conversely, any pair of tensors which satisfy the conditions of Theorem 2 are so related.*

Proof. It is clear that $\hat{h}$ satisfies the algebraic conditions and is homogeneous of degree -1. We have

$$(\nabla \hat{h})_{ij} = f(\nabla h)_{ij} + \Gamma(f) h_{ij} = 0.$$

Conversely, if $h_{ij} y^j = 0 = \hat{h}_{ij} y^j$ then $\hat{h}$ must be a scalar multiple of h, say $\hat{h} = fh$. Then by homogeneity we must have $\Delta(f) = 0$, and if $\nabla \hat{h} = 0 = \nabla h$ then $\Gamma(f) = 0$ also. □

The results above are y-local. But even if there is a y-global multiplier this by itself is not enough to guarantee the existence of a y-global positively-homogeneous function F on $T^\circ M$ of which it is the Hessian, because the Poincaré Lemma doesn't hold. One could imagine working with polar coordinates (r, θ) in each fibre. Each component h_{ij} of h is periodic in θ; but because F is obtained by integrating (twice), there is no guarantee that an F can be found which is periodic.

We can give conditions on the h_{ij}, expressed in terms of r and θ, for the existence of a periodic F as follows. We have $y^1 = r\cos\theta$, $y^2 = r\sin\theta$, so the h_{ij} satisfy

$$h_{11}\cos\theta + h_{12}\sin\theta = 0,$$
$$h_{21}\cos\theta + h_{22}\sin\theta = 0,$$

whence $h_{12} = -(h_{11} + h_{22})\sin\theta\cos\theta$. It will in fact be most convenient to work in terms of the trace $h_{11} + h_{22}$. Now the h_{ij} are homogeneous of degree -1, which means that $r(h_{11} + h_{22})$ is a function of θ alone: we denote it by $\tau(\theta)$. It is of course periodic. We have $h_{11} = \tau\sin^2\theta/r$, $h_{22} = \tau\cos^2\theta/r$, $h_{12} = -\tau\sin\theta\cos\theta/r$. Then

$$h_{11}\,dy^1 + h_{12}\,dy^2 = -(\tau\sin\theta)\,d\theta,$$
$$h_{21}\,dy^1 + h_{22}\,dy^2 = (\tau\cos\theta)\,d\theta.$$

Necessary and sufficient conditions for these two 1-forms to be exact are

$$\int_0^{2\pi} (\tau\sin\theta)\,d\theta = \int_0^{2\pi} (\tau\cos\theta)\,d\theta = 0.$$

This doesn't completely answer the question, however, because there is another integration to carry out. It turns out to be better to start from scratch. Let us compute the Hessian of a positively-homogeneous function F in polar coordinates. We may set $F(r, \theta) = r\varphi(\theta)$, where φ is periodic. Then by straightforward calculations

$$h_{11} = \frac{1}{r}\big(\varphi'' + \varphi\big)\sin^2\theta,$$
$$h_{22} = \frac{1}{r}\big(\varphi'' + \varphi\big)\cos^2\theta,$$
$$h_{12} = -\frac{1}{r}\big(\varphi'' + \varphi\big)\sin\theta\cos\theta.$$

So $\varphi'' + \varphi = \tau$, and the question is whether this equation has a periodic solution φ for given periodic τ. We expect that $\int_0^{2\pi}(\tau\sin\theta)\,d\theta = \int_0^{2\pi}(\tau\cos\theta)\,d\theta = 0$ should be necessary conditions for the existence of a periodic solution; and indeed

$$\big(\varphi'' + \varphi\big)\sin\theta = \frac{d}{d\theta}\big(\varphi'\sin\theta - \varphi\cos\theta\big), \qquad \big(\varphi'' + \varphi\big)\cos\theta = \frac{d}{d\theta}\big(\varphi'\cos\theta + \varphi\sin\theta\big)$$

and the integrals of these functions evidently vanish if φ is periodic. The conditions $\int_0^{2\pi}(\tau\sin\theta)\,d\theta = \int_0^{2\pi}(\tau\cos\theta)\,d\theta = 0$ are in fact sufficient, as can be seen as follows. Define $u(\theta)$, $v(\theta)$ by

$$u(\theta) = -\int_0^{\theta}\big(\tau(\phi)\sin\phi\big)\,d\phi, \quad v(\theta) = \int_0^{\theta}\big(\tau(\phi)\cos\phi\big)\,d\phi.$$

