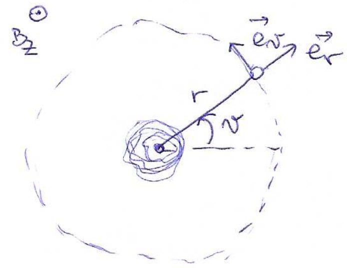
Př. 2 → GL rovnice ve válcových souřadnicích



$\psi(\vec{r}) = f(r)\, e^{im\vartheta}$ $\quad$ $f(r)$ reálna funkce

$\vec{A}(\vec{r}) = A(r)\,\vec{e}_\vartheta$ $\quad$ $\vec{e}_\vartheta$ jednotkový vektor odp. souř. ϑ

$\vec{j}(\vec{r}) = j(r)\,\vec{e}_\vartheta$

$$\vec{B} = \nabla \times \vec{A} = (0,0,B_z) \Rightarrow B_z = \frac{1}{r}\frac{d(rA)}{dr} \Rightarrow A(r) = \frac{1}{r}\int_0^r B_z(r')\, r'\, dr'$$

$j(r) = j_0$

$\nabla \times \vec{B} = \mu_0 \vec{j}$ $\quad$ Maxwellova rovnica

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1GL

$$-\xi^2\left(\nabla + i\frac{2e}{\hbar}\vec{A}\right)^2\psi - \psi + |\psi|^2\psi = 0$$

$\psi = f(r)\, e^{im\vartheta}$

$|\psi|^2\psi = f^2(r) \cdot f(r)\, e^{im\vartheta}$

$$-\xi^2\left(\nabla + i\frac{2e}{\hbar}\vec{A}\right)^2\psi = -\xi^2\left(\nabla^2 + 2i\frac{2e}{\hbar}\vec{A}\nabla + \left(\frac{i2e}{\hbar}\right)^2 A^2\right) f(r)\, e^{im\vartheta}$$

$$\nabla^2\left(f(r)\, e^{im\vartheta}\right) = \Delta f(r)\, e^{im\vartheta} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) e^{im\vartheta} + \frac{1}{r^2} f(r) \frac{\partial^2 e^{im\vartheta}}{\partial \vartheta^2} =$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) e^{im\vartheta} - \frac{m^2}{r^2} e^{im\vartheta} f(r)$$

$\vec{e}_\vartheta \cdot \vec{e}_r = 0$ $\qquad$ $\vec{e}_\vartheta \cdot \vec{e}_\vartheta = 1$

$$2i\frac{2e}{\hbar} A(\vec{r})\,\vec{e}_\vartheta\left[\nabla\left[f(r)\, e^{im\vartheta}\right]\right] = 2i\frac{2e}{\hbar} A\,\vec{e}_\vartheta\left[\vec{e}_r \frac{\partial f}{\partial r} e^{im\vartheta} + \frac{\vec{e}_\vartheta}{r} f\,(im)\, e^{im\vartheta}\right] =$$

$$= 2i\frac{2e}{\hbar r} A\,(im)\, f\, e^{im\vartheta} = -2\frac{2em}{\hbar r} A f e^{im\vartheta}$$

$$\left(\frac{i2e}{\hbar}\right)^2 A^2 f(r)\, e^{im\vartheta} = -\left(\frac{2e}{\hbar}\right)^2 A^2 f(r)\, e^{im\vartheta}$$

$$-\xi^2\left(\nabla + i\frac{2e}{\hbar}\vec{A}\right)^2\psi = -\xi^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) e^{im\vartheta} - \frac{m^2}{r^2} e^{im\vartheta} f(r) - 2\frac{2em}{\hbar r} A f e^{im\vartheta} - \left(\frac{2e}{\hbar}\right)^2 A^2 f\, e^{im\vartheta}\right]$$

$$\xi^2\left[\left(\frac{m}{r} + \frac{2e}{\hbar}A\right)^2 f - \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)\right] - f + f^3 = 0$$

2.GL

$$\underbrace{\nabla\times\vec{B}}_{\mu_0\vec{j}} = \frac{1}{\lambda^2}\left[\frac{i\hbar}{4e}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - |\psi|^2\vec{A}\right]$$

$$\vec{j} = j(r)\,\vec{e}_\vartheta$$

$$\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right] = f(r)\,e^{-in\vartheta}\,\nabla\left(f(r)\,e^{in\vartheta}\right) - f(r)\,e^{in\vartheta}\,\nabla\left(f(r)\,e^{-in\vartheta}\right) =$$

$$= f(r)\,e^{-in\vartheta}\left[\vec{e}_r\frac{\partial f}{\partial r}e^{in\vartheta} + \frac{\vec{e}_\vartheta}{r}f(in)\,e^{in\vartheta}\right] - f(r)\,e^{in\vartheta}\left[\vec{e}_r\frac{\partial f}{\partial r}e^{-in\vartheta} - \frac{\vec{e}_\vartheta}{r}f(in)\,e^{-in\vartheta}\right] =$$

