# Bilinear and Quadratic Forms

$U$ vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

**First** Linear form

$$f : U \longrightarrow \mathbb{K}$$

such that $\quad f(au_1 + bu_2) = a f(u_1) + b f(u_2)$

for all $u_1, u_2 \in U$ and all $a, b \in \mathbb{K}$.

**Bilinear form** $\quad f : U \times U \longrightarrow \mathbb{K}$

which is linear in both variables:

$$f(au_1 + bu_2, v) = a f(u_1, v) + b f(u_2, v)$$

$$f(u, av_1 + bv_2) = a f(u, v_1) + b f(u, v)$$

**Examples**

1. $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$f(x, y) = \sum_{i,j=1}^{n} a_{ij} \cdot x_i \cdot y_j = \sum_{i=1}^{n} x_i \left( \sum_{j=1}^{n} a_{ij} \cdot y_j \right)$$

$$= (x_1, x_2, \dots x_n) \begin{pmatrix} \sum_j a_{1j} \cdot y_j \\ \vdots \\ \sum_j a_{nj} \cdot y_j \end{pmatrix} = (x_1, \dots x_n) \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$= x^T A y \quad \text{where } x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \; A = (a_{ij}), \; y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

2 $f : \mathbb{R}_4[x] \times \mathbb{R}_4[x] \longrightarrow \mathbb{R}$

$$f(p,q) = p'(1) \cdot q''(2)$$

$p'(1)$ ... the 1-st derivation in 1 of a polynomial $p$
$q''(2)$ ... the 2-nd derivation in 2 of a polynomial $q$

3 $C[a,b]$ .... continuous functions on the interval $[a,b]$.

$x \in C[a,b]$, $y \in C[a,b]$

$$f(x,y) = \int_a^b x(t) \cdot y(t)\, dt$$

is a bilinear form

MATRIX OF A BILINEAR FORM $f$ IN A BASIS

$U$ vector space with a basis $\alpha = (u_1, \dots u_n)$

$f : U \times U \longrightarrow \mathbb{K}$ bilinear form

$$A = (a_{ij})_{i,j=1}^{n} \qquad a_{ij} = f(u_i, u_j)$$

is called the matrix of the bilinear form $f$ in the basis $\alpha$.

$A = (a_{ij})$ ... matrix of $f$ in a basis $\alpha = (u_1, u_2, .. u_n)$

$B = (b_{ij})$ .... matrix of $f$ in a basis $\beta = (v_1, v_2, .., v_n)$

Then

$$f(u,v) = x^T A y = \bar{x}^T B \bar{y}$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ are coordinates of the vector $u$ in the basis $\alpha$

$$u = x_1 u_1 + x_2 u_2 + \dots + x_n u_n$$

similarly for $y$ and $v$,

and $\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix}$ are coordinates of the vector $u$ in the basis $\beta$

$$u = \bar{x}_1 v_1 + \bar{x}_2 v_2 + \dots + \bar{x}_n v_n.$$

Transition matrix $P$

$$x = P\bar{x}, \quad y = P\bar{y}$$

is denoted in my lectures as

$$(id)_{\alpha, \beta}$$

and defined by the formula

$$(v_1, v_2, \ldots v_m) = (u_1, u_2, \ldots u_n)\,(id)_{\alpha, \beta}$$

Vectors of the basis $\beta$ are expressed as linear combinations of the vectors of the basis $\alpha$.

Now:

$$\bar{x}^T B \bar{y} = f(u,v) = x^T A y = (P\bar{x})^T A (P\bar{y}) =$$
$$= \bar{x}^T (P^T A P)\, \bar{y}$$

which implies

$$B = P^T A P.$$

Formula expressing the matrix of $f$ in the basis $\beta$ with the help of the matrix of $f$ in the basis $\alpha$ and the transition matrix.

Symmetric bilinear form

$$f(u,v) = f(v,u)$$

Antisymetric bilinear form

$$f(u,v) = -f(v,u).$$

The matrix of a symmetric bilinear form is symmetric, i.e $A = A^T$, while the matrix of an antisymmetric bilinear form is antisymmetric, i.e. $A = -A^T$.

Quadratic form is a function

$$g : U \to \mathbb{K}$$

such that there is a symmetric bilinear form $f : U \times U \to \mathbb{K}$ and

$$g(u) = f(u,u).$$

Example: $g : \mathbb{R}^3 \to \mathbb{R}$, $g(x) = x_1^2 + 2x_1x_2 - 8x_1x_3 + 2x_3^2$ is a quadratic form defined by a bilinear form

$$f(x,y) = x_1y_1 + x_1y_2 + x_2y_1 - 4x_1y_3 - 4x_3y_1 + 2x_3y_3$$