



Hilbert forms for a Finsler metrizable projective class of sprays

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ABSTRACT

The projective Finsler metrizable problem deals with the question whether a projective-equivalence class of sprays is the geodesic class of a (locally- or globally-defined) Finsler function. In this paper we use Hilbert-type forms to state a number of different ways of specifying necessary and sufficient conditions for this to be the case, and we show that they are equivalent. We also address several related issues of interest including path spaces, Jacobi fields, totally-geodesic submanifolds of a spray space, and the equivalence of path geometries and projective-equivalence classes of sprays.

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1. Introduction

A Finsler function can in many ways be regarded as a singular Lagrangian. As such, there are many sprays whose base integral curves are solutions of the Euler–Lagrange equations of a given Finsler function. These sprays are all projectively equivalent and together they constitute the geodesic class of sprays of the given Finsler function. It is therefore natural to ask whether or not a given projective-equivalence class (or projective class, for short) of sprays is the geodesic class of some Finsler function, or, in the terminology of this paper, whether or not a projective class of sprays is projectively Finsler metrizable.

One may think of (at least) three approaches to formulating the necessary and sufficient conditions for this to be the case. They differ with respect to what kind of geometric object the conditions are expressed in terms of:

1. a multiplier, that is, a symmetric twice covariant tensor along the tangent bundle projection τ , leading to Helmholtz-like conditions;

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2. a semi-basic 1-form, leading to the conditions given by Bucataru and Muzsnay [5] for such a form to be a Hilbert 1-form;
3. a 2-form, leading to conditions for such a form to be a Hilbert 2-form.

The third item can be further subdivided:

- 3.1. the 2-form is given on the slit tangent bundle, leading to conditions similar to those given for the ‘ordinary’ inverse problem of the calculus of variations by the first author as long ago as 1981 [6];
- 3.2. the 2-form is given on a certain manifold on which is defined an almost Grassmann structure associated with the projective class, leading to conditions formulated by the first and third authors in [10];
- 3.3. the 2-form is given on path space, leading to conditions discussed by Álvarez Paiva in [2].

Note that unlike Álvarez Paiva, who in [2] deals only with reversible paths, that is, paths which have no preferred orientation, we cover in this paper the more general case of oriented paths, or sprays in the fully general sense.

We have discussed the multiplier approach in detail in [9]. In this paper we deal with the versions of the conditions involving forms, that is, items 2 and 3.1–3.3 of the lists above.

It might be argued that there are two additional approaches that should be taken into account. One is the use of the Rapcsák conditions (which are discussed in [15,16] for example). We prefer to think of these conditions as just being reformulations of the Euler–Lagrange equations. They do play a significant role in our analysis of the multiplier problem, and have been discussed in [9]. The other is the holonomy method described in [7]. This approach is well-suited to the problem of determining whether a given spray is the canonical spray of a Finsler function, that is, the one whose integral curves are parametrized (up to affine transformations) by arc-length. However, it is not easily adapted to the projective problem which is the subject of this paper. We do not consider it further here therefore.

To the best of our knowledge, this paper states for the first time the metrizable conditions in terms of 2-forms on the slit tangent bundle. We also address the global aspects of the problem. The main purpose of this paper, however, is to discuss the relationship between the various approaches enumerated above, and in particular to show that they are equivalent. Such a discussion is in particular needed because comparison of the different results in the literature is far from obvious. To give just one example: whereas most authors consider the projective class of sprays as the main object under investigation, others, in particular Álvarez Paiva, give priority to the paths. We have therefore considered it desirable to discuss the relationship between what is called by Álvarez Paiva in [2] a path geometry, and a projective class of sprays. In the course of the discussion it will also be necessary to address a number of issues related to Finsler geometry and the projective geometry of sprays which are of interest in their own right, including Jacobi fields and totally-geodesic submanifolds of a spray space. We express our results as far as possible in projectively-invariant terms; in particular, this means that throughout we use the Finsler function rather than the energy, and avoid reference to the canonical geodesic spray. In the terminology introduced in [17] we deal entirely with the problem of metrizable in the broad sense.

The paper begins with a version of Álvarez Paiva’s definition of a path geometry adapted to the concerns of this paper. We show that in fact there is no loss of generality in working with sprays.

In Section 3 we give a summary of the relevant results on the multiplier problem from [9]. In Section 4 we quote the theorem of Bucataru and Muzsnay mentioned in item 2 above, and show that the conditions it contains are equivalent to those that must be satisfied by a multiplier. In Section 5 we give the most straightforward of the formulations of the conditions in terms of the existence of a 2-form with certain properties, and in the following section the somewhat more sophisticated version in which the 2-form is specified on a certain manifold which carries an almost Grassmann structure associated with a given projective class of sprays.

All three of the versions of the conditions discussed in Sections 4–6 involve closed 2-forms of which the involutive distribution \mathcal{D} determined by the projective class (see the next section) is the characteristic distribution. A natural further step therefore is to quotient out by \mathcal{D} , as one might say. Where this is possible the manifold obtained is called the path space, since each of its points represents a geodesic path of the projective class. The 2-form in question passes to the quotient to define a symplectic form there. In Section 7 we elaborate on this construction and begin the discussion of the further properties of the symplectic structure. As we show in Section 8, tangent vectors to path space can be thought of as Jacobi fields. Using this insight we reformulate the positive quasi-definiteness property of the multiplier required for the local existence of a Finsler function.

One much discussed special case of the projective metrizable problem is that raised by the Finslerian version of Hilbert’s fourth problem; this indeed is the main subject of [2]. In Álvarez Paiva’s analysis an important role is played by 2-planes in \mathbb{R}^n . From the more general point of view adopted here what is significant about planes in \mathbb{R}^n is that they are totally-geodesic submanifolds. We develop a theory of totally-geodesic submanifolds of spray manifolds in Section 9, and use it to give a modest generalization of one of the results of [2]. The paper ends with an illustrative example.

2. Path geometries and sprays

We first recall some basic concepts from spray and Finsler geometry, mainly to fix notations.

We shall always assume that the base manifold M is smooth and paracompact. Unless it is explicitly stated otherwise, we assume that $\dim M \geq 3$. The slit tangent bundle of M is the tangent bundle with the zero section removed. We shall denote it by $\tau : T^\circ M \rightarrow M$.

A spray is a vector field on $T^\circ M$ such that $\tau_* \Gamma_{(x,y)} = y$ for any $x \in M$ and $y \in T_x M$, $y \neq 0$, and such that $[\Delta, \Gamma] = \Gamma$ where Δ is the Liouville field. It is locally of the form

$$\Gamma = y^i \frac{\partial}{\partial x^i} - 2\Gamma^i \frac{\partial}{\partial y^i}$$

and it determines a horizontal distribution, spanned by the vector fields

$$H_i = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial y^j}, \quad \Gamma_j^i = \frac{\partial \Gamma^i}{\partial y^j}.$$

We shall also write V_i for the vertical vector fields $\partial/\partial y^i$. Horizontal and vertical lifts of a vector field X on M are denoted by X^H and X^V , respectively.

Two sprays are said to be *projectively equivalent* if their geodesics (base integral curves) are the same up to an orientation-preserving reparametrization. The geodesics of projectively equivalent sprays, in other words, define oriented paths in M . Projective equivalence is an equivalence relation on sprays; an equivalence class is called a projective class of sprays. If Γ is a spray, then any member of its projective class takes the form $\Gamma - 2P\Delta$ for some function P on $T^\circ M$ which satisfies $\Delta(P) = P$. Note that a projective class of sprays determines an involutive two-dimensional distribution \mathcal{D} on $T^\circ M$, which is spanned by Δ and any spray Γ of the class. This distribution plays an important role in our analysis. We refer to e.g. [15] for further reading on the geometry of sprays.

We shall work throughout with projective classes of sprays. It might however be regarded as more natural from the geometrical point of view to see a projective class of sprays as merely a surrogate for the collection of its geodesic paths, and to think of the metrizable problem as the question of whether a collection of oriented paths, suitably specified, is the set of geodesic paths of a Finsler structure. Álvarez Paiva for example, in [2, Section 4], has taken such an idea as basic and formalized it into the concept of a path geometry. In this section we give a definition of path geometry based on Álvarez Paiva's, but differing from his in that it deals with oriented paths; and we show that there is no loss of generality in working with sprays.

For any smooth manifold M we denote by $\sigma : STM \rightarrow M$ its sphere bundle, that is, the quotient of $T^\circ M$ by the action induced by Δ , so that a point s of STM is an equivalence class $[y]$ of vectors y in $T_x M$, $x = \sigma(s)$, where the equivalence relation is multiplication by a positive scalar. A *path geometry* on M is a smooth foliation of STM by oriented one-dimensional submanifolds \mathfrak{S} which satisfy what one might call the second-order property, namely that if \mathfrak{S}_s is the submanifold through s , the (oriented) tangent space to $\sigma(\mathfrak{S}_s)$ at x coincides with $[y]$ (where $s = [y]$).

We define a distribution \mathcal{D} on $T^\circ M$ as follows: $v \in \mathcal{D}_y \subset T_y T^\circ M$ if the projection of v to STM is tangent to $\mathfrak{S}_{[y]}$. Then \mathcal{D} is an involutive two-dimensional smooth distribution on $T^\circ M$, containing Δ . We shall show that \mathcal{D} is the distribution corresponding to a projective class of sprays on $T^\circ M$.

Theorem 1. *For any given path geometry on STM , there is a projective class of sprays on $T^\circ M$ such that the distribution \mathcal{D} is spanned by Δ and any spray of the class.*

Proof. We have to construct a suitable spray Γ .

There is a covering of $T^\circ M$ by open sets U , which we may assume to be connected, such that on U there is a smooth vector field Z_U such that $\mathcal{D}|_U$ is the span of Δ and Z_U . The projection of Z_U to STM is tangent to the foliation, and never vanishes. We may assume that it is oriented positively with respect to the foliation. Then for every $(x, y) \in U$, $\tau_* Z_U(x, y)$ is a positive scalar multiple of y , say $\tau_* Z_U(x, y) = \zeta(x, y)y$ where ζ is a positive smooth function on U . Set $\tilde{T}_U = (1/\zeta)Z_U$; then \tilde{T}_U is a second-order differential equation field on U , and $\mathcal{D}|_U$ is the span of Δ and \tilde{T}_U .