$$= f(r)\,e^{-in\vartheta}e^{in\vartheta}\left[\vec{e}_r\frac{\partial f}{\partial r} + \frac{\vec{e}_\vartheta}{r}f(in)\right] - f(r)\,e^{in\vartheta}e^{-in\vartheta}\left[\vec{e}_r\frac{\partial f}{\partial r} - \frac{\vec{e}_\vartheta}{r}f(in)\right] =$$

$$= f(r)\underbrace{\left[\vec{e}_r\frac{\partial f}{\partial r} - \vec{e}_r\frac{\partial f}{\partial r}\right]}_{0} + f(r)\left[\frac{\vec{e}_\vartheta}{r}f(in) + \frac{\vec{e}_\vartheta}{r}f(in)\right] = \frac{2in\,\vec{e}_\vartheta}{r}f^2$$

$$|\psi|^2\vec{A} = f^2 A(r)\,\vec{e}_\vartheta$$

$$\frac{1}{\lambda^2}\left[\frac{i\hbar}{4e}\,\frac{2in\,\vec{e}_\vartheta f^2}{r} - f^2A(r)\,\vec{e}_\vartheta\right] = \frac{1}{\lambda^2}\left[-\frac{\hbar n}{2er} - A(r)\right]f^2\,\vec{e}_\vartheta$$

$$\mu_0 j(r)\,\vec{e}_\vartheta = \frac{1}{\lambda^2}\left[-\frac{\hbar n}{2er} - A(r)\right]f^2\,\vec{e}_\vartheta$$

$$\boxed{\mu_0 j = \frac{1}{\lambda^2}\left[-\frac{\hbar n}{2er} - A\right]f^2}$$

Př 3   Struktura víru v supravodiči typu II

riešenie GL rovníc $\xi^2\left[\left(\frac{m}{r}+\frac{2e}{\hbar}A\right)^2 f - \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - f + f^3 = 0$

$$\mu_0 j = \frac{\hbar f^2}{2e\lambda^2}\frac{m}{r} - \frac{f^2}{\lambda^2}A$$

okrajové podmínky

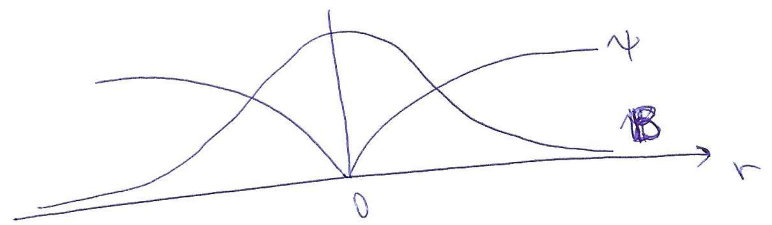
$\psi \to 1$ pro $r \to \infty$

$B_z \to 0$ pro $r \to \infty$

a) riešenie rovníc v blízkosti počátku

$A(r)$ a $f(r)$ do mocninnej rady v okolí počátku – jaké členy se v rozvoji vypadnú?

Určete koeficienty u prvních dvou nenulových členů rozvoje funkce f.



$$f(r) = f_0 + f_1 r + f_2 r^2 + f_3 r^3 \dots \quad \Rightarrow \quad f_0 = 0$$

$$B_z(r) = b_0 + b_1 r + b_2 r^2 + b_3 r^3 \dots$$

$$A(r) = \frac{1}{r}\int_0^r B_z(r')\, r'\, dr' = \frac{1}{r}\int_0^r (b_0 + b_1 r' + b_2 r'^2 + b_3 r'^3 \dots)\, r'\, dr' =$$

$$\frac{1}{r}\left[\frac{1}{2}b_0 r^2 + \frac{1}{3}b_1 r^3 + \frac{1}{4}b_2 r^4 + \frac{1}{5}b_3 r^5\right] = \frac{1}{2}b_0 r + \frac{1}{3}b_1 r^2 + \frac{1}{4}b_2 r^3 + \frac{1}{3}b_3 r^4 \dots \equiv$$

$$a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4 \dots$$

Dosademe do 1. GL

$$\xi^2\Big[\Big(\frac{m^2}{r^2} + \frac{m4e}{r\hbar}(a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4 \dots) + \left(\frac{2e}{\hbar}\right)^2 (a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4)^2\Big)(f_1 r + f_2 r^2 + f_3 r^3 \dots) -$$

$$- \frac{1}{r}\frac{d}{dr}\Big(r\frac{d}{dr}(f_1 r + f_2 r^2 + f_3 r^3 + f_4 r^4 \dots)\Big)\Big] - (f_1 r + f_2 r^2 + f_3 r^3 \dots) + (f_1 r + f_2 r^2 + f_3 r^3 \dots)^3 = 0$$

