Př: Charakterizace sférických křivek

Nechť $g(s)$ je parametrizace obloukem křivky bez inflexních bodů s křivostí $\kappa(s)$ a torzí $\tau(s) \neq 0$.

Dokažte, že tato křivka leží na nějaké sféře $\iff$

$$\boxed{\frac{\tau}{\kappa} + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)' = 0}$$

- $g(s) = \lambda(s)\, e_1(s) + \mu(s)\, e_2(s) + \nu(s)\, e_3(s)$ pro nějaké funkce λ, μ, ν

$$g'(s) = \lambda' e_1 + \lambda \kappa e_2$$
$$+ \mu' e_2 + \mu(-\kappa e_1 + \tau e_3)$$
$$+ \nu' e_3 + \nu(-\tau e_2)$$

$$= [\lambda' - \mu\varkappa]\, e_1 + [\mu' + \lambda\varkappa - \nu\tau]\, e_2 + [\nu' + \mu\tau]\, e_3 = e_1$$

$$\Rightarrow \quad \lambda' - \mu\varkappa = 1$$
$$\lambda\varkappa - \nu\tau + \mu' = 0 \quad /\cdot\mu$$
$$\mu\tau + \nu' = 0 \quad /\cdot\nu$$

"$\Longrightarrow$" Předp., že $g(s)$ leží na sféře se středem v počátku

$$\rightsquigarrow \lambda^2 + \mu^2 + \nu^2 = r^2 \quad / \frac{d}{ds}$$

↳ poloměr sféry

$$2\,(\underbrace{\lambda\lambda' + \mu\mu' + \nu\nu'}_{=0}) = 0$$

$$\underbrace{uu' + vv'}_{-\lambda\lambda'} = -\lambda u \varkappa$$

$$-\lambda\lambda' = -\lambda u \varkappa$$

$$0 = \lambda(\underbrace{\lambda' - u\varkappa}_{=1})$$

$$\Rightarrow \boxed{\lambda = 0}$$

$$\boxed{\begin{aligned} u\varkappa &= -1 \\ v\tilde{\tau} &= u' \\ u\tilde{\tau} &= -v' \end{aligned}}$$

$$\frac{1}{\varkappa} = -u$$

$$\left(\frac{1}{\varkappa}\right)' = -u'$$

$$\frac{1}{\tilde{\tau}} \cdot \left(\frac{1}{\varkappa}\right)' = \underbrace{\frac{v}{u'}}_{=\frac{1}{\tilde{\tau}}} \cdot (-u') = -v$$

$$\frac{\tilde{\tau}}{u\varkappa} = -\tilde{\tau}$$

$$\frac{\tilde{\tau}}{\varkappa} = -u\tilde{\tau} = v'$$

$$\frac{\tilde{\tau}}{\varkappa} + \left(\frac{1}{\tilde{\tau}} \cdot \left(\frac{1}{\varkappa}\right)'\right)' = v' - v' = 0$$

"$\Leftarrow$" kdyby křivka ležela na sféře, tak by střed byl

$$g(s) - \left(\lambda(s)e_1(s) + \mu(s)e_2(s) + \nu(s)e_3(s)\right)$$

$$\parallel$$

$h(s)$ $\rightsquigarrow$ pevný bod

"Uhádneme" střed sféry jako

$$h(s) = g(s) - \left[-\frac{1}{\kappa(s)}e_2(s) - \frac{1}{\tau}\left(\frac{1}{\kappa}\right)' e_3\right]$$

$\rightsquigarrow$ pak $\underline{h'(s) = 0}$

$$h' = \cancel{e_1} + \cancel{\left(\frac{1}{\kappa}\right)' e_2} + \frac{1}{\kappa}\left(-\kappa e_1 + \tau e_3\right) +$$

$$+ \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)' e_3 + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)' \cancel{\left(-\tau e_2\right)}$$

$$= \underbrace{\left[\frac{\tau}{\kappa} + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)'\right]}_{=0} e_3 = 0$$