The manifold M , which is assumed to be paracompact, admits a global Riemannian metric, say g . Denote by G the function on $T^\circ M$ given by the Riemannian norm, so that $G(x, y) = \sqrt{g_x(y, y)}$. Note that $\Delta(G) = G$. We can change the local basis of $\mathcal{D}|_U$ by adding some scalar multiple of Δ to \tilde{T}_U , and we can do so in such a way that the new vector field $\Gamma_U = \tilde{T}_U + f\Delta$ satisfies $\Gamma_U(G) = 0$: just take $f = -\tilde{T}_U(G)/G$. Of course, Γ_U is also a second-order differential equation field. It is moreover uniquely determined by the properties that it is a second-order differential equation field in $\mathcal{D}|_U$ and satisfies $\Gamma_U(G) = 0$: for if Γ'_U also has those properties then $\Gamma_U - \Gamma'_U$ is vertical, in $\mathcal{D}|_U$, and therefore a scalar multiple of Δ ; but since $\Gamma_U(G) - \Gamma'_U(G) = 0$, while $\Delta(G) = G$, the scalar factor must be zero.

It follows that there is a globally-defined vector field Γ , which is a second-order differential equation field in \mathcal{D} satisfying $\Gamma(G) = 0$, such that $\Gamma_U = \Gamma|_U$. For if Γ_U and $\Gamma_{U'}$ are the unique local vector fields with those properties on U and U' then by uniqueness they must agree on $U \cap U'$.

Finally, we show that Γ is a spray, that is, that it satisfies $[\Delta, \Gamma] = \Gamma$. Now $[\Delta, \Gamma] - \Gamma$ is certainly vertical, simply because Γ is a second-order differential equation field. Thus $[\Delta, \Gamma] = \Gamma + f\Delta$ for some function f on $T^\circ M$. But $\Gamma(G) = 0$, and $[\Delta, \Gamma](G) = \Delta(\Gamma(G)) - \Gamma(\Delta(G)) = -\Gamma(G) = 0$; but $\Delta(G) = G$, and so $f = 0$. \square

3. Some results on the multiplier problem

In order to keep the paper more or less self-contained, we shall quote here some results from [9].

A *Finsler function* is a smooth function on $T^\circ M$, which is positive, positively (but not necessarily absolutely) homogeneous, and strongly convex. The last property means that the matrix of functions

$$g_{ij} = \frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

must be positive definite. The *Hilbert 1- and 2-forms* on $T^\circ M$ are given, respectively, by

$$\theta = \frac{\partial F}{\partial y^i} dx^i \quad \text{and} \quad d\theta.$$

We shall say that Γ is a geodesic spray for F if its base integral curves are solutions of the Euler–Lagrange equations of F . The set of geodesic sprays for F forms a projective class. A modern introduction to Finsler geometry can be found in [3].

We shall use the term *multiplier* for a $(0, 2)$ tensor field h along the slit tangent bundle projection τ . A multiplier will also be called a tensor or tensor field for short, and we shall often denote it simply by its components $h_{ij}(x, y)$.

The conditions on a multiplier that form the basis of the analysis in [9] are these:

$$\begin{aligned} h_{ji} &= h_{ij}, \\ h_{ij}y^j &= 0, \\ \frac{\partial h_{ij}}{\partial y^k} &= \frac{\partial h_{ik}}{\partial y^j}, \\ (\nabla h)_{ij} &= 0, \\ h_{ik}W_j^k &= h_{jk}W_i^k. \end{aligned}$$

Here ∇ stands for the dynamical covariant derivative operator of any choice of spray Γ in the projective class. The action of this operator on tensors h is given by $(\nabla h)_{ij} = \Gamma(h_{ij}) - \Gamma_i^k h_{kj} - \Gamma_j^k h_{ik}$. The functions W_j^k are the components of the (projectively-invariant) Weyl tensor. A result of [9] states that the last condition can equivalently be replaced by the condition $\bigoplus R_{jk}^l h_{il} = 0$, where \bigoplus stands for a cyclic sum and where R_{jk}^l are the curvature components of the horizontal distribution (with $[H_i, H_j] = -R_{ij}^l V_l$), or by the condition $h_{ik}R_j^k = h_{jk}R_i^k$, where $R_j^k = R_{jl}^k y^l$ are the components of the Jacobi endomorphism (Riemann curvature).

The conditions displayed above, though expressed in coordinate form, are tensorial in nature. They play the same role in relation to the projective Finsler metrization problem as the Helmholtz conditions do for the general inverse problem of the calculus of variations; though it is not strictly accurate, for ease of reference we shall call them the Helmholtz conditions in this paper (in [9] we referred to them as Helmholtz-like conditions).

A tensor h_{ij} is said to be *positive quasi-definite* if $h_{ij}(y)v^i v^j \geq 0$, with equality only if v is a scalar multiple of y . We shall say that a multiplier h is *quasi-regular* if $h_{ij}(y)v^j = 0$ if and only if $v^i = ky^i$ for some scalar k . We shall call a positively-homogeneous function F whose Hessian with respect to fibre coordinates is quasi-regular a *pseudo-Finsler function*. We summarize the relevant results from [9] in the following theorem (they occur as Theorems 2, 3 and 4 in [9]).

Theorem 2.

- (1) Given a projective class of sprays over a manifold M , and any contractible coordinate neighbourhood $U \subset M$, there is a positively-homogeneous function F on $T^\circ U$ such that every spray in the class satisfies the Euler–Lagrange equations for F if and only if there are functions h_{ij} on $T^\circ U$ which satisfy the Helmholtz conditions.
- (2) If F is a (global) Finsler function on $T^\circ M$ then its Hessian h satisfies the Helmholtz conditions for the sprays of its geodesic class, and is in addition positive quasi-definite. Conversely, suppose given a projective class of sprays on $T^\circ M$. If there is a tensor field h on $T^\circ M$ which satisfies the Helmholtz conditions everywhere and is in addition positive quasi-definite, and if $H^2(M) = 0$, then the projective class is the geodesic class of a global pseudo-Finsler function, and of a local Finsler function over a neighbourhood of any point of M .
- (3) The projective class of a reversible spray on $T^\circ M$ is the geodesic class of a globally-defined absolutely-homogeneous Finsler function if and only if there is a tensor field h which satisfies the Helmholtz conditions everywhere and is in addition positive quasi-definite.

4. The theorem of Bucataru and Muzsnay

The following theorem appears in [5].

Theorem 3 (Bucataru and Muzsnay). *A spray Γ is projectively metrizable if and only if there exists a semi-basic 1-form θ on $T^\circ M$ such that*

$$\begin{aligned} \text{rank}(d\theta) &= 2n - 2, & i_\Gamma \theta &> 0, \\ \mathcal{L}_\Delta \theta &= 0, & d_J \theta &= 0, & d_H \theta &= 0. \end{aligned}$$

We have modified the notation to fit ours. Here J is the tangent structure and H the horizontal projector, both type $(1, 1)$ tensor fields on $T^\circ M$:

$$J = V_i \otimes dx^i, \quad H = H_i \otimes dx^i.$$

The conditions $d_J \theta = 0$ and $d_H \theta = 0$ amount to

$$d\theta(X^V, Y^H) + d\theta(X^H, Y^V) = 0, \quad d\theta(X^H, Y^H) = 0$$

respectively, where X and Y are any vector fields on M ; or in terms of the basis fields,

$$V_i(\theta_j) = V_j(\theta_i), \quad H_i(\theta_j) = H_j(\theta_i), \quad \text{where } \theta = \theta_i dx^i.$$

We call the conditions in the first line of the theorem the algebraic conditions, those in the second line the differential conditions, on θ . We show first that the differential conditions are equivalent to the Helmholtz conditions.

Theorem 4. *Suppose that, for a given spray Γ , there is a semi-basic 1-form θ satisfying the differential conditions of Theorem 3. Then $h_{ij} = V_i(\theta_j)$ satisfies the Helmholtz conditions. Conversely, suppose that the tensor h_{ij} satisfies the Helmholtz conditions. Then there is a semi-basic 1-form θ which satisfies the differential conditions of Theorem 3, and $h_{ij} = V_i(\theta_j)$.*

Proof. Suppose that the semi-basic 1-form θ satisfies the differential conditions of Theorem 3. Set $h_{ij} = V_i(\theta_j)$. This is a tensor field along τ of the indicated type. Since $V_i(\theta_j) = V_j(\theta_i)$, as we pointed out above, h_{ij} is symmetric. Moreover $V_k(h_{ij}) = V_k V_j(\theta_i) = V_j V_k(\theta_i) = V_j(h_{ik})$, since V_j and V_k commute. Furthermore $\mathcal{L}_\Delta \theta = y^j V_j(\theta_i) dx^i = h_{ij} y^j dx^i = 0$. Now $H_i(\theta_j) = H_j(\theta_i)$, from which it follows that

$$\Gamma(\theta_i) = H_i(\theta_k) y^k = H_i(\theta_k y^k) + \Gamma_i^k \theta_k.$$

Now apply V_j and use $[V_j, \Gamma] = H_j - \Gamma_j^k V_k$ to obtain

$$V_j \Gamma(\theta_i) = \Gamma(h_{ij}) + H_j(\theta_i) - \Gamma_j^k h_{ik} = V_j H_i(\theta_k y^k) + \Gamma_{ij}^k \theta_k + \Gamma_i^k h_{jk},$$

where $\Gamma_j^k = V_j(\Gamma_i^k)$. But $[V_j, H_i] = -\Gamma_j^k V_k$, and $V_j(\theta_k y^k) = h_{jk} y^k + \theta_j = \theta_j$. Thus $V_j H_i(\theta_k y^k) + \Gamma_{ij}^k \theta_k = H_i(\theta_j)$, and so

$$\Gamma(h_{ij}) - \Gamma_j^k h_{ik} - \Gamma_i^k h_{jk} = (\nabla h)_{ij} = 0.$$

Finally, note that $[H_j, H_k](\theta_i) = -R_{jk}^l h_{il}$: but then

$$\bigoplus [H_j, H_k](\theta_i) = -\bigoplus R_{jk}^l h_{il} = 0$$

in virtue of the fact that $H_i(\theta_j) = H_j(\theta_i)$. But as we remarked above, the vanishing of $\bigoplus R_{jk}^l h_{il}$ is equivalent to $h_{ik} W_j^k = h_{jk} W_i^k$. Thus h_{ij} satisfies the Helmholtz conditions.

