$f: A_2 \to A_2$

$$x' = 8x - 5y - 2$$
$$y' = -x + 4y - 2$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 8 & -5 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

V repéru $\langle P, \vec{e}_1, \vec{e}_2 \rangle$ $P = [0,0]$

$\vec{e}_1 = (1,0)$, $\vec{e}_2 = (0,1)$

Ztransformujte rovnice af. zobrazení do nového repéru $\langle Q, \vec{d}_1, \vec{d}_2 \rangle$, kde $Q = [1,1]$, $\vec{d}_1 = (-5,1)$, $\vec{d}_2 = (1,1)$.

Postupujeme podle příkladu 2.2.3

Transformační rovnice

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -5 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \bar{x}' \\ \bar{y}' \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Dosadíme do rovnic af. zobrazení

$$\underline{\begin{pmatrix} -5 & 1 \\ 1 & 1 \end{pmatrix}} \begin{pmatrix} \bar{x}' \\ \bar{y}' \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 & -5 \\ -1 & 4 \end{pmatrix} \left[\begin{pmatrix} -5 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] + \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

potřebujeme vyjádřit v závislosti na $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$

$$\begin{pmatrix} -5 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = -\frac{1}{6} \begin{pmatrix} 1 & -1 \\ -1 & -5 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -1 & 1 \\ 1 & 5 \end{pmatrix}$$

$$\underline{\frac{1}{6} \begin{pmatrix} -1 & 1 \\ 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} -5 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \bar{x}' \\ \bar{y}' \end{pmatrix}} + \underline{\frac{1}{6} \begin{pmatrix} -1 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \underline{\frac{1}{6} \begin{pmatrix} -1 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 8 & -5 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} -5 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}} +$$
$$+ \underline{\frac{1}{6} \begin{pmatrix} -1 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 8 & -5 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}} + \underline{\frac{1}{6} \begin{pmatrix} -1 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ -2 \end{pmatrix}}$$

$$\begin{pmatrix} \bar{x}' \\ \bar{y}' \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} \bar{x}' \\ \bar{y}' \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

Proč? 1.2.1, str. 12

$f: A_2 \to A_2 \quad : \quad A=[0,0] \to A'=[1,1]$

$B=[1,0] \to B'=[-1,0]$

$C=[0,1] \to C'=[0,0]$

$h: \quad [0,0] \to [2,1]$

$[1,0] \to [0,0]$

$[0,1] \to [-1,1]$

$g: A_2 \to A_2 \quad : \quad A'=[1,1] \to A''=[2,1]$

$B'=[-1,0] \to B''=[0,0]$

$C'=[0,0] \to C''=[-1,1]$

$f: \quad \left(\begin{array}{ccc|cc} 0&0&1&1&1\\ 1&0&1&-1&0\\ 0&1&1&0&0 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1&0&1&-1&0\\ 0&1&1&0&0\\ 0&0&1&1&1 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1&0&0&-2&-1\\ 0&1&0&-1&-1\\ 0&0&1&1&1 \end{array}\right)$

$f: \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$g: \quad \left(\begin{array}{ccc|cc} 1&1&1&2&1\\ -1&0&1&0&0\\ 0&0&1&-1&1 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1&1&1&2&1\\ 0&1&2&2&1\\ 0&0&1&-1&1 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1&1&0&3&0\\ 0&1&0&4&-1\\ 0&0&1&-1&1 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1&0&0&-1&1\\ 0&1&0&4&-1\\ 0&0&1&-1&1 \end{array}\right)$

$g: \quad \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$h: g \circ f \qquad \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix} \left[\begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] + \begin{pmatrix} -1 \\ 1 \end{pmatrix} =$

$= \begin{pmatrix} -2 & -3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$m(f) = \begin{vmatrix} -2 & -1 \\ -1 & -1 \end{vmatrix} = 1 \qquad m(g) = \begin{vmatrix} -1 & 4 \\ 1 & -1 \end{vmatrix} = -3 \qquad m(g \circ f) = \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} = -3$

$\left(\begin{array}{ccc|cc} 0&0&1&2&1\\ 1&0&1&0&0\\ 0&1&1&-1&1 \end{array}\right) \sim \ldots \sim \left(\begin{array}{ccc|cc} 1&0&0&-2&-1\\ 0&1&0&-3&0\\ 0&0&1&2&1 \end{array}\right)$

$x'' = -2x - 3y + 2$

$y'' = -x + 1$

$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Mějme v A_2 $\triangle ABC$. Kolik existuje afinit, kde
$A \to B,\ B \to C,\ C \to A$? Určete jejich samodružné body a samodružné směry.