Then $\varphi(\theta) = u(\theta)\cos\theta + v(\theta)\sin\theta$ is a particular solution of the equation $\varphi'' + \varphi = \tau$, as can easily be seen by direct calculation (it is in fact the solution obtained by the method of variation of parameters). Now

$$u(\theta+2\pi) - u(\theta) = -\int_0^{2\pi}\big(\tau(\phi)\sin\phi\big)\,d\phi, \qquad v(\theta+2\pi) - v(\theta) = \int_0^{2\pi}\big(\tau(\phi)\cos\phi\big)\,d\phi;$$

so if these integrals vanish then the equation has periodic solutions. Notice that the addition to the particular solution of a term $a\cos\theta + b\sin\theta$ (the complementary function) corresponds merely to the addition of a term linear in y to F.

If $\tau = k_1\cos\theta + k_2\sin\theta$ there is no periodic F whose Hessian is h; and indeed this is the key case, since for any τ there are constants k_1 and k_2 (more exactly, functions on M) such that $\tau - (k_1\cos\theta + k_2\sin\theta)$ does lead to a periodic F. We can nevertheless solve the equation $\varphi'' + \varphi = k_1\cos\theta + k_2\sin\theta$: a particular solution is

$$\varphi(\theta) = \frac{1}{2}(k_1\sin\theta - k_2\cos\theta)\theta.$$

The expressions for the corresponding h_{ij} are

$$h_{11} = \frac{(k_1y^1 + k_2y^2)(y^2)^2}{r^4},$$
$$h_{22} = \frac{(k_1y^1 + k_2y^2)(y^1)^2}{r^4},$$
$$h_{12} = -\frac{(k_1y^1 + k_2y^2)y^1y^2}{r^4}.$$

When faced with a non-periodic F one possible course of action, which can certainly be carried out over a coordinate neighbourhood $U \subset M$, is to replace the fibre by its universal covering space (or in other words simply to ignore the fact that F is not periodic). It appears that in this way one would obtain, on restricting to a level set of F, an example of what Bryant calls a generalized Finsler structure (see [5]).

8. Example

The purpose of the example described below is to illustrate how a spray may belong to the geodesic class of a globally-defined pseudo-Finsler function, where the pseudo-Finsler function may be made into a Finsler function only locally.

Consider the projective class of the spray

$$\Gamma = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \sqrt{u^2+v^2+w^2}\left(-v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}\right)$$

defined on $T^\circ\mathbb{R}^3$. The geodesics of Γ are spirals with axis parallel to the z-axis, together with straight lines parallel to the z-axis and circles in the planes $z =$ constant. To see this, note first that both $\sqrt{u^2+v^2} = \mu$ and w are constant; and therefore (or directly) $\sqrt{u^2+v^2+w^2} = \lambda$ is also constant. The geodesics are solutions of

$$\ddot{x} = -\lambda\dot{y}, \qquad \ddot{y} = \lambda\dot{x}, \qquad \ddot{z} = 0.$$

Integrating the first two we get

$$\dot{x} = -\lambda(y-\eta), \qquad \dot{y} = \lambda(x-\xi)$$

with constants ξ, η, whence

$$(x-\xi)^2 + (y-\eta)^2 = (\mu/\lambda)^2.$$

So the projections of the geodesics on the xy-plane are circles of center (ξ,η) and radius $r = \mu/\lambda$: note that $0 \leqslant r \leqslant 1$, the circle degenerating to a point when $r = 0$. The explicit parametrization of the geodesics is

$$x(t) = \xi + r\cos(\lambda t + \vartheta), \qquad y(t) = \eta + r\sin(\lambda t + \vartheta), \qquad z(t) = wt + z_0,$$

where ξ, η, r, λ, ϑ, w and z_0 are constants, with $w^2 = \lambda^2(1-r^2)$. So for $w/\lambda \neq 0, \pm 1$ the geodesics are spirals, with axis the line parallel to the z-axis through $(\xi, \eta, 0)$. The case $r = 0$ corresponds to $w/\lambda = \pm 1$ and the geodesics are straight lines parallel to the z-axis (in both directions). The case $r = 1$ ($w = 0$) gives circles of unit radius in the planes $z = z_0$.