Conversely, suppose that h_{ij} satisfies the Helmholtz conditions. Since $V_k(h_{ij}) = V_j(h_{ik})$ there are locally-defined functions $\bar{\theta}_i$, determined up to the addition of arbitrary functions of the x^i alone, such that $h_{ij} = V_j(\bar{\theta}_i)$; and $V_j(\bar{\theta}_i) = V_i(\bar{\theta}_j)$. We next show that for any choice of the $\bar{\theta}_i$, the functions $H_i(\bar{\theta}_j) - H_j(\bar{\theta}_i)$ are independent of the y^k . Now

$$\frac{\partial}{\partial y^k} (H_i(\bar{\theta}_j)) = H_i(h_{jk}) - \Gamma_{ik}^l h_{jl}.$$

It is a simple and well-known consequence of the assumptions that $(\nabla h)_{ij} = 0$ and $V_k(h_{ij}) = V_j(h_{ik})$ that

$$H_i(h_{jk}) - \Gamma_{ik}^l h_{jl} = H_j(h_{ik}) - \Gamma_{jk}^l h_{il},$$

whence $H_i(\bar{\theta}_j) - H_j(\bar{\theta}_i)$ is independent of the y^k . Thus

$$\chi = (H_i(\bar{\theta}_j) - H_j(\bar{\theta}_i)) dx^i \wedge dx^j$$

is a basic 2-form. We show that χ is closed. In computing $d\chi$ we may replace the partial derivative with respect to x^k with H_k . We have $\bigoplus H_k(H_i(\bar{\theta}_j) - H_j(\bar{\theta}_i)) = \bigoplus [H_j, H_k](\bar{\theta}_i) = -\bigoplus R_{jk}^i h_{il}$. But this vanishes if $h_{ik}W_j^k = h_{jk}W_i^k$. So χ is closed, and hence (locally) exact. If now $\chi = d\psi$ with $\psi = \psi_i dx^i$, and $\theta_i = \bar{\theta}_i - \psi_i$, then

$$(H_i(\theta_j) - H_j(\theta_i)) dx^i \wedge dx^j = \chi - d\psi = 0.$$

Set $\theta = \theta_i dx^i$. Then $V_i(\theta_j) = h_{ij}$, $d_j\theta = 0$ and $d_H\theta = 0$; moreover $\mathcal{L}_\Delta\theta = h_{ij}y^j dx^i = 0$; so θ satisfies the differential conditions of [Theorem 3](#). \square

The condition on the rank of $d\theta$ gives the following corollary.

Corollary 1. *The projective class of sprays containing Γ is the geodesic class of a pseudo-Finsler function if and only if there is a semi-basic 1-form θ on $T^\circ M$ which satisfies the differential conditions of [Theorem 3](#), and in addition $\text{rank}(d\theta) = 2n - 2$.*

Proof. Let $\{dx^i, \phi^i\}$ be the local basis of 1-forms dual to the local basis of vector fields $\{H_i, V_i\}$ corresponding to the horizontal distribution determined by Γ . Then

$$d\theta = H_i(\theta_j) dx^i \wedge dx^j - V_i(\theta_j) dx^i \wedge \phi^j = -h_{ij} dx^i \wedge \phi^j.$$

It follows from the fact that $h_{ij}y^j = 0$ that $i_\Gamma d\theta = i_\Delta d\theta = 0$; thus in general $\text{rank}(d\theta) \leq 2n - 2$, and $\text{rank}(d\theta) = 2n - 2$ if and only if h_{ij} is quasi-regular. \square

The condition $i_\Gamma\theta > 0$ now comes into its own in ensuring that the pseudo-Finsler function is actually a Finsler function: θ (if it exists with the given properties) is the Hilbert 1-form, and $i_\Gamma\theta = F$, so this condition, together with the rank condition on $d\theta$, say that there is a positive pseudo-Finsler function. But it can be shown that a pseudo-Finsler function which takes only positive values is a Finsler function, a result originally due to Lovas [12] which is quoted in [5].

It is worth remarking, with reference to the relation between [Theorem 3](#) and [Theorem 5](#) below, that if one adds to θ the pull-back of any closed 1-form on M then $d\theta$ is unchanged; and this operation corresponds exactly to adding a total derivative to F . So in a sense the inequality condition in [Theorem 3](#) requires that there must be, among all of the pseudo-Finsler functions with a given Hilbert 2-form, determined up to the addition of a total derivative, one (at least) which is everywhere positive. The result of the analysis leading to [Theorem 1](#) in [9] suggests however that to expect this positivity condition to hold globally over M is somewhat ambitious.

5. Formulations in terms of 2-forms

Let Γ be a (semi-)spray and $\{dx^i, \phi^i\}$ the local basis of 1-forms corresponding to its horizontal distribution. A symmetric tensor $h = h_{ij}(y) dx^i \otimes dx^j$ can always be lifted to a 2-form $\omega = h_{ij}(y) dx^i \wedge \phi^j$ on $T^\circ M$. This procedure was called the Kähler lift of h in [13], since ω is clearly a generalization of the Kähler form of a Riemannian metric.

Recall that for a given projective class of sprays we denote by \mathcal{D} the distribution on $T^\circ M$ spanned by Δ and any spray of the class; it is involutive.

Lemma 1. *Suppose given a projective class of sprays, and a symmetric tensor h_{ij} such that $h_{ij}y^j = 0$. Let Γ be any spray of the class, and $\omega = h_{ij} dx^i \wedge \phi^j$ the corresponding Kähler lift of h . Then ω is a concomitant of the class, that is, it is the same whichever spray in the class is used to define it. Moreover, the characteristic distribution of any such 2-form ω contains the distribution \mathcal{D} defined by the class.*

Proof. Any other member of the projective class is of the form $\tilde{\Gamma} = \Gamma - 2P\Delta$, where P is a positively-homogeneous function on $T^\circ M$. For the local basis $\{dx^i, \phi^i\}$ corresponding to $\tilde{\Gamma}$ we have

$$\tilde{\phi}^i = \phi^i + P dx^i + y^i V_j(P) dx^j,$$

from which the first result readily follows. Clearly $i_\Gamma\omega = i_\Delta\omega = 0$, as a consequence of the fact that $h_{ij}y^j = 0$. \square

Theorem 5. *Let Γ be a spray, and let ω be a 2-form on $T^\circ M$ such that*

1. *the characteristic distribution of ω contains \mathcal{D} , the distribution spanned by the projective class of Γ ;*
2. $\mathcal{L}_\Gamma\omega = 0$;
3. *for any pair of vertical vector fields V_1, V_2 , $\omega(V_1, V_2) = 0$;*
4. *for any horizontal vector field H and any pair of vertical vector fields V_1, V_2 , $d\omega(H, V_1, V_2) = 0$.*

Then over any coordinate neighbourhood $U \subset M$, $\omega = h_{ij} dx^i \wedge dx^j$ where h_{ij} satisfies the Helmholtz conditions. Conversely, if h_{ij} satisfies the Helmholtz conditions then for any spray Γ in the projective class the 2-form $\omega = h_{ij} dx^i \wedge dx^j$ on $\tau^{-1}U$ has the foregoing properties.

Assumptions 3 and 4 may be stated as follows: for every $(x, y) \in T^\circ M$ the vertical subspace of $T_{(x,y)}T^\circ M$ is isotropic for ω and $i_H d\omega$.

Proof of Theorem 5. We may express ω in terms of the basis $\{dx^i, \phi^j\}$ defined by Γ . It has no term in $\phi^i \wedge \phi^j$ because of assumption 3. Thus we may write

$$\omega = a_{ij} dx^i \wedge dx^j + h_{ij} dx^i \wedge \phi^j$$

where $a_{ji} = -a_{ij}$. A straightforward calculation yields

$$\mathcal{L}_\Gamma \omega = (\Gamma(a_{ij}) - 2a_{ik}\Gamma_j^k - h_{ik}R_j^k) dx^i \wedge dx^j + ((\nabla h)_{ij} + 2a_{ij}) dx^i \wedge \phi^j + h_{ij} \phi^i \wedge \phi^j.$$

Since this must vanish, it follows (working from right to left) that h_{ij} is symmetric; that $(\nabla h)_{ij} = a_{ij} = 0$ because one is symmetric, the other skew; and that $h_{ik}R_j^k$ is symmetric in i and j . In particular, $\omega = h_{ij} dx^i \wedge dx^j$; it then follows from the first assumption that $h_{ij}y^j = 0$. Now

$$d\omega = V_k(h_{ij}) dx^i \wedge \phi^j \wedge \phi^k \pmod{dx^i \wedge dx^j},$$

so that

$$d\omega(H_i, V_j, V_k) = V_k(h_{ij}) - V_j(h_{ik}) = 0.$$

Thus the coefficients h_{ij} satisfy the Helmholtz conditions.

The converse is straightforward. \square

Corollary 2. If a 2-form ω has the properties stated in Theorem 5 then

1. $\omega(X^H, Y^H) = 0$ and $\omega(X^H, Y^V) = \omega(Y^H, X^V)$ for any vector fields X, Y on M ;
2. ω is closed;
3. $\mathcal{L}_Z \omega = 0$ for any vector field Z in \mathcal{D} .

Proof. 1. These follow from the explicit form of ω .

2. A straightforward calculation gives

$$d\omega = \frac{1}{2} h_{il} R_{jk}^l dx^i \wedge dx^j \wedge dx^k + (H_i(h_{jk}) + h_{il} \Gamma_{jk}^l) dx^i \wedge dx^j \wedge \phi^k.$$

The first term vanishes because $\bigoplus h_{il} R_{jk}^l = 0$, the second because $H_i(h_{jk}) + h_{il} \Gamma_{jk}^l$ is symmetric in i and j , as we established in the proof of Theorem 4.

3. For any $Z \in \mathcal{D}$, $\mathcal{L}_Z \omega = d(i_Z \omega) + i_Z d\omega = 0$. \square

Corollary 3. A projective class of sprays is the geodesic class of a locally-defined pseudo-Finsler function (that is, one defined over a coordinate neighbourhood $U \subset M$) if and only if there is a 2-form ω on $\tau^{-1}U$ with the properties stated in Theorem 5, such that the characteristic distribution of ω is precisely the distribution \mathcal{D} spanned by Δ and any spray of the class.

We next describe how the positive quasi-definiteness condition on h may be specified as a condition on ω . For any chosen Γ of the class, for $x \in M$ and $y \in T_x^\circ M$ we define a quadratic form $q_{(x,y)}$ on $T_x M$ by $q_{(x,y)}(v) = \omega_{(x,y)}(v^H, v^V)$. Notice that $q_{(x,y)}(y) = \omega_{(x,y)}(\Gamma, \Delta) = 0$. This definition may appear to depend on a choice of Γ from the projective class. The two-dimensional subspaces of $T_y T^\circ M$ of the form $\langle v^H, v^V \rangle$ are well defined for a given Γ , but change if Γ is changed to a different member of the projective class. But if we change Γ to $\Gamma - 2P\Delta$ then v^H changes to $v^H - Pv^V - v^V(P)\Delta_{(x,y)}$, and this makes no difference to the value of $\omega_{(x,y)}(v^H, v^V)$. So the quadratic form $q_{(x,y)}$ is in fact a concomitant of the class.

Corollary 4. If the quadratic form q is positive quasi-definite everywhere on $T^\circ M$ then in a neighbourhood of any point in M there is a local Finsler function of which the projective class of sprays is the geodesic class.

Proof. This follows directly from the explicit form of ω . \square

Putting these local results together with Theorem 2 we obtain the following global theorem.

