Lineární zobrazení

$\varphi: V \to W$, $\forall \vec{u}, \vec{v} \in V$ a lib. $t \in T$ platí

1) $\varphi(\vec{u}+\vec{v}) = \varphi(\vec{u}) + \varphi(\vec{v})$

2) $\varphi(t \cdot \vec{u}) = t\, \varphi(\vec{u})$

Př. Rozhodněte, zda jde o LZ

a) $\varphi: \mathbb{R}^3 \to \mathbb{R}^2$, $\varphi((x_1, x_2, x_3)) = (2x_1 + 3x_2 \,,\, 4x_3 + 5)$ $\quad (0,0,0) \to (0,5)$ NE

b) $\varphi: \mathbb{R}^2 \to \mathbb{R}^1$, $\varphi((x_1, x_2)) = (x_1 \cdot x_2)$ $\quad (0,0) \to (0)$ $\quad (1,1) \to (1)$ $\quad 2(1,1) = (2,2) \to (4)$ NE

c) $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi((x_1, x_2)) = (x_1 + x_2 \,,\, x_1 + 2x_2)$

1) $\varphi((x_1, x_2)) = (x_1 + x_2, x_1 + 2x_2)$

$\varphi((y_1, y_2)) = (y_1 + y_2, y_1 + 2y_2)$

$$\varphi((x_1,x_2)+(y_1,y_2)) = \varphi((x_1+y_1, x_2+y_2)) = (x_1+y_1+x_2+y_2 \,,\, x_1+y_1+2(x_2+y_2)) =$$

$$= ((x_1+x_2)+(y_1+y_2), (x_1+2x_2)+(y_1+2y_2)) = (x_1+x_2, x_1+2x_2) + (y_1+y_2, y_1+2y_2)$$

$$= \varphi((x_1,x_2)) + \varphi((y_1,y_2))$$

2) $\underline{\varphi((x_1,x_2)) = (x_1+x_2, x_1+2x_2)}$

$$\varphi(t(x_1,x_2)) = \varphi((tx_1, tx_2)) = (tx_1 + tx_2 \,,\, tx_1 + 2tx_2) = t(x_1+x_2, x_1+2x_2)$$

$$t\,\varphi((x_1,x_2))$$

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_1 + 2x_2 \end{aligned} \qquad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}}_{A_\varphi} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Jádro ?

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + 2x_2 &= 0 \end{aligned} \qquad \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{aligned} x_1 + x_2 &= 0 \\ x_2 &= 0 \end{aligned} \qquad x_1 = 0$$

$\ker \varphi = \{\vec{o}\}$ $\qquad \varphi$ je prosté zobr.

Je surjektivní ? $\quad \forall (y_1, y_2) \in \mathbb{R}^2 \;\; \exists (x_1, x_2) \in \mathbb{R}^2$, tak že $\varphi((x_1,x_2)) = (y_1, y_2)$

$$\begin{aligned} x_1 + x_2 &= y_1 \\ x_1 + 2x_2 &= y_2 \end{aligned} \qquad \left(\begin{array}{cc|c} 1 & 1 & y_1 \\ 1 & 2 & y_2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & y_1 \\ 0 & 1 & y_2 - y_1 \end{array}\right) \qquad \begin{aligned} x_2 &= y_2 - y_1 \\ x_1 &= y_1 \end{aligned}$$

$$\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = 3\cdot(-1) - 1\cdot 2 = -6$$

$$\begin{vmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 0 & 2 & 1 \end{vmatrix} = 1\cdot1\cdot1 + (-1)\cdot2\cdot0 + 0\cdot2\cdot2 - \left(0\cdot1\cdot0 + 2\cdot2\cdot1 + 1\cdot2\cdot(-1)\right) = -1$$

$$\begin{matrix} 1 & 2 & 0 \\ -1 & 1 & 2 \end{matrix}$$

$$= 1\cdot\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 2\cdot\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1$$

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} \qquad |A| = -3+1 = -2 \checkmark$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

$$A = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$A = -\frac{1}{2}\begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \qquad |B| = 5 \qquad B^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$$