In fact Γ belongs to the geodesic class of the pseudo-Finsler function

$$F(x, y, z, u, v, w) = \sqrt{u^2 + v^2 + w^2} + \frac{1}{2}yu - \frac{1}{2}xv.$$

This is globally well defined but only locally a Finsler function: it is positive only for $x^2 + y^2 < 4$. It is globally pseudo-Finsler, however. To obtain a Finsler function in a neighbourhood of an arbitrary point (x_0, y_0, z_0) we can make a simple modification to

$$\tilde{F}(x, y, z, u, v, w) = \sqrt{u^2 + v^2 + w^2} + \frac{1}{2}(y - y_0)u - \frac{1}{2}(x - x_0)v;$$

this is positive for $(x - x_0)^2 + (y - y_0)^2 < 4$. Note that it differs from F by a total derivative.

The planes $z = $ constant have the property that a geodesic initially tangent to such a plane (so that $w = 0$ initially) remains always in the plane: that is to say, such planes are totally-geodesic submanifolds. The restriction of Γ to the submanifold $z = 0$, $w = 0$ of $T^\circ\mathbb{R}^3$ is the spray

$$u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - v\sqrt{u^2 + v^2}\frac{\partial}{\partial u} + u\sqrt{u^2 + v^2}\frac{\partial}{\partial v}$$

of Shen's circle example from [16]. We consider this as a spray defined on $T^\circ\mathbb{R}^2$. It has for its geodesics all circles in $\mathbb{R}^2$ of radius 1, traversed counter-clockwise. Again, this spray is locally projectively metrizable. One local Finsler function is the restriction of the one given for the spiral example, namely

$$F(x, y, u, v) = \sqrt{u^2 + v^2} + \frac{1}{2}yu - \frac{1}{2}xv.$$

Again, F is only locally defined as a Finsler function, though it is global as a pseudo-Finsler function.

Finally we shall compare the geodesics of the Finsler function

$$F(x, y) = \sqrt{g_{ij}(x)y^i y^j} + \beta_i(x)y^i = \alpha(x, y) + \beta(x, y)$$

of Randers type (to which class the examples above belong) with motion under the Lagrangian

$$L(x, y) = \frac{1}{2}g_{ij}(x)y^i y^j + \beta_i(x)y^i.$$

Here g is a Riemannian metric. In two- or three-dimensional flat space this is the Lagrangian for the motion of a classical charged particle, of unit charge, in the magnetic field determined by $d(\beta_i\,dx^i)$.

We have

$$\frac{\partial L}{\partial y^i} = g_{ij}y^j + \beta_i = y_i + \beta_i.$$

The energy E_L is given by

$$E_L = \Delta(L) - L = g_{ij}y^i y^j + \beta_i y^i - L = \frac{1}{2}g_{ij}y^i y^j = \alpha.$$

Thus motion under L is with constant Riemannian speed. The Euler–Lagrange field Γ_L is determined by

$$\Gamma_L(y_i) = \alpha\frac{\partial\alpha}{\partial x^i} - y^j\left(\frac{\partial\beta_i}{\partial x^j} - \frac{\partial\beta_j}{\partial x^i}\right).$$

On the other hand

$$\frac{\partial F}{\partial y^i} = \frac{y_i}{\alpha} + \beta_i,$$

and any geodesic spray Γ_F of F satisfies (but is not determined by)

$$\Gamma_F(y_i) = \Gamma_F(\alpha)\frac{y_i}{\alpha} + \alpha\frac{\partial\alpha}{\partial x^i} - \alpha y^j\left(\frac{\partial\beta_i}{\partial x^j} - \frac{\partial\beta_j}{\partial x^i}\right).$$

We can fix the spray Γ_F by the requirement that $\Gamma_F(\alpha) = 0$, so that motion is with constant Riemannian speed (this is not of course the canonical spray of F). Then for motion with unit Riemannian speed, motion under the Lagrangian L is the same as motion along the geodesics of the fixed spray Γ_F. (For another speed one would have to replace β by a constant multiple of it.)