Theorem 6. *If F is a (global) Finsler function on $T^\circ M$ then its Hilbert 2-form $\omega = d\theta$ satisfies the conditions of Theorem 5 for the sprays of its geodesic class, and in addition the corresponding quadratic form q is positive quasi-definite everywhere. Conversely, suppose given a projective class of sprays on $T^\circ M$. If there is a 2-form ω on $T^\circ M$ which everywhere satisfies the conditions of Theorem 5 and whose corresponding quadratic form q is everywhere positive quasi-definite, and if $H^2(M) = 0$, then the projective class is the geodesic class of a global pseudo-Finsler function, and of a local Finsler function over a neighbourhood of any point of M .*

We can illustrate the role of the cohomology condition in this theorem, in a way a little different from the way it appears in the proof of Theorem 2 in [9], by examining the obstructions to the existence of a global semi-basic 1-form θ whose exterior derivative is the closed 2-form ω .

We first prove the local version of the result. We shall make use of the obvious fact that a form (of any degree) on $T^\circ M$ which is semi-basic and closed is basic (and closed).

Lemma 2. *Let ω be a 2-form on $T^\circ M$ which is closed and which vanishes when both of its arguments are vertical. Then for any contractible coordinate neighbourhood $U \subset M$ there is a semi-basic 1-form θ on $\tau^{-1}U$ such that $\omega = d\theta$.*

Proof. Set $\omega = a_{ij} dx^i \wedge dx^j + b_{ij} dx^i \wedge dy^j$. The $dx \wedge dy \wedge dy$ term in $d\omega$ is $V_k(b_{ij}) dx^i \wedge dy^j \wedge dy^k$. This must vanish, whence $V_k(b_{ij}) = V_j(b_{ik})$. Assuming that $\dim M \geq 3$ it follows that there are functions $b_i(x, y)$ on $\tau^{-1}U$ such that $b_{ij}(x, y) = V_j(b_i)(x, y)$. Then

$$\omega + d(b_i dx^i) = \left(a_{ij} + \frac{\partial b_j}{\partial x^i} \right) dx^i \wedge dx^j.$$

The right-hand side is semi-basic and closed, so basic and closed, so there is a 1-form ψ on U such that $\omega + d(b_i dx^i) = d\psi$. Thus $\omega = d\theta$ with $\theta = \psi - b_i dx^i$, which is semi-basic. \square

To derive the global theorem we shall need the following concepts and results.

An open covering $\mathfrak{U} = \{U_\lambda : \lambda \in \Lambda\}$ of M which has the property that every U_λ , and every non-empty intersection of finitely many of the U_λ , is contractible is known as a good covering. It can be shown (see [9]) that every manifold over which is defined a spray admits good open coverings by coordinate patches.

The Čech cohomology group $\check{H}^2(\mathfrak{U}, \mathcal{R})$ of a good open covering \mathfrak{U} of M is isomorphic to the de Rham cohomology group $H^2(M)$. In particular, if $H^2(M) = 0$ then $\check{H}^2(\mathfrak{U}, \mathcal{R}) = 0$; it is this form of the assumption that we shall actually use in the proof.

Suppose that for a given good open covering \mathfrak{U} of M by coordinate neighbourhoods, for each $\lambda, \mu \in \Lambda$ for which $U_\lambda \cap U_\mu$ is non-empty there is defined on $U_\lambda \cap U_\mu$ a function $\phi_{\lambda\mu}$, and that these functions satisfy the cocycle condition $\phi_{\mu\nu} - \phi_{\lambda\nu} + \phi_{\lambda\mu} = 0$ on $U_\lambda \cap U_\mu \cap U_\nu$ (assuming it to be non-empty). Then there is a locally finite refinement $\mathfrak{V} = \{V_\alpha : \alpha \in A\}$ of \mathfrak{U} , and for each α a function ψ_α defined on V_α , such that on $V_\alpha \cap V_\beta$ (assuming it to be non-empty) $\phi_{\alpha\beta} = \psi_\alpha - \psi_\beta$, where $\phi_{\alpha\beta}$ is defined from some $\phi_{\lambda\mu}$ by restriction. This result, which is proved using a partition of unity argument in [9], is a particular case of the fact that Čech cohomology is a sheaf cohomology theory (see [18]).

Theorem 7. *Let ω be a 2-form on $T^\circ M$ which is closed and which vanishes when both of its arguments are vertical. Suppose that $H^2(M) = 0$. Then there is a semi-basic 1-form θ on $T^\circ M$ such that $d\theta = \omega$.*

Proof. Let \mathfrak{U} be a good open covering of M by coordinate neighbourhoods. On each U_λ there is a semi-basic 1-form θ_λ such that $\omega = d\theta_\lambda$. On $U_\lambda \cap U_\mu$, $d(\theta_\lambda - \theta_\mu) = 0$; that is, $\theta_\lambda - \theta_\mu$ is semi-basic and closed, so there is a function $\phi_{\lambda\mu}$ on $U_\lambda \cap U_\mu$ such that $\theta_\lambda - \theta_\mu = d\phi_{\lambda\mu}$. On $U_\lambda \cap U_\mu \cap U_\nu$, $d(\phi_{\mu\nu} - \phi_{\lambda\nu} + \phi_{\lambda\mu}) = 0$, so $\phi_{\mu\nu} - \phi_{\lambda\nu} + \phi_{\lambda\mu}$ is a constant, say $k_{\lambda\mu\nu}$. For any four members $U_\kappa, U_\lambda, U_\mu, U_\nu$ of \mathfrak{U} whose intersections in threes are non-empty

$$k_{\lambda\mu\nu} - k_{\kappa\mu\nu} + k_{\kappa\lambda\nu} - k_{\kappa\lambda\mu} = 0.$$

That is to say, k is a 2-cocycle in the Čech cochain complex for the covering \mathfrak{U} with values in the constant sheaf $M \times \mathbb{R}$. Under the assumption that $H^2(M) = \check{H}^2(\mathfrak{U}, \mathcal{R}) = 0$, it must be a coboundary. Thus we can modify each $\phi_{\lambda\mu}$ by the addition of a constant, so that (after modification) $\phi_{\mu\nu} - \phi_{\lambda\nu} + \phi_{\lambda\mu} = 0$. There is thus a refinement $\mathfrak{V} = \{V_\alpha : \alpha \in A\}$ of \mathfrak{U} , and for each α a function ψ_α defined on V_α , such that on $V_\alpha \cap V_\beta$ (assuming it to be non-empty) $\phi_{\alpha\beta} = \psi_\alpha - \psi_\beta$. But then $\theta_\alpha - d\psi_\alpha = \theta_\beta - d\psi_\beta$ on $V_\alpha \cap V_\beta$. So if we set $\theta = \theta_\alpha - d\psi_\alpha$ on V_α , θ is a well-defined semi-basic 1-form on $T^\circ M$ such that $d\theta = \omega$. \square

We have shown that when $H^2(M) = 0$ there is a globally-defined semi-basic 1-form θ such that $d\theta = \omega$. In virtue of the other conditions on ω , this 1-form must satisfy the differential conditions of the theorem of Bucataru and Muzsnay.

6. Almost Grassmann structures

We now make a detour to discuss another approach to the construction of a 2-form indicating Finsler metrizable, which gives a new geometrical interpretation of the vertical subspaces, on the one hand, and the two-dimensional subspaces of the form $\langle v^H, v^V \rangle$, on the other, which play an important role in the conditions for the existence of a Finsler function discussed in the previous section. This approach necessitates the use of an almost Grassmann structure [1].

Formally, an almost Grassmann structure on a manifold N of dimension pq , $p \geq 2$, $q \geq 2$, may be regarded as a Cartan geometry modelled on the Grassmannian of p -dimensional subspaces of \mathbb{R}^{p+q} [14]. One way to define such a structure is by specifying a class of local bases of 1-forms $\{\theta_\alpha^i\}$, any two such local bases of the class being related by a formula $\hat{\theta}_\alpha^i = B_j^i A_\alpha^\beta \theta_\beta^j$ where (A_α^β) and (B_j^i) are local matrix-valued functions, respectively $p \times p$ and $q \times q$, both non-singular.

Given an almost Grassmann structure, we denote the local basis of vector fields dual to a local basis of 1-forms $\{\theta_\alpha^i\}$ in the structure by $\{E_i^\alpha\}$, so that any vector $v \in T_x N$ may be written as $v_\alpha^i E_i^\alpha(x)$. Of special interest are those vectors v for which the coefficient matrix (v_α^i) has rank 1; the set of such v forms a cone in $T_x N$ called the Segre cone. That is to say, the Segre cone at $x \in N$ consists of those elements of $T_x N$ that can be expressed in the form $s_\alpha t^i E_i^\alpha(x)$ with respect to one, and hence any, basis $\{E_i^\alpha\}$ defined by the structure, where $(s_\alpha) \in \mathbb{R}^p$ and $(t^i) \in \mathbb{R}^q$. For fixed non-zero (t^i) , as (s_α) varies over \mathbb{R}^p we obtain a p -dimensional subspace of $T_x N$ contained in the Segre cone; we call it a p -dimensional plane generator of the Segre cone. The p -dimensional plane generators of Segre cones are parametrized by the points of the projective space P^{q-1} . Similarly, on fixing non-zero (s_α) , as (t^i) varies over \mathbb{R}^q we obtain a q -dimensional plane generator of the Segre cone.

There is an almost Grassmann structure of type $(2, n)$ associated with each projective class of sprays on the $2n$ -dimensional manifold $T^\circ M$. This structure is not, however, defined on $T^\circ M$ itself, but on a related $2n$ -dimensional bundle $\mathcal{T}^\circ M \rightarrow M$ obtained from a vector bundle $\mathcal{T}M \rightarrow M$ by deleting the zero section.

We may construct $\mathcal{T}M$ using a technique described in [10]. We let $\mathcal{V}M$ be the manifold of equivalence classes $[\pm\theta]$ of non-zero volume elements $\theta \in \wedge^n T^*M$, and let $\nu : \mathcal{V}M \rightarrow M$ be given by $\nu([\pm\theta]) = x$ where $\theta, -\theta \in \wedge^n T_x^*M$. Given coordinates x^i on M , define the map ν by

$$\theta = \nu(\theta)(dx^1 \wedge \dots \wedge dx^n)_x$$

and let $x^0 = |\nu|^{1/(n+1)}$ be a fibre coordinate on ν . In this way $\nu : \mathcal{V}M \rightarrow M$ becomes a principal \mathbb{R}_+ bundle with fundamental vector field $\Upsilon = x^0 \partial/\partial x^0$. Now consider the tangent bundle $T\mathcal{V}M \rightarrow \mathcal{V}M$ and the vector fields

$$\Upsilon^V = x^0 \frac{\partial}{\partial y^0}, \quad \tilde{\Upsilon} = \Upsilon^C - \tilde{\Delta} = x^0 \frac{\partial}{\partial x^0} - y^i \frac{\partial}{\partial y^i}$$

where $\tilde{\Delta}$ is the dilation field on $T\mathcal{V}M$. The distribution spanned by these two vector fields is integrable, and the quotient is a manifold $\mathcal{T}M$ which does not project to $\mathcal{V}M$ but does define a vector bundle over M . The fibre coordinates (u^i) on the new bundle are defined in terms of the fibre coordinates (y^i) of $\mathcal{T}M$ by $u^i = x^0 y^i$; the quotient manifold may be thought of as the tensor product of the ordinary tangent bundle with the bundle of scalar densities of weight $1/(n+1)$.