$$\xi^2 \frac{m^2}{r^2}(f_1 r + f_2 r^2 + f_3 r^3 + f_4 r^4 \dots) + \xi^2 \frac{m4e}{\hbar r}(f_1 a_1 r^2 + a_1 f_2 r^3 + a_2 f_1 r^3 + a_2 f_2 r^4 + \dots) +$$

$$+ \xi^2\left(\frac{2e}{\hbar}\right)^2 (a_1^2 r^2 + 2a_1 a_2 r^3 + a_2^2 r^4 + \dots)(f_1 r + f_2 r^2 + f_3 r^3) - \xi^2 \frac{1}{r}\frac{d}{dr}(f_1 r + 2f_2 r^2 + 3f_3 r^3 + 4f_4 r^4 \dots)$$

$$- (f_1 r + f_2 r^2 + f_3 r^3 \dots) + (f_1^3 r^3 + \dots) = 0$$

$$\xi^2 n^2\left(\frac{f_1}{r} + f_2 + f_3 r + f_4 r^2 \dots\right) + \xi^2 \frac{n 4e}{\hbar}\left(f_1 a_1 r + a_1 f_2 r^2 + a_2 f_1 r^2 + \dots\right) +$$

$$+ \xi^2 \left(\frac{2e}{\hbar}\right)^2 \left(a_1^2 f_1 r^3 + \dots\right) - \xi^2\left(\frac{f_1}{r} + 4 f_2 + 9 f_3 r + 16 f_4 r^2 \dots\right) - \left(f_1 r + f_2 r^2 + f_3 r^3\right)$$

$$+ \left(f_1^3 r^3 + \dots\right) = 0$$

$\frac{1}{r}$: $\quad \xi^2 n^2 \frac{f_1}{r} - \xi^2 \frac{f_1}{r} = 0 \Rightarrow \frac{\xi^2 f_1}{r}\left(n^2 - 1\right) = 0 \qquad f_1 = 0$ alebo $n = 1$

vír je nestabilný ak $n > 1$

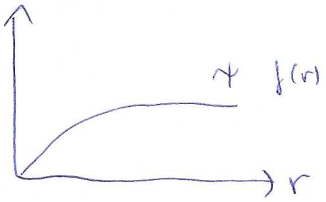
r^0: $\quad \xi^2 n^2 f_2 - \xi^2 4 f_2 = \xi^2 f_2 \left(n^2 - 4\right) = 0 \qquad f_2 = 0$ alebo $n = 2$

r^1: $\quad \xi^2 n^2 f_3 r + \xi^2 \frac{n 4e}{\hbar} f_1 a_1 r - \xi^2 9 f_3 r - f_1 r = 0$

$$r\left[\xi^2 f_3 + \xi^2 \frac{4e}{\hbar} f_1 a_1 - \xi^2 9 f_3 - f_1\right] = 0$$

$$f_3 - 9 f_3 = \frac{f_1}{\xi^2} - \frac{4e}{\hbar} f_1 a_1$$

$$f_3 = -\frac{f_1}{8}\left[\frac{1}{\xi^2} - \frac{4e}{\hbar} a_1\right]$$



$$f(r) = f_1 r + f_3 r^3 = f_1 r + \frac{f_1}{8}\left[\frac{4e}{\hbar} a_1 - \frac{1}{\xi^2}\right] r^3 =$$

$$= f_1 r \left[1 + \frac{r^2}{8}\left[\frac{4e}{\hbar} a_1 - \frac{1}{\xi^2}\right]\right] = f_1 r\left[1 - \frac{r^2}{8}\left(\frac{2 \cdot 2\pi}{\Phi} \frac{1}{2} B_z(0) + \frac{1}{\xi^2}\right)\right]$$

Dosadenie do 2. GL

$$\mu_0 j = \frac{-\hbar f^2}{2e \lambda^2} \frac{n}{r} - \frac{f^2}{\lambda^2} A$$

$$\nabla \times \vec{B} = -\frac{dB_z}{dr} = -\frac{d}{dr}\left(\frac{1}{r}\frac{d(rA)}{dr}\right) = -\frac{d}{dr}\left(\frac{1}{r}\left(\frac{d}{dr}\left(a_1 r^2 + a_2 r^3 + a_3 r^4 \dots\right)\right)\right) =$$

$$= -\frac{d}{dr}\left(\frac{1}{r}\left(2 a_1 r + 3 a_2 r^2 + 4 a_3 r^3 + \dots\right)\right) = -\left(3 a_2 + 8 a_3 r + 15 a_4 r^2 + \dots\right)$$