It has been claimed [19] that the derivation of the equations of motion from F rather than L is an example of Maupertuis's principle in action. Be that as it may, this analysis does reveal that the spiral example above can be related to the motion of a charged particle in a constant magnetic field along the z-axis. It is known that in this regime circular motion is possible in planes perpendicular to the direction of the magnetic field, the radius of such circles being a constant whose value depends on the strength of the field and the charge (amongst other things); it is known as the gyroradius, Larmor radius or cyclotron radius. This is of course Shen's circle example.

9. Concluding remarks

It is pointed out in [16] that there cannot be a globally-defined Finsler function for the circle example above, for the following reason. In a Finsler space whose geodesic spray is positively complete, that is, for which every geodesic is defined on $[0, \infty)$, every pair of points in M can be joined by a geodesic. This is a conclusion of the Hopf–Rinow Theorem of Finsler geometry. In the circle example the geodesics are evidently positively complete, but equally evidently there are pairs of points which cannot be joined by a geodesic. The same is true of the spiral example.

This observation raises an interesting point about the different levels at which global questions enter the problem. We leave aside the two-dimensional case, which as we have seen is atypical. The differential conditions on a multiplier stated in Theorem 2 are merely the starting point for further analysis which generally proceeds as follows. These differential conditions are regarded as partial differential equations for the unknowns h_{ij}. In principle these differential equations generate integrability conditions, which are further conditions on the coefficients Γ^i and their derivatives. In favourable cases, for example when the spray is isotropic, such integrability conditions are satisfied, and one can assert the local existence of a multiplier satisfying the differential conditions. But there is no guarantee that such a multiplier is even defined y-globally.

If there is in fact a y-global multiplier, and it is everywhere positive quasi-definite, then there is always a Finsler function which is local in M, that is, defined y-globally over an open subset of M. There is a cohomological obstruction on M to combining such locally-defined Finsler functions; but even when this obstruction vanishes the result is globally only a pseudo-Finsler function in general, and can be turned into a genuine Finsler function (by addition of a total derivative) only locally. This is the situation exemplified in the previous section.

It seems that to make further progress one would have to impose some conditions on the global properties of the sprays of the projective class. The most important question is how one incorporates the observation above about completeness and the Hopf–Rinow property into the story — a question, we suspect, of some subtlety because ideally it should be answered in a projectively-invariant manner.

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Appendix A. Some basic formulas

Let Γ be a spray

$$\Gamma = y^i \frac{\partial}{\partial x^i} - 2\Gamma^i \frac{\partial}{\partial y^i},$$

with corresponding horizontal and vertical local vector field basis $\{H_i, V_i\}$ where

$$H_i = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial y^j}, \qquad \Gamma_j^i = \frac{\partial \Gamma^i}{\partial y^j}.$$

We denote by R_i^j the components of the Riemann curvature or Jacobi endomorphism of Γ. We have the following well-known formulas, which serve to define R_i^j:

$$[\Gamma, H_i] = \Gamma_i^j H_j + R_i^j V_j, \qquad [\Gamma, V_i] = -H_i + \Gamma_i^j V_j,$$

Of course $[V_i, V_j] = 0$; we have

$$[V_i, H_j] = -\Gamma_{ij}^k V_k = [V_j, H_i], \qquad \Gamma_{ij}^k = \frac{\partial \Gamma_i^k}{\partial y^j}.$$

We set

$$R^i_{jk} = \frac{1}{3}\left(\frac{\partial R^i_j}{\partial y^k} - \frac{\partial R^i_k}{\partial y^j}\right);$$

then $R^i_j = R^i_{jk} y^k$, and the bracket of horizontal fields is determined by

$$[H_j, H_k] = -R^i_{jk}\frac{\partial}{\partial y^i}.$$

Associated with any spray there is an operator ∇ on tensors along τ which is a form of covariant derivative, and indeed is often called the dynamical covariant derivative. We need it mainly for its action on tensors h with components h_{ij}, when

$$(\nabla h)_{ij} = \Gamma(h_{ij}) - \Gamma^k_i h_{kj} - \Gamma^k_j h_{ik}.$$

If Γ is a spray, another spray $\tilde{\Gamma}$ is projectively equivalent to Γ if there is a function P on $T^\circ M$, necessarily of homogeneity degree 1, such that $\tilde{\Gamma} = \Gamma - 2P\Delta$. Then

$$\tilde{\Gamma}^i_j = \Gamma^i_j + P\delta^i_j + P_j y^i, \quad P_j = \frac{\partial P}{\partial y^j}.$$

