$$H = \sum_k \left[A_k a_k^+ a_k - \frac{1}{2} B_k (a_k a_{-k} + a_k^+ a_{-k}^+) \right]$$

$$[a_k, a_{k'}^+] = \delta_{kk'} \quad ; \quad [a_k, a_{k'}] = [a_k^+, a_{k'}^+] = 0$$

$$\boxed{A_k = A_{-k} \quad , \quad B_k = B_{-k}}$$

$$H = \sum_k \left[\frac{1}{2} A_k a_k^+ a_k + \frac{1}{2} A_{-k} a_{-k}^+ a_{-k} - \frac{1}{2} B_k (a_{-k} a_k + a_k^+ a_{-k}^+) \right]$$

$$= \sum_k \frac{1}{2} \left[A_k a_k^+ a_k + A_{-k} a_{-k} a_{-k}^+ - \frac{1}{2} B_k (a_{-k} a_k + a_k^+ a_{-k}^+) \right] - \frac{1}{2} \sum_k A_k$$

$$= \frac{1}{2} \sum_k \vec{\psi}_k^{+T} \hat{H}_k \vec{\psi}_k - \frac{1}{2} \sum_k A_k$$

$$\vec{\psi}_k^+ = \begin{pmatrix} a_k^+ \\ a_{-k} \end{pmatrix} \quad ; \quad \vec{\psi}_k = \begin{pmatrix} a_k \\ a_{-k}^+ \end{pmatrix}$$

$$\hat{H}_k = \begin{pmatrix} A_k & -B_k \\ -B_k & A_{-k} \end{pmatrix}$$

$$[\psi_k^\alpha, \psi_k^{\beta +}] = \cancel{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} K^{\alpha\beta}$$

$$\hat{K} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\vec{\alpha}_k^\alpha, \vec{\alpha}_k^{\beta +}] = \cancel{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} K^{\alpha\beta}$$

$$\boxed{\psi_k^\alpha = T^\alpha_\beta \vec{\alpha}_k^\beta} \qquad T = \begin{pmatrix} u_k & v_k \\ v_k^* & u_k \end{pmatrix}$$

$\cancel{[\psi^{\alpha}_k, \psi^{\beta+}_k] =}$

$$[\alpha^{\alpha}_k, \alpha^{\beta+}_k] = [(T^{-1})^{\alpha}_{\alpha'}\psi^{\alpha'}_k, (T^{-1})^{\beta*}_{\beta'}\psi^{\beta'+}_k] =$$

$$= (T^{-1})^{\alpha}_{\alpha'}[\psi^{\alpha'}_k, \psi^{\beta'+}_k](T^{-1})^{\beta*}_{\beta'} =$$

$$= (T^{-1})^{\alpha}_{\alpha'} K^{\alpha'\beta'}(T^{-1})^{\beta*}_{\beta'} = K^{\alpha\beta}$$

$$T^{-1} = \frac{1}{\det T}\begin{pmatrix} u_k^* & -v_k \\ -v_k^* & u_k \end{pmatrix}$$

$$\frac{1}{|\det T|^2}\begin{pmatrix} u_k^* & -v_k \\ -v_k & u_k \end{pmatrix}\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u_k & -v_k^* \\ -v_k & u_k^* \end{pmatrix} =$$

$$= \frac{1}{|\det T|^2}\begin{pmatrix} u_k^* & -v_k \\ -v_k & u_k \end{pmatrix}\begin{pmatrix} u_k & -v_k^* \\ v_k & -u_k^* \end{pmatrix} = \cancel{\left(|u_k|^2 - |v_k|^2\right.}$$

$$= \frac{1}{|\det T|^2}\begin{pmatrix} |u_k|^2 - |v_k|^2 & v_k^* u_k - v_k u_k^* \\ u_k^* v_k - v_k^* u_k & |v_k|^2 - |u_k|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$v_k^* u_k - v_k u_k^* = 0 \Leftarrow \boxed{u_k, v_k \in \mathbb{R}}$$

$$\frac{u_k^2 - v_k^2}{(u_k^2 - v_k^2)^2} = 1 \Rightarrow \boxed{u_k^2 - v_k^2 = 1} \Leftrightarrow \det T = 1$$

$$H' = \sum_k \omega_k \alpha_k^+ \alpha_k + \sum_k \frac{1}{2}(\omega_k - A_k) =$$

$$= \sum_k \frac{1}{2}\omega_k \alpha_k^+ \alpha_k + \frac{1}{2}\omega_{-k}\alpha_{-k}\alpha_{-k}^+ - \frac{1}{2}\omega_{-k} + \sum_k \frac{1}{2}(\omega_k - A_k)$$