Tedy $h(s)$ je pevný bod

Potřebujeme ověřit, že vektor

$$\frac{1}{\varkappa} e_2 + \frac{1}{\tau}\left(\frac{1}{\varkappa}\right)' e_3 \text{ má konst. délku}$$

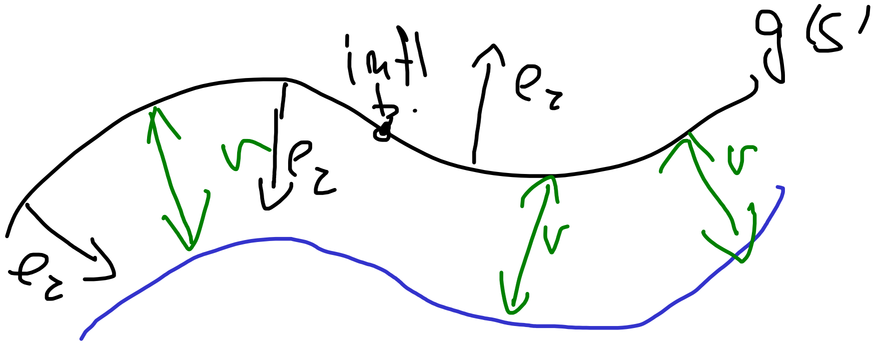
$$\frac{d}{ds}\left(\left(\frac{1}{\varkappa}\right)^2 + \left(\frac{1}{\tau}\left(\frac{1}{\varkappa}\right)'\right)^2\right) =$$

$$= 2\frac{1}{\varkappa}\cdot\left(\frac{1}{\varkappa}\right)' + 2\,\frac{1}{\tau}\left(\frac{1}{\varkappa}\right)'\cdot\left(\frac{1}{\tau}\cdot\left(\frac{1}{\varkappa}\right)'\right)'$$

$$= 2\left(\frac{1}{\varkappa}\right)'\cdot\frac{1}{\tau}\Big[\underbrace{\frac{\tau}{\varkappa} + \left(\frac{1}{\tau}\cdot\left(\frac{1}{\varkappa}\right)'\right)'}_{=0}\Big] = 0$$