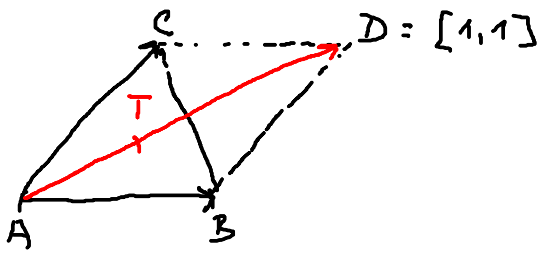
$A=[0,0]$, $\vec{u}_1 = \vec{AB}$, $\vec{u}_2 = \vec{AC}$ $\Rightarrow$ $B=[1,0]$, $C=[0,1]$

$A'=[1,0]$, $B'=[0,1]$, $C'=[0,0]$

$$\left(\begin{array}{ccc|cc} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{aligned} x' &= -x-y+1 \\ y' &= x \end{aligned}$$

$[x,y] \to [x,y]$ $\quad \begin{aligned} x &= -x-y+1 \\ y &= x \end{aligned} \quad \to \quad \begin{aligned} -2x-y+1 &= 0 \\ -2x-x+1 &= 0 \end{aligned} \quad \begin{aligned} x &= \tfrac{1}{3} \\ y &= \tfrac{1}{3} \end{aligned} \quad S=[\tfrac{1}{3}, \tfrac{1}{3}]$

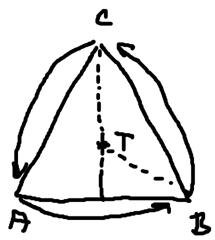


$$A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} -1-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = 0$$

$\lambda + \lambda^2 + 1 = 0 \qquad \lambda^2 + \lambda + 1 = 0$

nejsou sam. směry



Najděte samodr. body a směry zobrazení

$f: [1,0,1] \to [4,-6,4]$, $(1,2,0) \longrightarrow (7,8,-14)$

$(2,-1,1) \longrightarrow (4,-13,10)$

$(-1,2,1) \longrightarrow (1,8,-11)$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 7 & 8 & -14 \\ 2 & -1 & 1 & 4 & -13 & 10 \\ -1 & 2 & 1 & 1 & 8 & -11 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 7 & 8 & -14 \\ 0 & -5 & 1 & -10 & -29 & 38 \\ 0 & 4 & 1 & 8 & 16 & -25 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 7 & 8 & -14 \\ 0 & -5 & 1 & -10 & -29 & 38 \\ 0 & 0 & 1 & 0 & -4 & 3 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 7 & 8 & -14 \\ 0 & -5 & 0 & -10 & -25 & 35 \\ 0 & 0 & 1 & 0 & -4 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 0 \\ 0 & 1 & 0 & 2 & 5 & -7 \\ 0 & 0 & 1 & 0 & -4 & 3 \end{array}\right) \longrightarrow A = \begin{pmatrix} 3 & 2 & 0 \\ -2 & 5 & -4 \\ 0 & -7 & 3 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad \begin{pmatrix} +4 \\ -6 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 0 \\ -2 & 5 & -4 \\ 0 & -7 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{pmatrix} +4 \\ -6 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad \begin{aligned} +4 &= 3 + b_1 \\ -6 &= -6 + b_2 \\ 4 &= 3 + b_3 \end{aligned}$$

$$\underline{\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 3 & 2 & 0 \\ -2 & 5 & -4 \\ 0 & -7 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$$

$$\begin{aligned} x' &= 3x + 2y + 1 \\ y' &= -2x + 5y - 4z \\ z' &= -7y + 3z + 1 \end{aligned} \qquad \begin{aligned} x &= 3x + 2y + 1 \\ y &= -2x + 5y - 4z \\ z &= -7y + 3z + 1 \end{aligned} \qquad \begin{aligned} 2x + 2y &= -1 \\ -2x + 4y - 4z &= 0 \\ -7y + 2z &= -1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & -\frac{1}{2} \\ -1 & 2 & -2 & 0 \\ 0 & -7 & 2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 3 & -2 & -\frac{1}{2} \\ 0 & -7 & 2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 3 & -2 & -\frac{1}{2} \\ 0 & -4 & 0 & -\frac{3}{2} \end{array}\right)$$

$-4y = -\frac{3}{2} \to y = \frac{3}{8}$

$3y - 2z = -\frac{1}{2}$

$\frac{9}{8} - 2z = -\frac{1}{2}$

$z = \frac{13}{16}$

$x + y = -\frac{1}{2}$

$x + \frac{3}{8} = -\frac{1}{2}$ $\quad x = -\frac{7}{8}$

$S = \left[-\frac{7}{8}, \frac{3}{8}, \frac{13}{16}\right]$

$$A = \begin{pmatrix} 3 & 2 & 0 \\ -2 & 5 & -4 \\ 0 & -7 & 3 \end{pmatrix} \qquad \begin{vmatrix} 3-\lambda & 2 & 0 \\ -2 & 5-\lambda & -4 \\ 0 & -7 & 3-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 5-\lambda & -4 \\ -7 & 3-\lambda \end{vmatrix} - (-2)\underbrace{\begin{vmatrix} 2 & 0 \\ -7 & 3-\lambda \end{vmatrix}}_{4(3-\lambda)} =$$

$$= (3-\lambda)\cdot\left[15 - 8\lambda + \lambda^2 - 28\right] + 2\,(6-2\lambda) =$$

$$= (3-\lambda)\,(\lambda^2 - 8\lambda + 9) = (3-\lambda)(\lambda+1)(\lambda-9) = 0$$

