$$x \perp 0$$

$$0 + \cancel{\langle x, 0 \rangle} = \langle x, 0 + 0 \rangle =$$

$$= \langle x, 0 \rangle + \cancel{\langle x, 0 \rangle}$$

$$\Rightarrow \langle x, 0 \rangle = 0$$

(b) $\langle x, x \rangle = 0 \quad \substack{\Longrightarrow \\ \Longleftarrow} \quad x = 0$

$\langle 0, 0 \rangle = 0$

(c) $\overset{x \perp y}{\langle x, y \rangle = 0} \iff \overline{\langle x, y \rangle} = \overline{0}$

$\iff \underset{y \perp x}{\langle y, x \rangle = 0}$

(a) $x \perp y,\ x \perp z,\ c, d \in \mathbb{R}$

$$\overset{?}{\Longrightarrow}\ x \perp (cy + dz)$$

$$\langle x, cy + dz\rangle = \underbrace{\underbrace{c\,\underbrace{\langle x, y\rangle}_{= 0}}_{0} + \underbrace{d \cdot \underbrace{\langle x, z\rangle}_{= 0}}_{0}}_{0}$$

(a) $\emptyset^\perp = \{0\}^\perp = V$

$\emptyset^\perp = \{y \in V \mid y \perp x \ \forall x \in \emptyset\}$

$= V$

$\{0\}^\perp = \{y \in V \mid y \perp 0\} = V$

$V^\perp = \{y \in V \mid y \perp x \ \forall x \in V\}$

$= \{0\}$

$y = 0 \Leftarrow \langle y, y \rangle = 0$

$$(b)\ X^\perp = [X]^\perp = [X^\perp]$$

$$X^\perp = \{y \in V : y \perp x \ \forall x \in X\}$$

$$\overset{?}{\parallel}$$

$$[X]^\perp = \{y \in V : y \perp r \ \forall r \in [X]\}$$

$$y \in X^\perp \Rightarrow y \perp x \ \forall x \in X \Rightarrow$$

$$y \perp r \ \forall r \in [X]$$

$$r = \sum_{i=1}^{m} c_i x_i \quad x_i \in X \qquad y \perp x_i \ \forall i \Rightarrow y \perp r$$

$$0 \in X^{\perp} = [X^{\perp}]$$ ✓

$$\forall y_1, y_2 \in X^{\perp}, \; c, d \in \mathbb{R}$$

$$y_i \perp x \quad \forall x \in X, \; 1 \leq i \leq 2$$

$$c y_1 + d y_2 \perp x \quad \forall x \in X$$

(c) $X \subseteq Y \Rightarrow Y^\perp \subseteq X^\perp$

$\pi \in Y^\perp \Rightarrow \pi \perp y \quad \forall y \in Y$

$$\begin{array}{c} Y \\ \rotatebox{90}{$\subseteq$} \\ X \end{array}$$

$\pi \in Y^\perp \Rightarrow \underbrace{\pi \perp x, \forall x \in X}_{\pi \in X^\perp}$

(a) $X \subseteq X^{\perp\perp}$

$x \in X \Rightarrow x \perp X^{\perp}$

$\Rightarrow x \in X^{\perp\perp}$

$X \perp Y \Leftrightarrow \forall x \in X, \forall y \in Y \; x \perp y$

(e) $X^{\perp} \textcircled{=} X^{\perp\perp\perp}$

$X \subseteq X^{\perp\perp} \Longrightarrow X^{\perp\perp\perp} \textcircled{$\subseteq$} X^{\perp}$

$X^{\perp\perp\perp} = (X^{\perp})^{\perp\perp} \textcircled{$\supseteq$} X^{\perp}$

(β)

$$y \in X \cap X^{\perp} \iff y \in X \quad y \perp x \ \forall x \in X$$

$$\implies y \perp y \implies \underline{\underline{y = 0}}$$

$(X\cup Y)^\perp \overset{?}{\underset{\subseteq}{=}} X^\perp \cap Y^\perp$

$X \subseteq X\cup Y \Rightarrow (X\cup Y)^\perp \subseteq X^\perp$

$\subseteq Y^\perp$

$\Rightarrow (X\cup Y)^\perp \subseteq X^\perp \cap Y^\perp$

$\pi \in X^\perp \cap Y^\perp \Rightarrow \pi \perp x \ \forall x \in X$

$\pi \perp y \ \forall y \in Y$

$\Rightarrow \pi \perp u, \forall u \in X\cup Y$

bře 26-14:42

$$(X \cup Y)^{\perp} = (X + Y)^{\perp}$$

kde X, Y jsou <u>VP</u>.