The construction of the almost Grassmann structure may also be found in [10]. For any spray

$$y^i \frac{\partial}{\partial x^i} - 2\Gamma^i \frac{\partial}{\partial y^i}$$

on $T^\circ M$ there is a well-defined horizontal distribution on $\mathcal{T}^\circ M$, spanned locally by the vector fields

$$\mathcal{K}_i = \frac{\partial}{\partial x^i} - \left(\Gamma_i^j - \frac{1}{n+1} u^j \Gamma_i \right) \frac{\partial}{\partial u^j},$$

where u^i are the natural fibre coordinates on $\mathcal{T}^\circ M$ and

$$\Gamma_i^j = \frac{\partial \Gamma^j}{\partial y^i}, \quad \Gamma_i = \frac{\partial \Gamma_k^k}{\partial y^i}.$$

If two sprays are related by a projective transformation with function P , the vector fields are modified according to the rule

$$\mathcal{K}_i \mapsto \mathcal{K}_i - P \frac{\partial}{\partial u^i}.$$

We shall write, for $v \in T_x M$,

$$v^H = v^i \mathcal{K}_i, \quad v^V = v^i \frac{\partial}{\partial u^i}.$$

Now suppose given a projective class of sprays. Choose a particular spray in the class; from the remarks above we see that in a coordinate patch with coordinates (x^i, u^i) the 1-forms

$$\theta_1^i = dx^i, \quad \theta_2^i = du^i + \left(\Gamma_j^i - \frac{1}{n+1} u^i \Gamma_j^i \right) dx^j$$

transform as

$$\hat{\theta}_1^i = J_j^i \theta_1^j, \quad \hat{\theta}_2^i = |J|^{-1/(n+1)} J_j^i \theta_2^j$$

under a coordinate transformation, where J_j^i is the Jacobian matrix of the transformation on M and $|J|$ is its determinant, and as

$$\hat{\theta}_1^i = \theta_1^i, \quad \hat{\theta}_2^i = \theta_2^i + P\theta_1^i$$

under a projective transformation. It follows that the set of locally-defined 1-forms $\{A_j^i A_\alpha^\beta \theta_\beta^j\}$, with $\alpha, \beta = 1, 2$, $i, j = 1, 2, \dots, n$, with (A_j^i) , (A_α^β) arbitrary local non-singular-matrix-valued functions, of size $n \times n$ and 2×2 respectively, is defined independently of the choice of coordinates and of the choice of spray within the projective class. These 1-forms therefore determine an almost Grassmann structure of type $(2, n)$ on $\mathcal{T}^\circ M$.

The Segre cone at a point (x, u) of $\mathcal{T}^\circ M$ consists of vectors of the form $av^H + bv^V$ for $a, b \in \mathbb{R}$ and $v \in T_x M$. The n -dimensional plane generators of the Segre cone are obtained by fixing a and b and letting v range over $T_x M$; they consist of the horizontal subspace with respect to each spray of the projective class together with the vertical subspace. For the two-dimensional plane generators we fix v and allow the coefficients to vary over \mathbb{R}^2 . Notice that $\mathcal{D}_{(x,u)}$, where \mathcal{D} is the involutive two-dimensional distribution spanned by Δ and any spray of the class, is a two-dimensional plane generator of the Segre cone at (x, u) .

Each Finsler geometry on $T^\circ M$ determines a projective class of sprays, and therefore determines an almost Grassmann structure on $\mathcal{T}^\circ M$. We may determine a relationship between the two structures using the fact that if ω is a closed form on $T^\circ M$ satisfying $i_\Delta \omega = 0$ then its pull-back $(\nu_*)^* \omega$ by $\nu_* : T^\circ(\mathcal{V}M) \rightarrow T^\circ M$ is projectable to a form on $\mathcal{T}^\circ M$, and apply this to the Hilbert 2-form.

Theorem 8. (See [10].) *To each Finsler function F on $T^\circ M$ there is associated a closed 2-form ϖ on $\mathcal{T}^\circ M$, such that the characteristic distribution of ϖ is the two-dimensional distribution \mathcal{D} corresponding to the geodesic sprays of F , and such that the n -dimensional plane generators of the Segre cones are isotropic with respect to ϖ .*

Conversely, suppose given a projective class of sprays on $T^\circ M$ and corresponding almost Grassmann structure. If there is a 2-form ϖ on $\mathcal{T}^\circ M$ such that

1. *the n -dimensional plane generators of the Segre cones are isotropic with respect to ϖ ;*
2. *the characteristic distribution of ϖ is \mathcal{D} ;*
3. *ϖ is closed*

then the projective class is the geodesic class of a locally-defined pseudo-Finsler function.

Proof. In fact ϖ must take the form $h_{ij} dx^i \wedge \theta_2^j$ with h_{ij} the Hessian, or putative Hessian, of F , much as in Theorem 5: see [10, Theorems 4 and 6] for the details. \square

We can now further refine this result. Let α be a 2-covector on some vector space V of dimension at least two, and W a two-dimensional subspace of V . Then either $\alpha|_W \equiv 0$, or $\alpha(w_1, w_2) = 0$ for $w_1, w_2 \in W$ only if w_1 and w_2 are linearly dependent. In the former case we say that α vanishes on W .

Corollary 5. *If the 2-form ϖ vanishes on a 2-plane generator of Segre cones only if it is a generator determined by \mathcal{D} then either h or $-h$ is positive quasi-definite, and there is a local Finsler function whose geodesic class is the given projective class.*

Proof. At any $(x, u) \in \mathcal{T}^\circ M$ we have, for $v \in T_x M$, $\varpi(v^H, v^V) = h_{ij} v^i v^j$. This cannot vanish unless v is a scalar multiple of u . Thus, at each point of $\mathcal{T}^\circ M$, h is either positive or negative quasi-definite. By continuity either h or $-h$ must be positive quasi-definite everywhere. \square

7. Path space

We take up the argument from where we left it in Section 5.

We now assume that we can quotient out by the foliation on $T^\circ M$ defined by the involutive distribution \mathcal{D} , that is, that there is a $(2n - 2)$ -dimensional manifold $P_{\mathcal{D}}$ – path space – such that $\pi : T^\circ M \rightarrow P_{\mathcal{D}}$ is a fibration whose fibres are the leaves of the foliation. So we have a double fibration

$$M \xleftarrow{\tau} T^\circ M \xrightarrow{\pi} P_{\mathcal{D}}.$$

(It has to be admitted that in general there is no reason for the path space of a projective class of sprays to be a smooth manifold. For the geodesic class of sprays of a Riemannian metric, for example, two well-known cases where the path space can be given the structure of a smooth manifold are the cases where the Riemannian manifold is either a Hadamard manifold (i.e. a complete simply connected Riemannian manifold of non-positive curvature) or a manifold with closed geodesics of the same length. These two cases are discussed in detail in e.g. Ferrand [11] and Besse [4], respectively.)

For any $x \in M$, denote by \hat{x} the submanifold $\pi(T_x^\circ M)$ of $P_{\mathcal{D}}$, that is, the image under π of the fibre of $T^\circ M$ over x : it is the submanifold consisting of all paths through x . It is of dimension $n - 1$, because Δ is vertical.

The following theorem is our version of Theorem 4.1 of [2].

Theorem 9. *A projective class of sprays is the geodesic class of a pseudo-Finsler function if and only if there is a symplectic 2-form Ω on $P_{\mathcal{D}}$ such that \hat{x} is a Lagrangian submanifold of $P_{\mathcal{D}}$ with respect to Ω , for every $x \in M$.*

Proof. Suppose that the projective class of sprays is derivable from a pseudo-Finsler function. Let $\omega = h_{ij} dx^i \wedge \phi^j$ be the Hilbert 2-form on $T^\circ M$. It satisfies the conditions of Theorem 5, is closed, has \mathcal{D} for its characteristic distribution, and satisfies $\mathcal{L}_Z \omega = 0$ for every vector field Z in \mathcal{D} . It therefore passes to the quotient, that is, there is a 2-form Ω on $P_{\mathcal{D}}$ such that $\pi^* \Omega = \omega$. Then Ω is non-singular. Moreover $\pi^* d\Omega = 0$; but π is surjective, so $d\Omega = 0$. Thus Ω is symplectic. Let $p \in \hat{x}$ and $\xi, \eta \in T_p \hat{x}$. Then there is $y \in T_x^\circ M$, and $v, w \in T_y T_x^\circ M$ (i.e. vertical vectors at y) such that $p = \pi(y)$, $\xi = \pi_* v$, $\eta = \pi_* w$, and

$$\Omega_p(\xi, \eta) = \Omega_p(\pi_* v, \pi_* w) = \pi^* \Omega_p(v, w) = \omega_y(v, w) = 0.$$

Conversely, suppose that there is a 2-form Ω on $P_{\mathcal{D}}$ with the stated properties. Set $\omega = \pi^* \Omega$. Then $d\omega = 0$, and the characteristic distribution of ω is \mathcal{D} . Evidently $\mathcal{L}_\Gamma \omega = 0$ for any spray in the class. Let $x \in M$, $y \in T_x^\circ M$, and $v, w \in T_y T_x^\circ M$ (i.e. any vertical vectors at y). Then

$$\omega_y(v, w) = \pi^* \Omega_p(v, w) = \Omega_p(\pi_* v, \pi_* w) = 0$$

because $\pi_* v, \pi_* w \in T_p \hat{x}$. Now apply Corollary 3. \square

When $\dim M = 2$ the dimension of $P_{\mathcal{D}}$ is also 2, so in this case there is essentially no condition in Theorem 9, because every 2-form is closed, and \hat{x} is one-dimensional. That is, every volume form (nowhere vanishing 2-form) on $P_{\mathcal{D}}$ satisfies the conditions of the theorem. Moreover, we see that the freedom in choice of Hessians of pseudo-Finsler functions is the same as the freedom in choice of volume forms in 2 dimensions, that is, multiplication by a function on $P_{\mathcal{D}}$. These observations give another interpretation of the results on the two-dimensional case in [9].

