$$\lambda_1 \quad [u_1] \quad \varphi([u_1]^{\perp}) \subseteq [u_1]^{\perp}$$

$$\lambda = a + ib \qquad u = u_1 + i u_2$$

$$\bar{\lambda} = a - ib \qquad \bar{u} = u_1 - i u_2$$

$$\bar{A}\left(\overline{u_1 + i u_2}\right) = \bar{\lambda}\left(\overline{u_1 + i u_2}\right)$$

A''

$$\bar{A}^T A = I_n$$

$$\lambda = a + ib$$

$$A(u_1 + i u_2) = \lambda (u_1 + i u_2)$$

$$\overline{}$$

$$A(u_1 - i u_2) = \bar{\lambda} (u_1 - i u_2)$$

$$\langle \hat{u}_1 + i u_2 , u_1 - \widehat{i u_2} \rangle = 0$$

$$\langle u_1, u_1 \rangle = \langle u_2, u_2 \rangle$$

$$- \langle u_1, u_2 \rangle + \langle u_2, u_1 \rangle = 0$$

$$Au_1 \in [u_1, u_2]$$

$$Au_2$$

$$\underline{Au_1} + i\,\underline{Au_2} = (a+ib)(u_1 + iu_2)$$

$$au_1 - bu_2$$

$$i(\underline{bu_1 + au_2})$$


π_1
π_2
π_n

$$A \qquad A^* = \bar{A}^T$$

$$u = \sum_{i=1}^{n} c_i u_i, \quad v = \sum_{j=1}^{n} d_j v_j.$$

$$\langle u, v \rangle = \sum_{i,j} c_i \bar{d}_j \langle u_i, u_j \rangle =$$

$$= \sum_{i=1}^{n} c_i \bar{d}_i$$

$$\langle \varphi(u), v \rangle = \langle u, \varphi(v) \rangle$$

$$\varphi(V) \subseteq V \overset{?}{\Longrightarrow} \varphi(V^\perp) \subseteq V^\perp$$

$$v \in V^\perp \Longrightarrow \varphi(v) \in V^\perp$$

$$u \in V \overset{?}{\Longrightarrow} \langle u, \varphi(v) \rangle = 0$$

$$\langle u, \varphi(v) \rangle = \langle \underbrace{\varphi(u)}_{\in V}, v \rangle = 0.$$

$\varphi(u) = \lambda u \qquad \lambda \overset{?}{\in} \mathbb{R}$

$u \neq 0$

$\langle \varphi(u), u \rangle = \langle u, \varphi(u) \rangle$

$\langle \lambda u, u \rangle = \langle u, \lambda u \rangle$

$\lambda \langle u, u \rangle = \overline{\lambda} \langle u, u \rangle$

$\lambda = \overline{\lambda}$

$$\varphi(u) = \lambda_1 u \qquad u, v \neq 0$$

$$\varphi(v) = \lambda_2 v \qquad \lambda_1 \neq \lambda_2$$

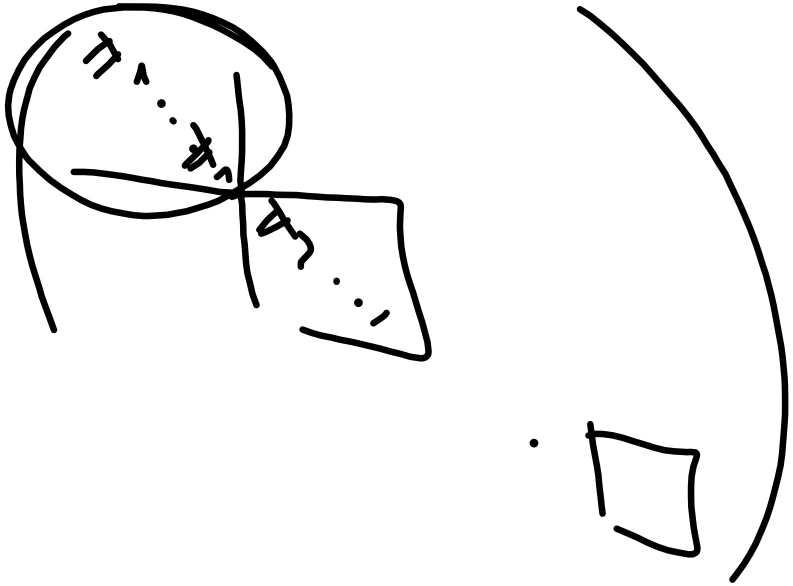
$$\langle \varphi(u), v \rangle = \lambda_1 \langle u, v \rangle$$

$$\|$$

$$\langle u, \varphi(v) \rangle = \lambda_2 \langle u, v \rangle$$

$$\underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \underbrace{\langle u, v \rangle}_{\Rightarrow = 0} = 0$$