$\square$

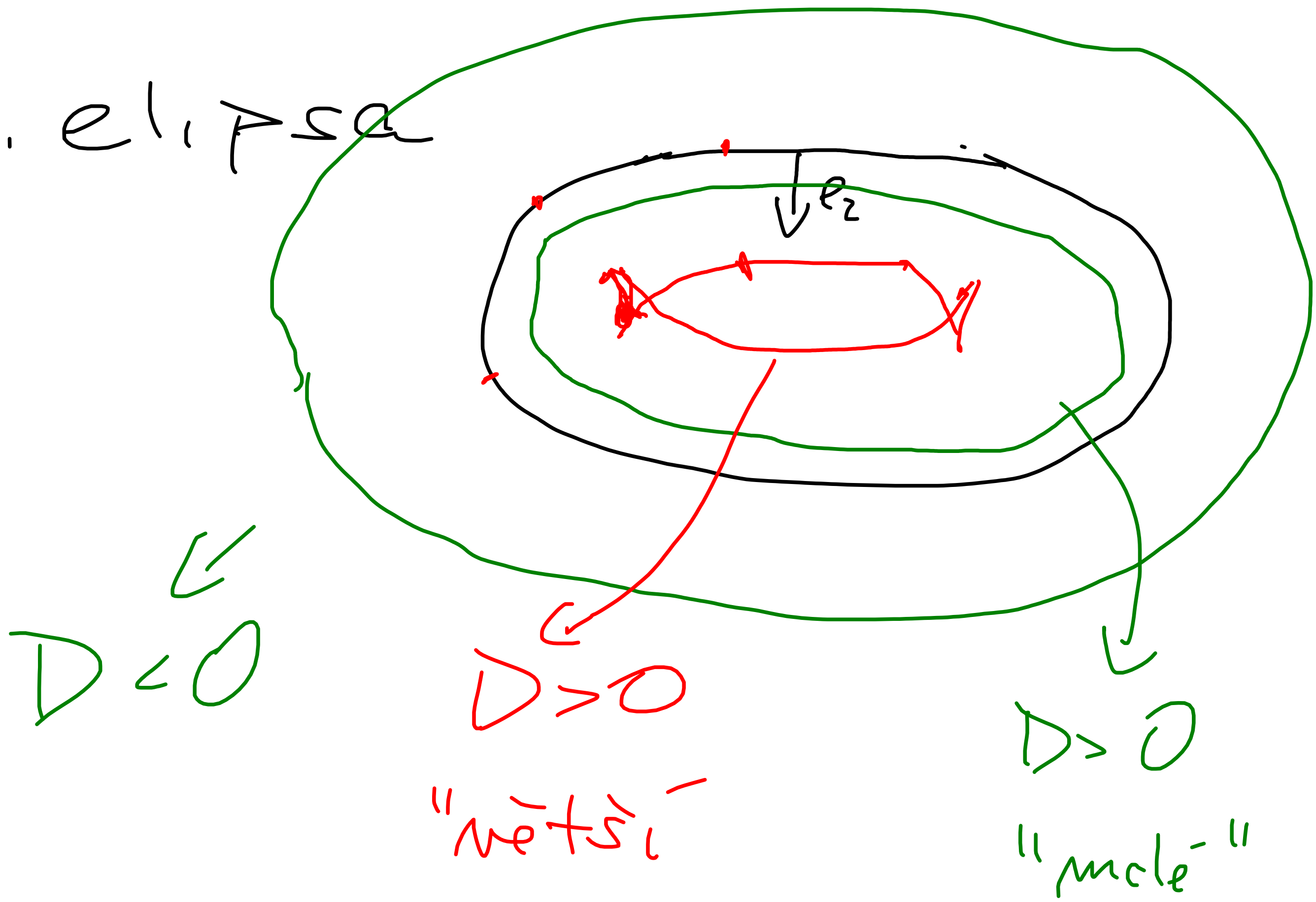
Př:



Nechť $g(s)$ je ~~pan.~~ rovinná křivky délk

bez infl. bodů. Pak křivku
$f(s) := g(s) + D e_2(s)$, $D \in \mathbb{R}$
nazveme paralelní křivka

Např. elipsa



(i) Určete, kdy je $f(s)$ skutečně křivka

(ii) Určete křivost $f(s)$

NB: $f(s)$ není par. obloukem

(i) $f'(s) = \left(g(s) + D e_2(s) \right)' =$

$= e_1(s) - D\varkappa(s) e_1(s)$

$= (1 - D\varkappa(s))\, e_1(s)$

$f'(s) \neq 0 \iff \varkappa(s) \neq \frac{1}{D}$

(ii) křivost $f(s)$ je

$$\omega(s) = \frac{|\det(f'(s), f''(s))|}{\|f'(s)\|^3} =$$

| $\varkappa$ – křivost $g(s)$ |
| ω – –"– $f(s)$ |

$f'(s) = (1 - D\varkappa(s))\, e_1(s)$

$f''(s) = (1 - D\varkappa'(s))\, e_1(s)$

$+ (1 - D\varkappa(s))\, \varkappa\, e_2(s)$

$$= \frac{(1 - D\varkappa(s))^2 \varkappa(s)}{|1 - D\varkappa(s)|^3} = \frac{\varkappa(s)}{|1 - D\varkappa(s)|}$$

$$\det(a e_1, b e_1 + c e_2) =$$
$$= \det(a e_1, c e_2) = ac \underbrace{\det(e_1, e_2)}_{=1}$$