8. Jacobi fields

Roughly speaking, a point in path space $P_{\mathcal{D}}$ represents a geodesic, and so a tangent vector to path space at a point in it is an ‘infinitesimal connecting vector to a nearby geodesic’, that is, a Jacobi field along the initial geodesic. This observation, when tidied up, gives another interpretation of the requirement that \hat{x} is a Lagrangian submanifold of $P_{\mathcal{D}}$ with respect to Ω .

In order to discuss Jacobi fields we have to fix the parametrization, that is, choose a specific spray Γ from the projective equivalence class. However, since the argument to be presented below leads merely to a reinterpretation of the conditions just mentioned, which we know from the previous section to be defined for the whole projective class, it clearly makes no difference which particular spray from the class we choose to work with.

Let $t \mapsto \gamma(t) \in M$ be a geodesic, that is, a base integral curve of Γ . Then $t \mapsto \bar{\gamma}(t) = (\gamma(t), \dot{\gamma}(t))$ is an integral curve of Γ in $T^\circ M$, and in coordinates

$$\ddot{\gamma}^i(t) + 2\Gamma^i(\gamma(t), \dot{\gamma}(t)) = 0.$$

Let Z be a vector field along $\bar{\gamma}$ such that $\mathcal{L}_\Gamma Z = 0$. We set $\zeta = \tau_* Z$, a vector field along the geodesic γ ; then the condition $\mathcal{L}_\Gamma Z = 0$ is equivalent to $Z = \zeta^H + (\nabla \zeta)^V$ where $\nabla^2 \zeta^i + R_j^i \zeta^j = 0$. That is to say, ζ is a Jacobi field along γ , and there is a 1–1 correspondence between Jacobi fields along γ and vector fields which are Lie transported along $\bar{\gamma}$. Evidently $\dot{\gamma}$ is a Jacobi field along γ , corresponding to the restriction of Γ to $\bar{\gamma}$. Moreover, $t \mapsto t\dot{\gamma}(t)$ is a Jacobi field along γ , corresponding to the restriction to $\bar{\gamma}$ of $t\Gamma + \Delta$. These Jacobi fields in the tangent direction of γ may be regarded as trivial. We denote by \mathfrak{J}_γ the space of Jacobi fields along γ . It is a $2n$ -dimensional real vector space. We denote by \mathfrak{J}_γ^0 the quotient of \mathfrak{J}_γ by the two-dimensional subspace consisting of the trivial Jacobi fields which lie in the direction tangent to γ .

There is a leaf of the involutive distribution \mathcal{D} containing $\bar{\gamma}$: call it \mathfrak{L}_γ . It consists of all points of $T^\circ M$ of the form $(\gamma(t), e^s \dot{\gamma}(t))$ for $(s, t) \in \mathbb{R}^2$ (assuming that γ is defined on \mathbb{R}). The leaf \mathfrak{L}_γ determines a point $p = \pi(\mathfrak{L}_\gamma) \in P_{\mathcal{D}}$. Now let Z be a vector field defined over \mathfrak{L}_γ (but not tangent to it; strictly speaking, Z is a vector field along the injection $\mathfrak{L}_\gamma \rightarrow T^\circ M$). The Lie derivative of such a vector field Z by any vector field in \mathcal{D} is well defined; and Z projects to an element of $T_p P_{\mathcal{D}}$

if and only if every such Lie derivative lies in $\mathcal{D}|_{\mathfrak{L}_\gamma}$. That is to say, for every vector field Z on \mathfrak{L}_γ such that $\mathcal{L}_\Delta Z \in \mathcal{D}$ and $\mathcal{L}_\Gamma Z \in \mathcal{D}$, $\pi_* Z$ is a well-defined element of $T_p P_{\mathcal{D}}$; and every element of $T_p P_{\mathcal{D}}$ is of this form for some such Z .

We shall show that there is an isomorphism of \mathfrak{J}_γ^0 with $T_p P_{\mathcal{D}}$. We know that any $\zeta \in \mathfrak{J}_\gamma$ lifts to a vector field Z along $\bar{\gamma}$ such that $\mathcal{L}_\Gamma Z = 0$. We shall first show that such a vector field Z can be extended to a vector field (also denoted by Z) on \mathfrak{L}_γ such that $\mathcal{L}_\Delta Z = \mathcal{L}_\Gamma Z = 0$.

Lemma 3. *Let $t \mapsto Z(t)$ be a vector field along $\bar{\gamma}$ such that $\mathcal{L}_\Gamma Z = 0$. Then there is a unique vector field $(s, t) \mapsto Z(s, t)$ on \mathfrak{L}_γ such that $\mathcal{L}_\Delta Z = \mathcal{L}_\Gamma Z = 0$ and $Z(t) = Z(0, t)$.*

Proof. Let δ_s be the 1-parameter group generated by Δ acting on \mathfrak{L}_γ , so that for any $(x, y) \in \mathfrak{L}_\gamma$, $\delta_s(x, y) = (x, e^s y)$. Let $Z(t)$ be any vector field along $\bar{\gamma}$ and set $Z(s, t) = \delta_{s*} Z(t)$. Then $\mathcal{L}_\Delta Z = 0$ and $Z(0, t) = Z(t)$; moreover $Z(s, t)$ is uniquely determined by these properties. Now $[\Delta, \Gamma] = \Gamma$, whence $\mathcal{L}_\Delta \mathcal{L}_\Gamma Z = \mathcal{L}_\Gamma \mathcal{L}_\Delta Z + \mathcal{L}_\Gamma Z = \mathcal{L}_\Gamma Z$. It follows that $(\mathcal{L}_\Gamma Z)(s, t) = e^s \delta_{s*} (\mathcal{L}_\Gamma Z)(0, t) = e^s \delta_{s*} (\mathcal{L}_\Gamma Z)(t)$. So if $(\mathcal{L}_\Gamma Z)(t) = 0$ then $(\mathcal{L}_\Gamma Z)(s, t) = 0$. \square

We define a linear map $j : \mathfrak{J}_\gamma \rightarrow T_p P_{\mathcal{D}}$ as follows. For $\zeta \in \mathfrak{J}_\gamma$ let $Z(t)$ be the corresponding vector field along $\bar{\gamma}$, and $Z(s, t)$ the vector field on \mathfrak{L}_γ whose existence is guaranteed by the lemma. Then $\pi_* Z$ is a well-defined element of $T_p P_{\mathcal{D}}$, and we set $\pi_* Z = j(\zeta)$. We shall show that the kernel of j is spanned by the trivial Jacobi fields $\dot{\gamma}$ and $t\dot{\gamma}$, whence $j : \mathfrak{J}_\gamma^0 \rightarrow T_p P_{\mathcal{D}}$ is an isomorphism by dimension.

Proposition 1. *The linear map $j : \mathfrak{J}_\gamma^0 \rightarrow T_p P_{\mathcal{D}}$ is an isomorphism.*

Proof. Let us denote by φ the map $\mathbb{R}^2 \rightarrow \mathfrak{L}_\gamma$ given by $\varphi(s, t) = (\gamma(t), e^s \dot{\gamma}(t))$. Then evidently

$$\varphi_* \left(\frac{\partial}{\partial s} \right) = \Delta_{\varphi(s,t)}.$$

Furthermore

$$\varphi_* \left(e^s \frac{\partial}{\partial t} \right) = e^s \left(\dot{\gamma}^i(t) \frac{\partial}{\partial x^i} + e^s \ddot{\gamma}(t)^i \frac{\partial}{\partial y^i} \right).$$

But $\ddot{\gamma}(t)^i = -2\Gamma^i(\gamma(t), \dot{\gamma}(t))$, and Γ^i is positively homogeneous of degree 2 in the fibre coordinates; thus

$$\begin{aligned} \varphi_* \left(e^s \frac{\partial}{\partial t} \right) &= e^s \dot{\gamma}^i(t) \frac{\partial}{\partial x^i} - 2e^{2s} \Gamma^i(\gamma(t), \dot{\gamma}(t)) \frac{\partial}{\partial y^i} \\ &= e^s \dot{\gamma}^i(t) \frac{\partial}{\partial x^i} - 2\Gamma^i(\gamma(t), e^s \dot{\gamma}(t)) \frac{\partial}{\partial y^i} \\ &= \Gamma_{\varphi(s,t)}. \end{aligned}$$

Thus we can use s and t as coordinates on \mathfrak{L}_γ , with

$$\Delta = \frac{\partial}{\partial s}, \quad \Gamma = e^s \frac{\partial}{\partial t}.$$

A vector field on \mathfrak{L}_γ which projects onto $0 \in T_p P_{\mathcal{D}}$ takes the form

$$Z(s, t) = \sigma(s, t) \frac{\partial}{\partial s} + \tau(s, t) \frac{\partial}{\partial t}.$$

Then $\mathcal{L}_\Delta Z = 0$ if and only if σ and τ are independent of s . Furthermore,

$$\mathcal{L}_\Gamma Z = e^s \left(\frac{\partial \sigma}{\partial t} \frac{\partial}{\partial s} + \left(\frac{\partial \tau}{\partial t} - \sigma \right) \frac{\partial}{\partial t} \right),$$

so that $\mathcal{L}_\Gamma Z = 0$ if and only if $\sigma = a$ is constant and $\tau(s, t) = at + b$ where b is constant. Then

$$Z(s, t) = a \Delta_{\varphi(s,t)} + (at + b) e^{-s} \Gamma_{\varphi(s,t)},$$

and in particular

$$Z(0, t) = a(t\Gamma_{\bar{\gamma}(t)} + \Delta_{\bar{\gamma}(t)}) + b\Gamma_{\bar{\gamma}(t)}.$$

That is to say, Z corresponds to a linear combination of trivial Jacobi fields, and so the kernel of j is spanned by the trivial Jacobi fields. \square

Now let γ be a geodesic of Γ through $x \in M$, with $\gamma(0) = x$ for convenience. Denote by $\mathfrak{J}_{\gamma,0}$ the space of Jacobi fields along γ which vanish at x , and $\mathfrak{J}_{\gamma,0}^0$ the quotient of $\mathfrak{J}_{\gamma,0}$ by the constant multiples of $t\dot{\gamma}(t)$. Then j maps $\mathfrak{J}_{\gamma,0}^0$ onto $T_p\hat{x}$, and is an isomorphism.

Let ω be a 2-form on $T^\circ M$ such that $\mathcal{L}_\Gamma \omega = 0$ (Theorem 5, assumption 2). Let ζ_1 and ζ_2 be Jacobi fields along γ , and Z_1 and Z_2 the corresponding vector fields on \mathfrak{L}_γ as given in Lemma 3. Then since $\mathcal{L}_\Gamma Z_1 = \mathcal{L}_\Gamma Z_2 = 0$, $\Gamma(\omega(Z_1, Z_2)) = 0$, that is to say, $\omega(Z_1, Z_2)$ is constant along every integral curve of Γ in \mathfrak{L}_γ . Suppose further that for $x \in M$, $\omega|_{T_x^\circ M} = 0$ (Theorem 5, assumption 3). If now $\zeta_1(0) = \zeta_2(0) = 0$, so that $Z_1(0)$ and $Z_2(0)$ are vertical, then $\omega(Z_1, Z_2) = 0$ on the ray $s \mapsto e^s \dot{\gamma}(0)^V$ in \mathfrak{L}_γ . But this is transversal to Γ in \mathfrak{L}_γ , so $\omega(Z_1, Z_2) = 0$ on \mathfrak{L}_γ .