$X_1 \ldots, X_n$



$$\langle (\varphi^* \circ \varphi)(u), v \rangle \stackrel{?}{=} \langle u, (\varphi^* \circ \varphi) v \rangle$$

$$\shortparallel$$

$$\varphi \mapsto \varphi^* \mapsto \varphi^{**} = \varphi$$

$$\langle \varphi(u), v \rangle = \underline{\langle u, \varphi^*(v)} \rangle$$

$$\langle \underline{\varphi^*} \underline{(v)}, u \rangle = \langle v, \varphi^{**}(u) \rangle$$

$$\langle \underline{\varphi}(u), v \rangle = \langle \underline{\varphi^{**}}(u), v \rangle$$

$$\langle (\varphi^* \circ \varphi)(u), v \rangle =$$

$$= \langle \varphi(u), \varphi^{**}(v) \rangle =$$

$$= \langle \varphi(u), \varphi(v) \rangle = \langle u, (\varphi^* \circ \varphi)(v) \rangle$$

$$\langle (\varphi^* \circ \varphi)(u), u \rangle =$$

$$= \langle \varphi(u), \varphi^{**}(u) \rangle =$$

$$= \langle \varphi(u), \varphi(u) \rangle \geq 0$$

$$\lambda, u \neq 0 \quad (\varphi^* \circ \varphi)(u) = \lambda u$$

$$\langle (\varphi^* \circ \varphi)(u), u \rangle = \langle \lambda u, u \rangle =$$

$$= \lambda \langle u, u \rangle \geq 0$$

$$\operatorname{Ker}(\varphi^* \circ \varphi) = \operatorname{Ker}\varphi$$

$$x \in \operatorname{Ker}\varphi \Rightarrow \varphi(x) = 0$$

$$\Rightarrow \varphi^* \varphi(x) = 0 \Rightarrow x \in \operatorname{Ker}(\varphi^* \circ \varphi)$$

$$x \in \operatorname{Ker}(\varphi^* \circ \varphi) \overset{?}{\Rightarrow} x \in \operatorname{Ker}\underline{\varphi}$$

$$\Downarrow \quad (\varphi^* \circ \varphi)(x) = 0$$

$$0 = \langle (\varphi^* \circ \varphi)(x), x \rangle = \langle \varphi(x), \varphi(x) \rangle$$

$$\Rightarrow \varphi(x) = 0.$$

$$A = P \cdot S \cdot Q^*$$

$$A^* A = Q S^* \underbrace{P^* P}_{= I_m} S Q^*$$

$$\left(\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right) \cdot \left(\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right)$$

$$= \begin{pmatrix} \sigma_1^2 & & & 0 \\ & \ddots & & \\ & & \sigma_r^2 & \\ 0 & & & 0 \end{pmatrix}$$

$$A v_1 = \sigma_1 u_1$$

$$r_1(AQ) = r_1(P \cdot S)$$

$$\parallel$$

$$A\, r_1(Q) = P\, r_1(S)$$

$$= \sigma_1 r_1(P)$$

$$\langle u_i, u_j \rangle = \left(\frac{1}{\sigma_i}(A \cdot v_i)\right)^T \frac{1}{\sigma_j} A v_j =$$

$$= \frac{1}{\sigma_i} \frac{1}{\sigma_j} \cdot v_i^T A^T \cdot A v_j \cdot =$$

$$= \frac{\sigma_j^2}{\sigma_i \sigma_j} \underbrace{v_i^T v_j}_{0} = 0$$

$$A=\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix} \quad A^T A=\begin{pmatrix}2 & 4\\ 4 & 8\end{pmatrix}$$

$$\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}=\begin{pmatrix}2 & 4\\ 4 & 8\end{pmatrix}$$

$$(2-\lambda)(8-\lambda)-16=-10\lambda+\lambda^2$$

$$\lambda_2=0,\ \lambda_1=10 \qquad S=\begin{pmatrix}\sqrt{10} & 0\\ 0 & 0\end{pmatrix}$$

$$\begin{pmatrix} 2-10 & 4 \\ 4 & 8-10 \end{pmatrix} \sim \begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix} \sim$$

$$\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \quad \frac{1}{\sqrt{5}}(1,2) = v_1$$

$$\frac{1}{\sqrt{5}}(-2,1) = v_2$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} \sqrt{5} \\ \sqrt{5} \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = u_1, \quad u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$