$$[X \cup Y] = X + Y$$

$$(X+Y)^{\perp} = [X \cup Y]^{\perp} = [(X \cup Y)^{\perp}] =$$

$$= [X^{\perp} \cap Y^{\perp}] = X^{\perp} \cap Y^{\perp}$$

(a) $S \subseteq V$ $\dim S = k$

$u_1, \dots, u_k$ je ON báze S

$x \in V$ $\quad x_S = \sum_{i=1}^{k} \langle x, u_i \rangle u_i$

$$\boxed{x - x_S \perp S}$$

$$x_S = \sum_{i=1}^{k} c_i u_i$$

$$x - \sum_{i=1}^{k} c_i u_i \perp u_j \quad \forall j, 1 \le j \le k$$

$$\langle x - \sum_{i=1}^{k} c_i u_i , u_j \rangle = 0$$

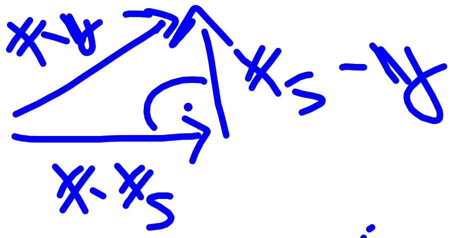
$$\langle x, u_j \rangle - \sum_{i=1}^{k} c_i \langle u_i, u_j \rangle = 0$$

$$\underline{\underline{\langle x, u_j \rangle}} = c_j \underbrace{\langle u_j, u_j \rangle}_{1} = c_j$$

(b) $\|x - x_S\| \le \|x - y\|$

$x_S \in S,$ $\forall y \in S$

$\Rightarrow x_S - y \in S$, $\underline{\underline{x - x_S}} \in S^\perp$



$$\|x - x_S\|^2 + \underbrace{\|x_S - y\|^2}_{\ge 0}$$

$$= \|x - y\|^2$$

$$\|x - y\|^{\cancel{2}} \ge \|x - x_S\|^{\cancel{2}}$$

$$\underline{\underline{\frac{|\langle \mathbf{x}, \mathbf{y} \rangle|}{\|\mathbf{x}\| \, \|\mathbf{y}\|}}} \leq \frac{\|\mathbf{x}_s\|}{\|\mathbf{x}\|}$$

$$\frac{\langle \mathbf{x}, \mathbf{x}_s \rangle}{\|\mathbf{x}\| \, \|\mathbf{x}_s\|} = \frac{\langle \mathbf{x}_s + (\mathbf{x} - \mathbf{x}_s), \mathbf{x}_s \rangle}{\|\mathbf{x}\| \, \|\mathbf{x}_s\|} =$$

$$\|\mathbf{x}_s\|^2 =$$

$$= \frac{\langle \mathbf{x}_s, \mathbf{x}_s \rangle + \overbrace{\langle \mathbf{x} - \mathbf{x}_s, \mathbf{x}_s \rangle}^{0}}{\|\mathbf{x}\| \quad \|\mathbf{x}_s\|} =$$

$$= \frac{\|\mathbf{x}_s\|}{\|\mathbf{x}\|}$$

$$y \in S \quad y \perp x - x_s$$
$$y \neq 0$$

$$\langle x, y \rangle = \langle x_s + (x - x_s), y \rangle =$$

$$= \langle x_s, y \rangle$$

$$\frac{|\langle x, y \rangle|}{\|x\| \, \|y\|} = \frac{|\langle x_s, y \rangle|}{\|x\| \, \|y\|} \underset{=}{\leq} \frac{\|x_s\| \cdot \cancel{\|y\|}}{\|x\| \cdot \cancel{\|y\|}} = \frac{\|x_s\|}{\|x\|}.$$

(a) $S = S^{\perp\perp}$, $V = S \oplus S^{\perp}$

$(S \cap T)^{\perp} = S^{\perp} + T^{\perp}$

$S \subseteq S^{\perp\perp}$, $S = [\underbrace{u_1, \dots, u_k}_{ON}]$

$r \in S^{\perp\perp} \overset{?}{\Longrightarrow} r \in S$

$$r = \underset{\in S}{r_S} + \underset{\in S^{\perp}}{(r - r_S)} \quad / u_i$$

$$\boxed{\|r\|^2} \underset{\leq}{=} \boxed{\|r_S\|^2} + \|\underset{\overset{?}{=}0}{r - r_S}\|_V^2$$

$$\langle \mathbb{R}, \mathbb{R} \rangle = \langle \mathbb{R}, \mathbb{R}_S + \underbrace{(\mathbb{R} - \mathbb{R}_S)}_{\in S^\perp} \rangle$$

$$\mathbb{R} \in S^{\perp\perp}, \; \mathbb{R} \perp S^\perp$$

$$= \langle \mathbb{R}, \mathbb{R}_S \rangle = \langle \mathbb{R}_S + \underbrace{(\mathbb{R} - \mathbb{R}_S)}_{\in S^\perp}, \underbrace{\mathbb{R}_S}_{\in S} \rangle$$