It is this property of ω which corresponds to the property that the submanifolds \hat{x} are Lagrangian in Theorem 9 (when ω satisfies the conditions of Theorem 5 and Corollary 2): note that $\pi(\mathfrak{L}_\gamma) = p \in \hat{x}$, and we have in effect shown that Ω_p (where Ω is the projection of ω) vanishes on any pair of vectors in $T_p\hat{x}$.

We consider next the positive quasi-definiteness condition. Take $x \in M$ and $y \in T_x^\circ M$, and let γ be the geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = y$. Let $v \in T_x M$: there are unique Jacobi fields $\zeta_1(t)$, $\zeta_2(t)$ along γ such that

$$\zeta_1(0) = v, \quad \nabla \zeta_1(0) = 0; \quad \zeta_2(0) = 0, \quad \nabla \zeta_2(0) = v.$$

Let Z_1, Z_2 be the corresponding vector fields along $\bar{\gamma}$ such that $\mathcal{L}_\Gamma Z_1 = \mathcal{L}_\Gamma Z_2 = 0$. Then the quadratic form $q_{(x,y)}$ on $T_x M$ defined in Section 5 is given by

$$q_{(x,y)}(v) = \omega_{(x,y)}(v^H, v^V) = \omega_{(x,y)}(Z_1, Z_2) = \Omega_p(\zeta_1, \zeta_2)$$

where $p = \pi(x, y)$ and $\zeta_1, \zeta_2 \in T_p P_{\mathcal{D}}$ are the elements determined by the Jacobi fields $\zeta_1(t)$, $\zeta_2(t)$ by means of j . We have the following version of Corollary 4.

Corollary 6. *A projective class of sprays is the geodesic class of a local Finsler function if and only if there is a symplectic 2-form Ω on $P_{\mathcal{D}}$ such that \hat{x} is a Lagrangian submanifold of $P_{\mathcal{D}}$ with respect to Ω , for every $x \in M$, and moreover $\Omega_p(\zeta_1, \zeta_2) > 0$ for all non-zero $\zeta_1, \zeta_2 \in T_p P_{\mathcal{D}}$ of the special form described above.*

9. Totally-geodesic submanifolds

In Álvarez Paiva’s analysis of Hilbert’s fourth problem ([2]; see also [8]) 2-planes in \mathbb{R}^n play an important role: see for example Theorem 4.5 of [2], which relates the positivity properties of an admissible 2-form ω (our Ω) to the pull-back of ω to each two-dimensional submanifold of path space consisting of all lines in a 2-plane. This can be generalized to the kind of situation discussed here if we change 2-planes to two-dimensional totally-geodesic submanifolds, as we now explain.

Let N be a proper embedded submanifold of M . We define a submanifold \hat{N} of $T^\circ M$, of twice the dimension, as follows: $\hat{N} = \{(x, y) \in T^\circ M : y \in T_x N\}$. Thus \hat{N} is $T^\circ N$ considered as a submanifold of $T^\circ M$. Evidently Δ is tangent to \hat{N} (if $y \in T_x N$ then also $e^t y \in T_x N$). Moreover, if v is any vector tangent to N then v^V is tangent to \hat{N} , since v_y^V is the tangent at $t = 0$ to the curve $t \mapsto y + tv$, and if $y \in T_x N$ and $v \in T_x N$ then $y + tv \in T_x N$ for all t .

We say that the submanifold N is *totally geodesic* with respect to the spray Γ if Γ is tangent to \hat{N} . Then every geodesic γ of Γ in M which starts at a point $x = \gamma(0)$ of N and is tangent to N there (so that $\dot{\gamma}(0) \in T_x N$) lies totally within N : it is the projection of the integral curve of Γ through $(\gamma(0), \dot{\gamma}(0))$, which lies in \hat{N} . Note that since Δ is tangent to \hat{N} , if Γ is tangent to \hat{N} so is any projectively-equivalent spray: that is, being totally geodesic is a projective property (as it should be, since it should be concerned with geodesic paths rather than parametrized geodesics).

Lemma 4. *If N is totally geodesic then if v is any vector tangent to N , v^H is tangent to \hat{N} .*

Proof. We can find coordinates on M such that N is given by $x^\alpha = 0$, $\alpha = \dim N + 1, \dots, n$. We use a, b for indices $1, \dots, \dim N$. Clearly \hat{N} is given by $x^\alpha = 0$, $y^\alpha = 0$. With

$$\Gamma = y^i \frac{\partial}{\partial x^i} - 2\Gamma^i \frac{\partial}{\partial y^i},$$

N is totally geodesic if and only if $\Gamma^\alpha(x^a, 0, y^a, 0) = 0$. Now

$$H_a = \frac{\partial}{\partial x^a} - \Gamma_a^i \frac{\partial}{\partial y^i}.$$

But on \hat{N}

$$\Gamma_a^\alpha(x^b, 0, y^b, 0) = \frac{\partial \Gamma^\alpha}{\partial y^a}(x^b, 0, y^b, 0) = 0.$$

Thus on \hat{N}

$$H_a = \frac{\partial}{\partial x^a} - \Gamma_a^b \frac{\partial}{\partial y^b},$$

which is tangent to \hat{N} . \square

For convenience, when speaking of vector fields in relation to a submanifold we shall use ‘on’ to mean not just ‘defined on’ but also ‘tangent to’.

When N is totally geodesic the space of vector fields on \hat{N} is spanned by the vector fields X^V, X^H , where X is any vector field on N ; notice that X^V and X^H coincide with Δ and Γ where $y = X(x)$, that is, on the image of the corresponding section. Now

$$[\Gamma, X^V] = -X^H + (\nabla X)^V, \quad [\Gamma, X^H] = (\nabla X)^H + \Phi(X)^V.$$

For a totally-geodesic submanifold these formulas make sense on \hat{N} with X any vector field on N . Then since Γ, X^V and X^H are all tangent to \hat{N} , so is $(\nabla X)^V$, and so is $\Phi(X)^V$. Of course $[\Delta, X^V] = -X^V$ and $[\Delta, X^H] = 0$.

We now consider two-dimensional totally-geodesic submanifolds. Any such submanifold N defines a two-dimensional submanifold \bar{N} of $P_{\mathcal{D}}$, whose points consist of geodesic paths in N ; alternatively, $\bar{N} = \pi(\hat{N})$ where π is the projection $T^\circ M \rightarrow P_{\mathcal{D}}$.

Suppose we have a closed 2-form ω on $T^\circ M$ whose characteristic distribution is spanned by Γ and Δ , as in Theorem 5. Then $\omega = h_{ij} dx^i \wedge \phi^j$. As we know, to determine whether h is positive quasi-definite at any $(x, y) \in T^\circ M$ we must consider $\omega_{(x,y)}(v^H, v^V)$ as v ranges over $T_x M$. Moreover, ω determines a symplectic form Ω on $P_{\mathcal{D}}$, according to Theorem 9.

Proposition 2. *Let N be a two-dimensional totally-geodesic submanifold of M , $\bar{N} = \pi(\hat{N})$. Then for any $p \in \bar{N}$ and any $\xi, \eta \in T_p \bar{N}$, $\Omega_p(\xi, \eta) = \pm \omega_{(x,y)}(v^H, v^V)$ where $(x, y) \in \hat{N}$ with $\pi(x, y) = p$, for some $v \in T_x N$.*

Proof. Let γ be a geodesic of Γ in N whose path projects to p ; set $x = \gamma(0) \in N$, $y = \dot{\gamma}(0) \in T_x N$. Now let $\nu(t)$ be a vector field along γ everywhere tangent to N and independent of $\dot{\gamma}$; then $\{\nu(t), \dot{\gamma}(t)\}$ is a basis of $T_{\gamma(t)} N$. Let $\xi, \eta \in T_p P_{\mathcal{D}}$: then there are Jacobi fields $\xi(t), \eta(t)$ along γ corresponding to ξ and η ; furthermore, the vector fields $X = \xi^H + (\nabla \xi)^V$, $Y = \eta^H + (\nabla \eta)^V$ satisfy $\mathcal{L}_\Gamma X = \mathcal{L}_\Gamma Y = 0$; and $\omega_{(\gamma(t), \dot{\gamma}(t))}(X(t), Y(t))$ is constant and equal to $\Omega_p(\xi, \eta)$. Since N is totally geodesic, if $\xi, \eta \in T_p \bar{N}$ then $\xi(t), \eta(t) \in T_{\gamma(t)} N$. We can express $\xi(t)$ and $\eta(t)$ in terms of the basis $\{\nu(t), \dot{\gamma}(t)\}$: say $\xi(t) = a(t)\nu(t) \pmod{\dot{\gamma}(t)}$, $\eta(t) = b(t)\nu(t) \pmod{\dot{\gamma}(t)}$. Then

$$\nabla \xi = \dot{a}\nu + a\nabla \nu, \quad \nabla \eta = \dot{b}\nu + b\nabla \nu \pmod{\dot{\gamma}(t)},$$

and so

$$\begin{aligned} \Omega_p(\xi, \eta) &= \omega(X, Y) \\ &= \omega(\xi^H + (\nabla \xi)^V, \eta^H + (\nabla \eta)^V) \\ &= \omega(\xi^H, (\nabla \eta)^V) - \omega(\eta^H, (\nabla \xi)^V) \\ &= \omega(a\nu^H, \dot{b}\nu^V + b(\nabla \nu)^V) - \omega(b\nu^H, \dot{a}\nu^V + a(\nabla \nu)^V) \\ &= (a\dot{b} - b\dot{a})\omega(\nu^H, \nu^V). \end{aligned}$$

But $\omega(X, Y)$ is constant along γ , so in the end

$$\Omega_p(\xi, \eta) = (a\dot{b} - b\dot{a})(0)\omega_{(x,y)}(\nu(0)^H, \nu(0)^V).$$

Now if $a(0)\dot{b}(0) - b(0)\dot{a}(0) = 0$ then $\xi(t)$ and $\eta(t)$ are linearly dependent, and so $\Omega_p(\xi, \eta) = 0$. Otherwise, one can scale ν to eliminate the overall scalar factor: that is, set

$$\nu = \frac{1}{\sqrt{|a(0)\dot{b}(0) - b(0)\dot{a}(0)|}} \nu(0). \quad \square$$

Now h is everywhere positive quasi-definite if and only if for every $(x, y) \in T^\circ M$ and for every $v \in T_x M$, $\omega_{(x,y)}(v^H, v^V) \geq 0$, and $\omega_{(x,y)}(v^H, v^V) = 0$ if and only if v is a scalar multiple of y . Suppose that Γ has the property that for every $x \in M$ and every two-dimensional subspace of $T_x M$ there is a totally-geodesic submanifold N through x with the given subspace as its tangent space. Then for every point $p \in P_{\mathcal{D}}$ and every two-dimensional subspace of $T_p P_{\mathcal{D}}$ there is a two-dimensional submanifold \bar{N} through p with the given subspace as its tangent subspace. We conclude therefore:

Proposition 3. Let Γ be such that for every $x \in M$ and every two-dimensional subspace of $T_x M$ there is a totally-geodesic submanifold through x with the given subspace as its tangent space. If h is everywhere positive quasi-definite, the pull-back of Ω to any submanifold N as above is non-vanishing (i.e. it is a volume form). Conversely, if Ω has this property then either h or $-h$ is everywhere positive quasi-definite.