$$+\left(3 a_2 + 8 a_3 r + 15 a_4 r^2 + \dots\right) = \frac{\hbar}{2e\lambda^2} \frac{\left(f_1^2 r^2 + 2 f_1 f_2 r^3 \dots\right)}{r} + \frac{\left(f_1^2 r^2 + 2 f_1 f_2 r^3 \dots\right)\left(a_1 r + a_2 r^2 \dots\right)}{\lambda^2}$$

$$3 a_2 + 8 a_3 r + 15 a_4 r^2 \dots = \frac{\hbar}{2e\lambda^2}\left(f_1^2 r + 2 f_1 f_2 r^2 \dots\right) + \frac{\left(f_1^2 a_1 r^3 \dots\right)}{\lambda^2}$$

r^0: $\quad 3 a_2 = 0 \qquad a_2 = 0 \Rightarrow b_1 = 0$ vypadne lineárny člen

r^1: $\quad 8 a_3 r = \frac{\hbar}{2e\lambda^2} f_1^2 r \Rightarrow a_3 = \frac{1}{8}\frac{\hbar}{2e}\frac{1}{\lambda^2} f_1^2 \qquad b_2 = 4 a_3 = \frac{4}{8}\frac{h}{2\pi\, 2e}\frac{1}{\lambda^2} f_1^2 =$

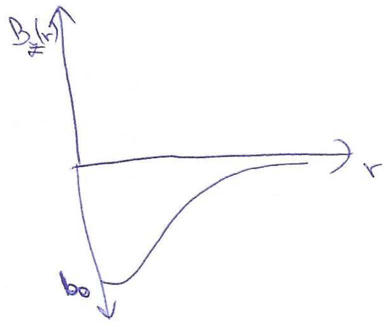
$$= \frac{1}{2}\frac{\Phi}{2\pi}\left(\frac{f_1}{\lambda}\right)^2$$

r^2: $\quad 15 a_4 r^2 = \frac{\hbar}{2e\lambda^2} 2 f_1 \underset{0}{\underset{\parallel}{f_2}} r^2 \Rightarrow a_4 = 0$

$b_0 = 1$?

$$a(r) = \frac{1}{2} b_0 r + a_3 r^3 = \frac{1}{2} b_0 r + \frac{1}{8}\frac{\hbar}{2e}\frac{1}{\lambda^2} f_1^2 r^3 = r\left[\frac{1}{2} + \frac{1}{8}\frac{\hbar}{2e}\frac{f_1^2}{\lambda^2} r^2\right]$$

$$B_z(r) = b_0 + b_2 r^2 = \underbrace{b_0}_{\text{záporné}} + \frac{\Phi}{4\pi} \frac{j_1^2}{\lambda^2} r^2$$



b) $f \approx 1$

na rovnici pro proudovou hustotu zapůsobíme operátorem rotace a obdržíme tak rovnici pro $B_z(r)$. Ukažte, že jejím řešením je Hankelova komplexní funkce ~~pozměná~~ argumentu a magnetické pole ji můžeme zapsat ve tvaru $B_z(r) = \frac{\Phi_0}{2\pi\lambda^2} K_0\left(\frac{r}{\lambda}\right)$

$$\text{rot}\,(\text{rot}\, B_z) = \text{rot}\left[\frac{\hbar f^2}{2e\lambda^2}\frac{n}{r} - \frac{f^2}{\lambda^2}A\right]\vec{e_r}$$

$$\text{rot}\left(0, -\frac{\partial B_z}{\partial r}, 0\right) = \frac{1}{r}\frac{d}{dr} r\left(-\frac{dB_z(r)}{dr}\right) = -\nabla^2 \vec{B}$$

$$\text{rot}\left[\frac{\hbar f^2}{2e\lambda^2}\frac{n}{r} - \frac{f^2}{\lambda^2}A\right]\vec{e_r} = \underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\hbar}{2e\lambda^2 r}\right)}_{0} - \frac{1}{\lambda^2}\underbrace{\frac{1}{r}\frac{d}{dr}(rA)}_{B_z(r)}$$

$$-\nabla^2\vec{B} = -\frac{1}{\lambda^2}B_z(r)$$

$$-\nabla^2\vec{B} = -\frac{1}{r}\frac{dB_z(r)}{dr} - \frac{1}{r}\cdot r\frac{d^2B_z(r)}{dr^2}$$

$$B_z''(r) + \frac{1}{r}B_z'(r) - \frac{1}{\lambda^2}B_z(r) = 0$$

Bessel function

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0 \qquad / x^2$$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{\alpha^2}{x^2}\right)y = 0 \Rightarrow J_\alpha(x),\ Y_\alpha(x)$$

$$J_0,\ Y_0\left(\frac{i}{\lambda}r\right)$$

$$B_z(r) = \frac{\Phi}{2\pi\lambda^2}K_0\left(\frac{r}{\lambda}\right)$$