Appendix B. Geodesic convexity and good open coverings

In Section 6 we stated that every manifold admits good open coverings by coordinate patches. The usual argument for this uses Whitehead's results in [21] on the existence of geodesically convex sets, together with the fact that on any paracompact manifold one can construct a Riemannian metric, which of course provides a source of geodesics.

Though it is convenient, it is not actually necessary to appeal to the existence of a Riemannian metric. In the first place the results on convexity in [21] apply to the geodesics of any affine connection. What seems not to be so well known is that they also apply to the geodesics of any spray, when due account is taken of the fact that in this case the geodesics will be not in general be reversible: that is to say, when one remembers always to speak of the geodesic path from x_1 to x_2 (which may not be the same as the geodesic path from x_2 to x_1). Whitehead himself acknowledged that the results of [21] can be extended in this way, in an addendum to that paper, [22], published the following year. Since in our work we always have a spray at our disposal it seems natural to use this second version of the convexity result. The fact that geodesically convex regions exist for spray spaces is of interest more widely in spray and Finsler geometry than just in relation to the construction of good open coverings. Whitehead's addendum [22] appears to be far less well known than the original paper [21]; and it seems fair to say that in it Whitehead was not as careful about the matter of the non-reversiblity of geodesics of sprays as he might have been. For these reasons we have thought it worthwhile to outline how the result is proved.

Let us be precise about what is to be proved. Following Whitehead we shall use the term *open region* for an open subset of a coordinate patch of a manifold M, *closed region* for the closure of an open region, and *region* for either. A region C is (geodesically) *convex* with respect to a spray Γ on $T^\circ M$ if for every $x_1, x_2 \in C$ there is at least one geodesic path of Γ from x_1 to x_2 lying entirely within C. A region C is (geodesically) *simple* if there is at most one geodesic path of Γ from x_1 to x_2 lying entirely within C. Whitehead proves (for affine sprays in [21], and for sprays in general in [22]) that for any manifold M equipped with a spray Γ, "any point in M is contained in a simple, convex region which can be made as small as we please".

The basic analytical tool is Picard's Theorem on the existence and uniqueness of solutions of two-point or boundary-value problems for systems of second-order ordinary differential equations. Using this theorem Whitehead is able to show that for any $x \in M$ there is a region C_1 containing x such that for $x_1, x_2 \in C_1$ there is a geodesic $s \mapsto \gamma(x_1, x_2, s)$ with $\gamma(x_1, x_2, 0) = x_1$, $\gamma(x_1, x_2, 1) = x_2$, and $\gamma(x_1, x_2, s) \in C_1$ for $0 \leqslant s \leqslant 1$; moreover γ is continuous in all of its arguments. Thus C_1 is convex. Whitehead shows further that there is a second region C_2 with $x \in C_2 \subset C_1$ which is simple. These results apply to any spray, not just to affine ones; it was Whitehead's realization of this point that led him to write his addendum.

Now any subregion of a simple region is simple. This is not true of a convex region however: a subregion of a convex region need not be convex. But suppose that we can find a subregion of a convex region C_1 whose interior is an open connected set C containing x, whose closure is compact, and whose boundary B is a smooth hypersurface (codimension 1 submanifold) with the property that any geodesic tangent to B passes outside $\bar{C} = C \cup B$ at least locally: that is to say, that if γ is a geodesic with $\gamma(0) \in B$ and $\dot{\gamma}(0)$ tangent to B, then $\gamma(s) \notin \bar{C}$ for s in some open interval $(0, t)$, $t > 0$. In this case a geodesic cannot touch B while remaining otherwise in C. Following a line of argument from [21], using this observation, we show that C must be convex.

