7.1 (i) $\int \frac{1}{\sqrt[3]{x}}\,dx = \int x^{-\frac{1}{3}}\,dx =$

$$\left| \int x^a\,dx = \frac{1}{a+1}x^{a+1} \right|$$

$$= \frac{1}{-\frac{1}{3}+1}x^{-\frac{1}{3}+1} + C = \frac{3}{2}x^{\frac{2}{3}} + \underset{\text{lit. konst.}}{\underline{C}}$$

(ii) $\int tg^2 x\,dx =$

$$= \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx$$

$$= \int\left(\frac{1}{\cos^2 x} - 1\right)dx =$$

$$= tg\,x - x + C$$

(iii) $\int \left(6\sin 5x + \cos\frac{x}{2} + 2e^{\frac{2x}{3}}\right)dx$

$$= -6\cdot\frac{1}{5}\cos 5x + 2\sin\frac{x}{2} +$$

$$+ 2\cdot\frac{3}{2}\cdot e^{\frac{2x}{3}} + C$$

(iv) $\int \frac{1}{\sqrt{4-x^2}}\,dx$

$$\left| \int \frac{1}{\sqrt{1-x^2}}\,dx = \right.$$
$$\left. = \arcsin x \right|$$

$$= \int \frac{1}{\sqrt{4\left(1-\left(\frac{x}{2}\right)^2\right)}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx =$$

$$= \frac{1}{2}\cdot 2\cdot \arcsin\left(\frac{x}{2}\right) + C = \arcsin\frac{x}{2} + C$$

(iii) $\int \frac{1}{x^2+3} =$

$$\left| \int \frac{1}{x^2+1} = \operatorname{arctg} x \right|$$

$$= \int \frac{1}{3\left(\left(\frac{x}{\sqrt{3}}\right)^2+1\right)}\,dx =$$

$$= \frac{1}{3}\int \frac{1}{\left(\frac{x}{\sqrt{3}}\right)^2+1}\,dx$$

$$= \frac{1}{3}\cdot\sqrt{3}\cdot \operatorname{arctg}\frac{x}{\sqrt{3}} + C$$

Pr 7.2

(i) $\int x \sin x \, dx$

Integração por partes

$\int (uv)' = uv$ $\qquad u = u(x)$, $\quad v = v(x)$

$\int (u'v + uv') = uv$

$$\boxed{\int u'v = uv - \int uv'}$$

$$\int x \sin x = \left| \begin{array}{ll} u' = \sin x & u = -\cos x \\ v = x & v' = 1 \end{array} \right|$$

$= -x\cos x - \int (-\cos x)\, dx$

$= -x\cos x + \int \cos x \, dx$

$= -x\cos x + \sin x + C$

(ii) $\int (x^2+1)e^{-x}\,dx =$

$$= \left| \begin{array}{ll} u' = e^{-x} & u = -e^{-x} \\ v = x^2+1 & v' = 2x \end{array} \right| =$$

$$= -(x^2+1)e^{-x} - \int(-2x\,e^{-x})\,dx$$

$$= -(x^2+1)e^{-x} + 2\int x\,e^{-x}\,dx$$

$$\left| \begin{array}{ll} u' = e^{-x} & u = -e^{-x} \\ v = x & v = 1 \end{array} \right|$$

$$= -(x^2+1)e^{-x} + 2\left[-x\,e^{-x} - \int(-e^{-x})\,dx\right]$$

$$= -(x^2+2x+1)e^{-x} + 2\int e^{-x}\,dx$$

$$= -(x^2+2x+1)e^{-x} - 2e^{-x}\,dx + C$$

$$= -(x^2+2x+3)e^{-x} + C$$

(iii) $\int \operatorname{arctg} x \, dx =$

$= \int 1 \cdot \operatorname{arctg} x \, dx =$

$\left| \begin{array}{ll} u' = 1 & u = x \\ v = \operatorname{arctg} x & v' = \frac{1}{1+x^2} \end{array} \right|$

$= x \operatorname{arctg} x - \int \frac{x}{1+x^2} dx$

$= x \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$

$\left| \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| \right|$

$= x \operatorname{arctg} x - \ln(1+x^2) + C$

(iv) $\int e^x \sin x \, dx =$

$\left| \begin{array}{ll} v = e^x & v' = e^x \\ u' = \sin x & u = -\cos x \end{array} \right|$

$$= -\cos x\, e^x - \int e^x(-\cos x)\,dx$$

$$= -\cos x\, e^x + \int e^x \cos x\, dx$$

$$= \left| \begin{matrix} v = e^x & v' = e^x \\ u' = \cos x & u = \sin x \end{matrix} \right|$$