$$= \langle \mathbb{R}_S, \mathbb{R}_S \rangle \Rightarrow \mathbb{R} - \mathbb{R}_S = 0 \Rightarrow S \ni \mathbb{R} = \mathbb{R}_S$$

$$V = S \oplus S^\perp$$

$$\forall x = x_S + \underbrace{(x - x_S)}_{\in S^\perp}$$

$S, S^\perp$ nadp. $\quad S \cap S^\perp = \{0\}$

$$(S \cap T)^\perp = S^\perp + T^\perp$$

$$(X+Y)^\perp = X^\perp \cap Y^\perp \quad , \quad S = S^{\perp\perp}$$

$$T = T^{\perp\perp}$$

$$\left((S^\perp)^\perp \cap (T^\perp)^\perp\right)^\perp = \left(S^\perp + T^\perp\right)^{\perp\perp}$$

$$\supseteq S^\perp + T^\perp$$

$$\underset{?}{=}$$

$$S^\perp + T^\perp \underset{\subseteq}{\overset{?}{=}} (S^\perp + T^\perp)^{\perp\perp}$$

$$\mathbf{x} \in S \cap T$$

$$\mathbf{x} = \mathbf{x}_{S^\perp + T^\perp} + \underbrace{(\mathbf{x} - \mathbf{x}_{S^\perp + T^\perp})}_{\in (S^\perp + T^\perp)^\perp}$$

$$(S^\perp + T^\perp)^\perp = S \cap T = S^{\perp\perp} \cap T^{\perp\perp}$$

$$\|\mathbf{x}\|^2 = \|\mathbf{x}_{S^\perp + T^\perp}\|^2 + \underbrace{\|\mathbf{x} - \mathbf{x}_{S^\perp + T^\perp}\|^2}_{= 0}$$

$$\Rightarrow \mathbf{x} = \mathbf{x}_{S^\perp + T^\perp} \in S^\perp + T^\perp$$

$$\langle \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{x}_{S^\perp+T^\perp} + (\mathbf{x} - \mathbf{x}_{S^\perp+T^\perp}) \rangle$$

$$\mathbf{x} \in (S^\perp + T^\perp)^{\perp\perp}$$

$$\mathbf{x} \perp (S^\perp + T^\perp)^\perp = S \cap T$$

$$= \langle \mathbf{x}, \mathbf{x}_{S^\perp+T^\perp} \rangle = \langle \mathbf{x}_{S^\perp+T^\perp} + \underline{(\mathbf{x} - \mathbf{x}_{S^\perp+T^\perp})}, \underline{\mathbf{x}_{S^\perp+T^\perp}} \rangle =$$

$$= \langle \mathbf{x}_{S^\perp+T^\perp}, \mathbf{x}_{S^\perp+T^\perp} \rangle \Rightarrow$$

(b) $p_S : V \to V$ je LZ

$(u_1, \ldots, u_k)$ ON S

$$p_S(x) = \sum_{i=1}^{k} \langle x, u_i \rangle \cdot u_i$$

$x, y \in V$, $c, d \in \mathbb{R}$

$$p_S(x+y) = \sum_{i=1}^{k} \langle x+y, u_i \rangle u_i =$$

$$= \sum_{i=1}^{k} \left(\langle x, u_i \rangle + \langle y, u_i \rangle \right) \cdot u_i =$$

$$= \sum_{i=1}^{k} \langle x, u_i \rangle u_i + \sum_{i=1}^{k} \langle y, u_i \rangle u_i = p_S(x) + p_S(y)$$

$$x \in S \Leftrightarrow \mathrm{pr}_S(x) = x$$

$$\Leftarrow$$

$$S \ni \mathrm{pr}_S(x) = x \underline{\in S}$$

$$\Rightarrow \quad x \in S, \quad x = \sum_{i=1}^{k} c_i u_i$$

$$\mathrm{pr}_S(x) = \sum_{j=1}^{k} \left\langle \sum_{i=1}^{k} c_i u_i, u_j \right\rangle u_j$$

$$= \sum_{j=1}^{k} c_j \underbrace{\langle u_j, u_j \rangle}_{=1} \cdot u_j = \sum_{j=1}^{k} c_j u_j = x$$

$\operatorname{Im} p_S = S, \ \operatorname{Ker} p_S = S^\perp$

$x \in S \Rightarrow x = p_1(x)$

$S \supseteq \operatorname{Im} p_S \supseteq S$

$x \in \operatorname{Ker} p_S \Leftrightarrow p_S(x) = 0$

$\Leftrightarrow x = \underbrace{x - p_S(x)}_{\in S^\perp} \Leftrightarrow x \in S^\perp$