Lin. transformace

$\varphi: V \to V$

Def 1.2.2 $\quad U \subseteq V$ invariantní vzhledem φ

$\forall \vec{u} \in U$ je $\varphi(\vec{u}) \in U$

Def 1.2.3 $\quad$ Charakteristický (vlastní) vektor zob. φ

$\varphi(\vec{u}) = \lambda \vec{u}$, λ ... char. číslo (vlastní číslo)

Věta 1.2.4 $\quad |A_\varphi - \lambda E_n| = 0$ $\quad$ char. rovnice, polynom n-tého řádu

Věta 1.2.6

Poznámka $\quad \varphi \ldots A_\varphi \quad \nu \to \nu'$

$\underline{B_\varphi = S^{-1} A_\varphi S}$, S je matice přechodu od $\nu \to \nu'$

Určete char. čísla a char. vektory LT φ

a) $A_\varphi = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$ $\quad \begin{vmatrix} 3-\lambda & 2 \\ 2 & 0-\lambda \end{vmatrix} = (3-\lambda)(-\lambda) - 4 = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$

$\lambda_1 = 4 \quad \lambda_2 = -1$

Pro $\lambda_1 = 4$ $\quad \begin{matrix} -1x_1 + 2x_2 = 0 \\ 2x_1 - 4x_2 = 0 \end{matrix} \to -x_1 + 2x_2 = 0 \quad \begin{matrix} x_2 = t \\ x_1 = 2t \end{matrix} \quad (2t, t) \sim \underline{\underline{(2,1)}}$

Pro $\lambda_2 = -1$ $\quad \begin{matrix} 4x_1 + 2x_2 = 0 \\ 2x_1 + x_2 = 0 \end{matrix} \quad 2x_1 + x_2 = 0 \quad \begin{matrix} x_2 = -2t \\ x_1 = t \end{matrix} \quad (t, -2t) \sim \underline{\underline{(1,-2)}}$

$A_\varphi = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix} \quad S = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \quad |S| = -5 \quad S^{-1} = -\frac{1}{5}\begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix}$

$$B_\varphi = -\frac{1}{5}\begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 8 & -1 \\ 4 & 2 \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -20 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$$

b) $A_\varphi = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $\quad \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)(3-\lambda) = 0$

$\lambda_1 = 3, \ \lambda_2 = 2, \ \lambda_3 = 1$

Pro $\lambda_1 = 3$

$$\begin{aligned} -x_1 + x_2 + 0x_3 &= 0 \\ 0x_1 - 2x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{aligned}$$

$-x_1 + x_2 = 0$

$-2x_2 = 0$ $\quad x_2 = 0, \ x_1 = 0 \quad x_3 = t$

$(0,0,t) \sim (0,0,1)$

Pro $\lambda_2 = 2$

$$\begin{aligned} 0x_1 + x_2 &= 0 \\ -x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

$(t,0,0) \sim (1,0,0)$

Pro $\lambda_3 = 1$

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_3 &= 0 \end{aligned}$$

$x_3 = 0 \quad x_2 = t \quad x_1 = -t \quad (-t,t,0) \sim (-1,1,0)$

B_φ pro bázi $(0,0,1), (1,0,0), (-1,1,0)$ $\quad \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

c) $A_4 = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ $\quad \begin{vmatrix} -\lambda & 1 & -1 \\ 1 & -\lambda & 1 \\ -1 & 1 & -\lambda \end{vmatrix} = \cdots = -\lambda^3 + 3\lambda - 2$

$$-\lambda^3 + (0+0+0)\lambda^2 - \left(\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \right)\lambda$$

$$+ |A_4|$$

$-\lambda^3 + 3\lambda - 2 = 0$

$\lambda^3 - 3\lambda + 2 = 0$

	1	0	-3	2
1	1	1	-2	0
1	1	2	0	

$\lambda_1 = 1, \ \lambda_2 = 1, \ \lambda_3 = -2$

$\lambda_{1,2} = 1, \ \lambda_3 = -2$

Pro $\lambda_{1,2} = 1$

$-x_1 + x_2 - x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$-x_1 + x_2 - x_3 = 0$

$\begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad -x_1 + x_2 - x_3 = 0$