Consider the set of points $(a, b) \in C \times C$ such that there is a geodesic in C from a to b: call it G. Firstly, G is not empty: for any $a \in C$ and for any geodesic γ starting at a, $\gamma(s)$ must lie in C for all s in some interval $[0, t)$ for t sufficiently small. Secondly, G is open by continuity of $\gamma(a, b, s)$. Thirdly, G is closed (as a subset of $C \times C$). Consider a point (a, b) of $C \times C$ which lies in the relative closure of G; that is, (a, b) is the limit of a sequence (a_n, b_n) of points of G. Each geodesic

$s \mapsto \gamma(a_n, b_n, s)$ lies in C for $0 \leqslant s \leqslant 1$. There is a geodesic $s \mapsto \gamma(a, b, s)$ from a to b, but we know only that it lies in C_1 (though of course its initial and final points a and b are in C). Now for each s with $0 \leqslant s \leqslant 1$, $\gamma(a, b, s)$ is the limit of the sequence $\gamma(a_n, b_n, s)$, so $\gamma(a, b, s)$ certainly lies in $\bar{C}$ for all s. But the geodesic $s \mapsto \gamma(a, b, s)$ lies in C initially, and can neither meet B transversely, nor be tangent to it, without subsequently passing out of $\bar{C}$. So $\gamma(a, b, s)$ lies in C for all s, $0 \leqslant s \leqslant 1$, and $(a, b) \in G$. Since C is connected by assumption, so is $C \times C$; G is a non-empty subset of $C \times C$ which is both open and closed, and so $G = C \times C$.

So if we can find a region C with these properties in C_2 it will be convex and simple. In fact the open Euclidean coordinate ball $\{(x^i)\colon\ \delta_{ij}x^i x^j < r^2\}$ for any sufficiently small positive r will do for C (we take x as origin of coordinates). Here B is the Euclidean coordinate sphere of radius r of course. For any positive r consider the function $V_r(x^i) = \delta_{ij}x^i x^j - r^2$. Consider a geodesic γ such that $V_r(\gamma(0)) = 0$ and

$$\frac{d}{ds}(V_r \circ \gamma)_{s=0} = 2\delta_{ij}\gamma^i(0)\dot{\gamma}^j(0) = 0;$$

it is tangent to B at $\gamma(0) \in B$. We have

$$\frac{d^2}{ds^2}(V_r \circ \gamma)_{s=0} = 2\delta_{ij}\dot{\gamma}^i(0)\dot{\gamma}^j(0) - 4\delta_{ij}\gamma^i(0)\Gamma^j\big(\gamma(0), \dot{\gamma}(0)\big).$$

We shall show that for r sufficiently small this is positive, which means that $V_r(\gamma(s)) > 0$ for s in some open interval about 0, but $s \neq 0$, from which it will follow that any geodesic tangent to B locally lies outside $\bar{C}$. Without essential loss of generality we may assume that the first term is 2. For any $r_0 > 0$ the set $\{(x, y)\colon\ \delta_{ij}x^i x^j \leqslant r_0^2,\ \delta_{ij}y^i y^j = 1\}$ is compact, so there is $K > 0$ such that $|\Gamma^i(x, y)| < K$ for all i and all (x, y) in that set. Take $r \leqslant r_0$. Then $|\gamma^i(0)| \leqslant r$ and $|\Gamma^j(\gamma(0), \dot{\gamma}(0))| < K$, whence

$$\big|\delta_{ij}\gamma^i(0)\Gamma^j\big(\gamma(0), \dot{\gamma}(0)\big)\big| < nrK.$$

So provided that $r < 1/(2nK)$,

$$\frac{d^2}{ds^2}(V_r \circ \gamma)_{s=0} > 0$$

as required. This argument is essentially the same as the one given by Whitehead in [22]. It actually shows that the final inequality holds for all geodesics, not just those tangent to B; and in fact Whitehead gives this as his requirement for C to be convex, though this seems to us to be a stronger condition than is really necessary. Be that as it may, we have shown that for every sufficiently small r the Euclidean coordinate ball of radius r about x is a simple, convex region.

A convex region C is path connected and so connected. It is also contractible on any point $x_0 \in C$, for the map $h : C \times [0, 1] : h(x, s) = \gamma(x, x_0, s)$ is a continuous homotopy of the identity with the constant map $x \mapsto x_0$. Moreover the intersection of simple, convex sets is simple, convex. Thus a covering by open simple, convex regions is a good open covering.

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