$$= -\cos x\, e^x + \left[e^x \sin x - \underline{\int e^x \sin x} \right]$$

$$= \int e^x \sin x\, dx + C$$

$$\Rightarrow 2\int e^x \sin x\, dx = -\cos e^x + \sin e^x + C$$

$$\int e^x \sin x\, dx = \frac{1}{2} e^x (\sin x - \cos x) + \frac{C}{2}$$

Pr 7.3 (i) $\int \cos^5 x \underbrace{\sin x \, dx}_{-du}$

$\int f(x)\, dx$, $y = y(x)$ substitute

$dy = ???\, dx$

$u = \cos x$

$du = -\sin x \, dx$

$= \int u^5 (-1)\, du =$

$= -\int u^5 \, du = -\frac{1}{6} u^6 + C$

$= -\frac{1}{6} \cos^6 x + C$

(ii) $\int \cos^5 x \sin^7 x \, dx =$

$u = \sin x$

$du = \cos x \, dx$

$$= \int \cos^4 x \cdot \sin^2 x \cdot \underbrace{\cos x \, dx}_{du}$$

$$= \int (1 - \sin^2 x)^2 \sin^2 x \underbrace{\cos x \, dx}_{du}$$

$$= \int (1 - u^2)^2 u^2 \, du$$

$$= \int (1 - 2u^2 + u^4) u^2 \, du$$

$$= \int (u^6 - 2u^4 + u^2) \, du$$

$$= \frac{1}{7} u^7 - \frac{2}{5} u^5 + \frac{1}{3} u^3 + C$$

$$= \frac{1}{7} \sin^7 x - \frac{2}{5} \sin^5 x + \frac{1}{3} \sin^3 x + C$$

(iii) $\displaystyle\int \frac{\sin^4 x}{\cos^4 x}\,dx$

$\left| \begin{array}{l} u = \operatorname{tg} x \\ du = \frac{1}{\cos^2 x}\,dx \end{array} \right|$

$\cos^2 x = \frac{1}{\operatorname{tg}^2 x + 1}$

$\sin^2 x = \ldots\ldots$

$= \cancel{\displaystyle\int \frac{\sin^4 x}{\cos^2 x} \frac{1}{\cos^2 x}\,dx}$

$= \cancel{\displaystyle\int \sin^2 x \quad \operatorname{tg}^2 x}$

$\left| \begin{array}{l} du = \frac{\sin^2 x + \cos^2 x}{\cos^2 x}\,dx \\ du = (\operatorname{tg}^2 x + 1)\,dx \\ du = (u^2+1)\,dx \end{array} \right|$

$= \displaystyle\int u^4 \cdot \frac{1}{u^2+1}\,du$

$= \displaystyle\int \frac{u^4}{u^2+1}\,du = \left| \begin{array}{l} \text{dělíme se} \\ \text{zbytkem} \\ u^4 : (u^2+1) \ldots\ldots \end{array} \right|$

$\left| \begin{array}{l} \dfrac{u^4}{u^2+1} = \dfrac{u^2(u^2+1) - u^2}{u^2+1} = \\ = u^2 - \dfrac{u^2+1-1}{u^2+1} = u^2 - 1 + \dfrac{1}{u^2+1} \end{array} \right|$

$$= \int \left(u^2 - 1 + \frac{1}{u^2+1}\right) du =$$

$$= \frac{1}{3}u^3 - u + \operatorname{arctg} u + C$$

$$= \frac{1}{3}\operatorname{tg}^3 x - \operatorname{tg} x + x + C$$

Pozn: najatecnija substitucija je $u = \operatorname{tg}\frac{x}{2}$

(iv) $$\int \frac{(7+\ln x)^7}{x}\,dx =$$

$$\left| \begin{array}{l} u = 7 + \ln x \\ du = \frac{1}{x}dx \end{array} \right|$$

$$= \int u^7\,du = \frac{1}{8}u^8 + C$$

$$= \frac{1}{8}(7+\ln x)^8 + C$$

Pozn.

$$\int \ln x \, dx = ?$$

$$\| $$

$$\int 1 \cdot \ln x \, dx = \dots$$

6.51: $\displaystyle\int \frac{x}{(x-1)^2(x^2+2x+2)}\,dx$

parciální zlomky

$$\frac{x}{(x-1)^2\underbrace{(x^2+2x+2)}_{\text{nejde rozložit nad } \mathbb{R}}} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2x+2} \quad / (x-1)^2(x^2+2x+2)$$

$$x = A(x-1)(x^2+2x+2) + B(x^2+2x+2) + (Cx+D)(x^2-2x+1)$$

$$x = A\left[x^3+x^2-2\right] + B\left[x^2+2x+2\right] + C\left[x^3-2x^2+x\right] + D\left[x^2-2x+1\right]$$

x^3: $0 = A + C$

x^2: $0 = A + B - 2C + D$

x: $1 = 2B + C - 2D$

x^0: $0 = -2A + 2B + D$

vyřešíme soustavu rovnic

$$\frac{x}{(x-1)^2(x^2+2x+2)} = \frac{1}{25(x-1)} + \frac{1}{5(x-1)^2} - \frac{x+8}{25(x^2+x+2)}$$

(dokončení příště)