The question arises, are there any spray spaces with this property – other than those covered by Hilbert’s fourth problem, namely those for which the paths are straight lines? For a two-dimensional totally-geodesic submanifold N , with $x \in N$ and $y, v \in T_x N$, $\Phi_y(v) \in T_x N$ also: that is, $\Phi_y(v)$ is a linear combination of y and v (if v is a multiple of y then $\Phi_y(v) = 0$). If this holds for all y and all $v \in T_{\tau(y)} M$ then the space must be isotropic: $\Phi_j^i = \lambda \delta_j^i + \mu_j y^i (= R_j^i)$. We don’t know, however, whether this is sufficient as well as necessary. But it is well known that every isotropic space is projectively metrizable, see e.g. [5,7].

10. Example

The following example, which is an extension of Shen’s circle example from [15], was introduced in [9]. Consider the projective class of the spray

$$\Gamma = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} + \sqrt{u^2 + v^2 + w^2} \left(-v \frac{\partial}{\partial u} + u \frac{\partial}{\partial v} \right)$$

defined on $T^{\circ} \mathbb{R}^3$. As we showed in [9], the geodesics of Γ are spirals with axis parallel to the z -axis, together with straight lines parallel to the z -axis and circles in the planes $z = \text{constant}$. Evidently both $\sqrt{u^2 + v^2} = \mu$ and w are constant; and therefore (or directly) $\sqrt{u^2 + v^2 + w^2} = \lambda$ is also constant. The geodesics are the solutions of $\ddot{x} = -\lambda \dot{y}$, $\ddot{y} = \lambda \dot{x}$, $\ddot{z} = 0$, which are

$$x(t) = \xi + r \cos(\lambda t + \vartheta), \quad y(t) = \eta + r \sin(\lambda t + \vartheta), \quad z(t) = wt + z_0,$$

where ξ, η, r, ϑ are constants, with $w^2 = \lambda^2(1 - r^2)$. The initial point on the geodesic (the point where $t = 0$) is (x_0, y_0, z_0) where $x_0 = \xi + r \cos \vartheta$, $y_0 = \eta + r \sin \vartheta$. The projections of the geodesics on the xy -plane are circles of centre (ξ, η) and radius $r = \mu/\lambda$: note that $0 \leq r \leq 1$, the circle degenerating to a point when $r = 0$. For $w/\lambda \neq 0, \pm 1$ the geodesics are spirals, with axis the line parallel to the z -axis through $(\xi, \eta, 0)$. The case $r = 0$ corresponds to $w/\lambda = \pm 1$ and the geodesics are straight lines parallel to the z -axis (in both directions). The case $r = 1$ ($w = 0$) gives circles of unit radius in the planes $z = z_0$.

Consider the genuine spirals, that is, take $r \neq 0$ and $w \neq 0$. Note first that the circle which is the spiral’s projection on the xy -plane is always traversed anticlockwise, though $z(t)$ may increase or decrease with increasing t , depending on the sign of w . Next, we may fix the origin of t so that $z_0 = 0$: then ϑ determines the point on the circle in the xy -plane where $t = 0$. Let us (in general) set $w/\lambda = v$: then v is constant with $-1 \leq v \leq 1$, and also is homogeneous of degree 0 as a function on $T^{\circ} \mathbb{R}^3$. We can eliminate t , to express the spiral paths ($v \neq 0$) as

$$x = \xi + \sqrt{1 - v^2} \cos(z/v + \vartheta), \quad y = \eta + \sqrt{1 - v^2} \sin(z/v + \vartheta).$$

Then $(\xi, \eta, v, \vartheta)$ smoothly parametrize the set of genuine spirals. (However, it is not possible to parametrize smoothly the full set of paths.) We have a map $(x, y, z, u, v, w) \mapsto (\xi, \eta, v, \vartheta)$ where

$$\xi = x - v/\lambda, \quad \eta = y + u/\lambda, \quad v = w/\lambda, \quad \vartheta = \arccos(v/\mu) - \lambda z/w.$$

Then

$$d\xi = dx - d(v/\lambda), \quad d\eta = dy + d(u/\lambda), \quad dv = d(w/\lambda).$$

It is simplest to compute $d\vartheta$ from the implicit definition, which can be written

$$\cos(z/v + \vartheta) = v/\mu, \quad \sin(z/v + \vartheta) = -u/\mu,$$

from which it follows that

$$d\vartheta = -d(z/v) - \mu^{-2}(v du - u dv) = -d(z/v) - (\lambda/\mu^2)(v d(u/\lambda) - u d(v/\lambda)).$$

The 1-forms $d\xi, d\eta, dv$ and $d\vartheta$ are evidently independent.

Consider the 2-form $\Omega = d\xi \wedge d\eta + v dv \wedge d\vartheta$. It is a symplectic form. The spiral paths through (x, y, z) map to the points $(\xi, \eta, v, \vartheta)$ for which

$$\xi = x - \sqrt{1 - v^2} \cos(z/v + \vartheta), \quad \eta = y - \sqrt{1 - v^2} \sin(z/v + \vartheta),$$

where we treat x , y and z as constants. On the 2-manifold so defined we have

$$d\xi = \frac{v}{\sqrt{1-v^2}} \cos(z/v + \vartheta) dv + \sqrt{1-v^2} \sin(z/v + \vartheta) (-(z/v^2) dv + d\vartheta),$$

$$d\eta = \frac{v}{\sqrt{1-v^2}} \sin(z/v + \vartheta) dv - \sqrt{1-v^2} \cos(z/v + \vartheta) (-(z/v^2) dv + d\vartheta),$$

whence

$$d\xi \wedge d\eta = -v dv \wedge d\vartheta.$$

That is, every such 2-manifold is Lagrangian for Ω .

We next compute the pull-back ω of Ω to $T^\circ\mathbb{R}^3$. We do so by using the formulas above for $d\xi$, $d\eta$ etc., but no longer treat x , y and z as constants. We have

$$d\xi \wedge d\eta = (dx - d(v/\lambda)) \wedge (dy + d(u/\lambda))$$

$$= dx \wedge dy + dx \wedge d(u/\lambda) + dy \wedge d(v/\lambda) + d(u/\lambda) \wedge d(v/\lambda).$$

On the other hand,

$$v dv \wedge d\vartheta = dz \wedge d(w/\lambda) - \mu^{-2} d(w/\lambda) \wedge ((vw) d(u/\lambda) - (uw) d(v/\lambda)).$$

But since $(u/\lambda)^2 + (v/\lambda)^2 + (w/\lambda)^2 = 1$

$$u d(u/\lambda) + v d(v/\lambda) + w d(w/\lambda) = 0,$$

whence

$$w d(w/\lambda) \wedge d(u/\lambda) = v d(u/\lambda) \wedge d(v/\lambda),$$

$$w d(w/\lambda) \wedge d(v/\lambda) = -u d(u/\lambda) \wedge d(v/\lambda),$$

and therefore

$$d(w/\lambda) \wedge ((vw) d(u/\lambda) - (uw) d(v/\lambda)) = (u^2 + v^2) d(u/\lambda) \wedge d(v/\lambda).$$

Finally, we have

$$\omega = dx \wedge dy + dx \wedge d(u/\lambda) + dy \wedge d(v/\lambda) + dz \wedge d(w/\lambda).$$

The 2-form ω satisfies the conditions of [Theorem 5](#) and [Corollary 3](#). Thus Γ should admit a pseudo-Finsler function. In fact Γ comes from the pseudo-Finsler function

$$F(x, y, z, u, v, w) = \sqrt{u^2 + v^2 + w^2} + \frac{1}{2}yu - \frac{1}{2}xv.$$

This is globally well defined but only locally a Finsler function. A straightforward calculation confirms that its Hilbert 2-form is ω (up to sign).

The function F is a Finsler function, that is, is positive, only for $x^2 + y^2 < 4$. It is globally pseudo-Finsler. To obtain a Finsler function in a neighbourhood of an arbitrary point (x_0, y_0, z_0) we can make a simple modification to

$$\tilde{F}(x, y, z, u, v, w) = \sqrt{u^2 + v^2 + w^2} + \frac{1}{2}(y - y_0)u - \frac{1}{2}(x - x_0)v;$$

this is positive for $(x - x_0)^2 + (y - y_0)^2 < 4$. Note that it differs from F by a total derivative.

The planes $z = \text{constant}$ are totally-geodesic submanifolds. Indeed, if we denote by N any such plane then $w = 0$ on the corresponding submanifold \hat{N} of $T^\circ\mathbb{R}^3$, and the restriction of Γ to \hat{N} is the spray

$$u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} - v \sqrt{u^2 + v^2} \frac{\partial}{\partial u} + u \sqrt{u^2 + v^2} \frac{\partial}{\partial v}$$

of Shen's circle example. We consider this as a spray defined on $T^\circ\mathbb{R}^2$. It has for its geodesics all circles in \mathbb{R}^2 of radius 1, traversed counter-clockwise. The path space is smoothly parametrized by the coordinates (ξ, η) of the circles' centres.

Again, this spray is locally projectively metrizable. One local Finsler function is the restriction of the one given for the spiral example, namely

$$F(x, y, u, v) = \sqrt{u^2 + v^2} + \frac{1}{2}yu - \frac{1}{2}xv.$$

A straightforward calculation leads to its Hilbert 2-form:

$$d\theta = -dx \wedge dy + \frac{1}{\mu^3}(v du - u dv) \wedge (v dx - u dy).$$

But this is just $-d\xi \wedge d\eta$. So in this case there is a globally-defined path space equipped with a global symplectic form. The Hilbert 2-form passes to the path space and coincides with this symplectic form there. Moreover, it does so globally, despite the fact that F is only locally defined as a Finsler function (though again it is global as a pseudo-Finsler function).

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