$$= \left| \omega_{-k} = \omega_k \right| = \frac{1}{2}\sum_k \bar{\alpha}_k^{+T} H'_k \bar{\alpha}_k - \frac{1}{2}\sum_k A_k$$

$$\bar{\alpha}_k^+ = \begin{pmatrix} \alpha_k^+ \\ \alpha_{-k} \end{pmatrix} \quad ; \quad \bar{\alpha}_k = \begin{pmatrix} \alpha_k \\ \alpha_{-k}^+ \end{pmatrix}$$

$$\hat{H}'_k = \begin{pmatrix} \omega_k & 0 \\ 0 & \omega_k \end{pmatrix}$$

$$\boxed{\hat{T}_k^+ \hat{H}_k \hat{T}_k = \hat{H}'_k}$$

$$\begin{pmatrix} \omega_k & 0 \\ 0 & \omega_k \end{pmatrix} \underset{A_k = A_{-k}}{\overset{\uparrow}{=}} \begin{pmatrix} u_k & v_k \\ v_k & u_k \end{pmatrix} \begin{pmatrix} A_k & -B_k \\ -B_k & A_k \end{pmatrix} \begin{pmatrix} u_k & v_k \\ v_k & u_k \end{pmatrix}$$

$$= \begin{pmatrix} u_k & v_k \\ v_k & u_k \end{pmatrix} \begin{pmatrix} A_k u_k - B_k v_k & A_k v_k - B_k u_k \\ A_k v_k - B_k u_k & A_k u_k - B_k v_k \end{pmatrix} =$$

$$= \begin{pmatrix} A_k v_k^2 + A_k u_k^2 - 2B_k v_k u_k & -B_k u_k^2 - B_k v_k^2 + 2A_k v_k u_k \\ \cdots & \cdots \end{pmatrix}$$

$$\omega_k = A_k(v_k^2+u_k^2) - 2B_k v_k u_k$$

$$0 = -B_k(u_k^2+v_k^2) + 2A_k u_k v_k$$

$$u_k^2 - v_k^2 = 1$$

- $$u_k v_k = \frac{B_k(u_k^2+v_k^2)}{2A_k} \qquad (1)$$

$$\Rightarrow \quad \omega_k = A_k(v_k^2+u_k^2) - \frac{B_k^2(u_k^2+v_k^2)}{A_k}$$

$$\Rightarrow \quad u_k^2+v_k^2 = \frac{\omega_k A_k}{A_k^2-B_k^2} \Rightarrow u_k^2 = \left[\frac{\omega_k A_k}{A_k^2-B_k^2} + 1\right]\frac{1}{2}$$

$$\cancel{= \frac{\omega_k A_k + A_k^2 - B_k^2}{A_k^2 - B_k^2}}$$

- $$u_k v_k = \frac{-\omega_k + A_k(v_k^2+u_k^2)}{2B_k}$$

$$0 = -B_k(u_k^2+v_k^2) + \frac{A_k^2}{B_k}(u_k^2+v_k^2) - \frac{\omega_k A_k}{B_k}$$

$$(u_k^2+v_k^2) = \frac{\omega_k A_k}{B_k}\,\frac{B_k}{A_k^2-B_k^2} \qquad (2)$$

$$\Rightarrow \quad v_k^2 = \left[\frac{\omega_k A_k}{A_k^2-B_k^2} - 1\right]\frac{1}{2}$$

• $$u_r v_r = \frac{B_r\left(u_r^2 + v_r^2\right)}{2A_r} \underset{(2)\to(1)}{=} \frac{\omega_r B_r}{2\left(A_r^2 - B_r^2\right)}$$

$$u_r^2 v_r^2 = \frac{\omega_r^2 B_r^2}{4\left(A_r^2 - B_r^2\right)^2} = \left[\frac{\omega_r A_r}{\left(A_r^2 - B_r^2\right)} + 1\right]\left[\frac{\omega_r A_r}{\left(A_r^2 - B_r^2\right)} - 1\right]\frac{1}{4}$$

$$\cancel{\frac{4\omega_r^2 B_r^4}{A_r^2 + B_r^2}} = \frac{\omega_r^2 B_r^2}{\left(A_r^2 - B_r^2\right)^2} = \frac{\omega_r^2 A_r^2}{\left(A_r^2 - B_r^2\right)^2} - 1$$

$$1 = \omega_r \frac{1}{A_r^2 - B_r^2} \Rightarrow \boxed{\omega_r^2 = A_r^2 - B_r^2}$$

$\omega_r = \sqrt{A_r^2 - B_r^2}$ ... volba

$$u_r = \frac{1}{\sqrt{2}}\sqrt{\frac{A_r}{\omega_r} + 1}$$

$$v_r = \frac{1}{\sqrt{2}}\sqrt{\frac{A_r}{\omega_r} - 1}\ \mathrm{sign}(B_r)$$