$\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 9$

Pro $\lambda_1 = -1$

$$\begin{pmatrix} 4 & 2 & 0 \\ -2 & 6 & -4 \\ 0 & -7 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -7 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 2 \\ 0 & 7 & -4 \\ 0 & -7 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 2 \\ 0 & 7 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

$x - 3y + 2z = 0$ $\quad z = t$ $\quad x = -\frac{2}{7}t$

$7y - 4z = 0$ $\quad y = \frac{4}{7}t$

$$\vec{u}_1 = (-2, 4, 7)$$

Pro $\lambda_2 = 3$

$$\begin{pmatrix} 0 & 2 & 0 \\ -2 & 2 & -4 \\ 0 & -7 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$y = 0$ $\quad -2x + 0 - 4t = 0$

$z = t$ $\quad x = -2t$

$$\vec{u}_2 = (-2, 0, 1)$$

Pro $\lambda_3 = 9$

$$\begin{pmatrix} -6 & 2 & 0 \\ -2 & -4 & -4 \\ 0 & -7 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 2 \\ 0 & 7 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

$x + 2y + 2z = 0$ $\quad z = t$

$7y + 6z = 0$ $\quad y = -\frac{6}{7}t$

$x = \frac{2}{7}t$

$$\vec{u}_3 = (2, -6, 7)$$

$f: A_2 \to A_2$ Ukažte, že rovnicemi $\begin{aligned} x' &= x+2y-3 \\ y' &= x+y-1 \end{aligned}$ je zadána afinita.

Rozhodněte, zda jde o přímou či nepřímou afinitu, najděte f^{-1}.

$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ $\quad |A| = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1 \neq 0$ $\quad$ je af., nepřímá

$X' = AX + B$ $\qquad X \to X'$

$X' - B = AX \quad / \cdot A^{-1}$

$A^{-1}(X' - B) = A^{-1}AX$

$A^{-1}X' - A^{-1}B = X$

$\underline{f^{-1}: X' = A^{-1}X - A^{-1}B}$

$A^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$

$f: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

$f^{-1}: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -3 \\ -1 \end{pmatrix}$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Napište rovnice stejnolehlosti se středem $S = [\underset{s_1}{1}, \underset{s_2}{2}]$, která zobrazí bod $\underset{X}{[2,3]} \to \underset{X'}{[-1,0]}$.

$\underline{\overrightarrow{SX'} = k\,\overrightarrow{SX}}$

$\underline{X' - S = k(X - S)}$

$\begin{aligned} x' - s_1 &= k(x - s_1) \\ y' - s_2 &= k(y - s_2) \end{aligned}$ $\qquad \begin{aligned} -1 - 1 &= k(2-1) \\ 0 - 2 &= k(3-2) \end{aligned}$ $\qquad \begin{aligned} -2 &= k \\ -2 &= k \end{aligned}$ $\qquad \underline{k = -2}$

$\begin{aligned} x' - 1 &= -2(x-1) \\ y' - 2 &= -2(y-2) \end{aligned}$ $\qquad \boxed{\begin{aligned} x' &= -2x + 3 \\ y' &= -2y + 6 \end{aligned}}$ $\qquad \begin{aligned} x' &= kx + (1-k)s_1 \\ y' &= ky + (1-k)s_2 \end{aligned}$

Je dána stejnolehlost f a translace (posunutí) g

f: $x' = 2x + 1$, $y' = 2y - 1$ g: $x' = x + 3$, $y' = y$ Určete středy stejnolehlosti $h_1: f \circ g$ a $h_2: g \circ f$.

$h_1: f \circ g$

$x' = 2(x+3) + 1 = 2x + 7$

$y' = 2y - 1 = 2y - 1$

$k = 2$

$x' = kx + (1-k)s_1$

$7 = (1-2)s_1 \longrightarrow s_1 = -7$

$y' = ky + (1-k)s_2$

$-1 = (1-2)s_2 \Rightarrow s_2 = 1$

$S[-7, 1]$

$h_2: g \circ f$

$x' = 2x + 1 + 3 = 2x + 4$

$y' = 2y - 1 = 2y - 1$

$k = 2 \longrightarrow S = [-4, 1]$

$4 = (1-2)s_1$

$-1 = (1-2)s_2$

$$f: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad g: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$h_1 = f \circ g \qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right] + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$