$x_3 = t$

$x_2 = s$

$x_1 = s - t$

$t = 0 \quad (s, s, 0) \sim (1,1,0)$

$s = 0 \quad (-t, 0, t) \sim (-1, 0, 1)$

Pro $\lambda_3 = -2$

$2x_1 + x_2 - x_3 = 0$

$x_1 + 2x_2 + x_3 = 0$

$-x_1 + x_2 + 2x_3 = 0$

$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ -1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$x_1 + 2x_2 + x_3 = 0$

$x_2 + x_3 = 0$

$x_3 = t, \ x_2 = -t$

$x_1 = t$

$(t, -t, t) \sim (1, -1, 1)$

d) $A_4 = \begin{pmatrix} -1 & 4 & 3 \\ -2 & 5 & 3 \\ 2 & -4 & -2 \end{pmatrix}$ $\quad \begin{vmatrix} -1-\lambda & 4 & 3 \\ -2 & 5-\lambda & 3 \\ 2 & -4 & -2-\lambda \end{vmatrix} = \cdots = -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda^2 - 2\lambda + 1) =$

$= -\lambda(\lambda - 1)^2$

Pro $\lambda_1 = 0$

$-x_1 + 4x_2 + 3x_3 = 0$

$-2x_1 + 5x_2 + 3x_3 = 0$

$2x_1 - 4x_2 - 2x_3 = 0$

$\begin{pmatrix} -1 & 4 & 3 \\ -2 & 5 & 3 \\ 2 & -4 & -2 \end{pmatrix} \sim \cdots \begin{pmatrix} -1 & 4 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$-x_1 + 4x_2 + 3x_3 = 0$

$x_2 + x_3 = 0$

$x_3 = t, \ x_2 = -t, \ x_1 = -t$

$(1, 1, -1)$

jádro

Pro $\lambda_{2,3} = 1$

$-2x_1 + 4x_2 + 3x_3 = 0$

$-2x_1 + 4x_2 + 3x_3 = 0$

$2x_1 - 4x_2 - 3x_3 = 0$

$\rightarrow -2x_1 + 4x_2 + 3x_3 = 0$

$x_3 = 2s, \ x_2 = t, \ x_1 = 3s + 2t$

$(3s + 2t, t, 2s) < \begin{matrix} t = 0 & (3, 0, 2) \\ s = 0 & (2, 1, 0) \end{matrix}$

e) $A_\varphi = \begin{pmatrix} 2 & 2 \\ -4 & -2 \end{pmatrix}$ $\begin{vmatrix} 2-\lambda & 2 \\ -4 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 8 = \dots = \lambda^2 + 4$

$\lambda_{1,2} = \pm 2i$ $\qquad \underset{\alpha}{0} \pm \underset{\beta}{2i}$

pro $\lambda_1 = 2i$ $\qquad (2-2i)x_1 + 2x_2 = 0$

$-4x_1 + (-2-2i)x_2 = 0$

$\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}^{(1+i)} \sim \begin{pmatrix} 2 & 1+i \\ -2 & -1-i \end{pmatrix} \sim$

$\sim \begin{pmatrix} 2 & 1+i \\ 0 & 0 \end{pmatrix}$ $\qquad 2x_1 + (1+i)x_2 = 0$

$x_2 = t \qquad 2x_1 = -(1+i)t$

$x_1 = -\frac{1+i}{2} t$

$\left(-\frac{1+i}{2} t, t\right) \sim \underline{\underline{(+1+i, -2)}}$

$(1+i, -2) = \underset{v_1}{(1, -2)} + i \underset{v_2}{(1, 0)}$

$\underline{\varphi(\vec{v}_1)} = \varphi((+1, -2)) = \begin{pmatrix} 2 & 2 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \underline{(-2, 0)}^T$

$= -2 \cdot v_2$

$\varphi(\vec{v}_2) = \begin{pmatrix} 2 & 2 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = 2 \cdot v_1$

$\varphi(\vec{v}_1) = 0\vec{v}_1 - 2\vec{v}_2$

$\varphi(\vec{v}_2) = 2v_1 + 0v_2$ $\qquad \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

D4'

$\begin{pmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{pmatrix}$ $\qquad \lambda_{1,2} = 1 \qquad (1,0,0), (0,1,1)$

$\lambda_3 = -2 \qquad (1,2,1)$

$\begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{pmatrix}$ $\qquad \lambda_1 = -1 \qquad (1, -3, 4)$

$\lambda_{2,3} = 2 \qquad (1,0,1)$