(e) $\mathbb{x} - \mathbb{x}_S$ je kolmý průmět do $S^\perp$

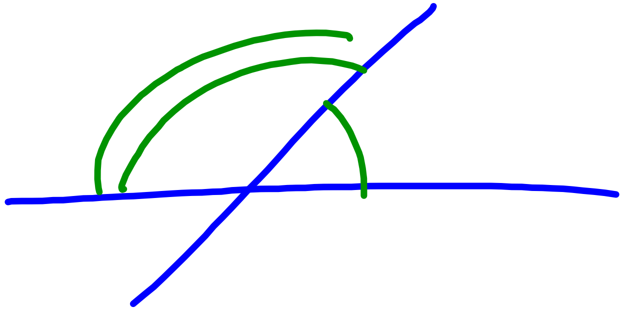
$$\mathbb{x} = x_S + (\mathbb{x} - \mathbb{x}_S)$$

$$\in S = (S^\perp)^\perp \qquad \in S^\perp$$

$$\mathbb{x}_S = \mathbb{x} - (\mathbb{x} - \mathbb{x}_S) \in S^{\perp\perp}$$

$$\frac{|\langle x, y\rangle|}{\|x\|\,\|y\|} \le \frac{\|x_s\|}{\|x\|}$$

$$\underline{\measuredangle(x_s, x)} \le \measuredangle(x, y)$$



$S \subseteq V$ $\quad \{u_1, \ldots, u_k\}$ báze

$x \in V$

$$x_S = \sum_{i=1}^{k} c_i u_i \qquad (u_1, \ldots, u_k) = \alpha$$

$$G(\alpha) \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} \langle x, u_1 \rangle \\ \vdots \\ \langle x, u_k \rangle \end{pmatrix}$$

$$x - x_S \perp S \iff x - x_S \perp u_i \quad \forall i = 1..k$$

$$\langle x - \sum_{j=1}^{k} c_j u_j , u_i \rangle = 0 \quad \forall i$$

$$\langle x, u_i \rangle - \sum_{j=1}^{2} c_j \langle u_j, u_i \rangle = 0$$

$$\langle x, u_i \rangle = \underset{\underset{r_i(G(\alpha))}{\parallel}}{r_i(G(\alpha))^T} \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_2 \end{pmatrix} \quad \forall i$$

$$G(\alpha) \begin{pmatrix} c_1 \\ \vdots \\ c_2 \end{pmatrix} = \begin{pmatrix} \langle x, u_1 \rangle \\ \vdots \\ \langle x, u_2 \rangle \end{pmatrix}.$$

pro α z ON : $G(\alpha) = id = I_2$.

$\mathbb{R}^4$, $x=(1,1,1,1)^T$

$S=[u,v]$, $u=(0,-1,0,1)^T$,

$v=(1,-2,1,-3)^T$

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 15 & -3 \\ c & d & \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -15 & 3 \\ 0 & 29 & -6 \end{array}\right)$$

$d = -\frac{6}{29}$

$$c = 3 + 15\left(-\frac{6}{29}\right) = \frac{87-90}{29} = -\frac{3}{29}$$

$$x_S = -\frac{3}{29}\begin{pmatrix}0\\-1\\0\\1\end{pmatrix} + \left(-\frac{6}{29}\right)\begin{pmatrix}1\\-2\\1\\-3\end{pmatrix} =$$

$$= \frac{1}{29}\begin{pmatrix}-6\\15\\-6\\15\end{pmatrix} = \frac{3}{29}\begin{pmatrix}-2\\5\\-2\\5\end{pmatrix}$$

$$\text{dist}(x, S) = \| x - x_S \| = \left\| \begin{pmatrix}35\\14\\35\\14\end{pmatrix} \right\| \cdot \frac{1}{29} =$$

$$= \frac{7}{29}\left\| \begin{pmatrix}5\\2\\5\\2\end{pmatrix} \right\| = \frac{7}{29}\sqrt{58}$$

$$\sin \measuredangle(x, S) = \frac{\| x - x_S \|}{\| x \|} =$$

$$= \frac{\frac{7}{29} \cdot \sqrt{58}}{2} = \frac{7}{58} \cdot \sqrt{58} = \frac{7}{\sqrt{58}}$$

$$\measuredangle(x, S) = 66^\circ 48' 5''$$

$$S = [s_1(A), \ldots, s_n(A)] \subseteq \mathbb{R}^m$$

$$\dim S = n$$

$$y \in S \iff \boxed{y = Ac} \text{ pro vhodné } c \in \mathbb{R}^n$$

$$G(A) \cdot c = (x^T A)^T = A^T x$$

$$c = G(A)^{-1} A^T x$$

$$G(A) = A^T A$$

$$m_S(x) = \boxed{A (A^T A)^{-1} A^T} x$$