<u>observation:</u> any additive functor preserves split short exact sequences (they are essentially biproducts)

$0 \to P_n \to R_n \to Q_n \to 0$ allways splits since Q_n is projective

$\Rightarrow 0 \to FP_n \to FR_n \to FQ_n \to 0$ is also exact.

$0 \to P_1 \to P_1 \oplus Q_1 \to Q_1 \to 0$ ... ↑continue in the same way

$\Rightarrow$ enough to find R_1 because P, Q could be chosen arbitrarily.

$$\begin{array}{ccccccccc} 0 & \to & \ker \varepsilon & \to & \ker(\varepsilon\ \tilde{\eta}) & \to & \ker \eta & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & P_0 & \to & P_0 + Q_0 & \to & Q_0 & \to & 0 \\ & & \varepsilon \downarrow & & (\varepsilon\ \tilde{\eta}) \downarrow & \overset{\tilde{\eta}}{\swarrow} & \downarrow \eta & & \\ 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \end{array}$$

← by snake lemma, since $\operatorname{coker} \varepsilon = 0$ (ε is surjective)

take kernels

$$\begin{array}{ccc} (x,y) & \longmapsto & y \\ \downarrow & & \downarrow \\ f\varepsilon x + f\tilde{\eta} y & & \eta y \\ & \searrow & \| \\ & & \underbrace{g f \varepsilon x}_{0} + \underbrace{g \tilde{\eta} y}_{\eta} \end{array}$$

- there exists $\tilde{\eta}: Q_0 \to B$ such that $g\tilde{\eta} = \eta$

We obtain

$$\begin{array}{ccccccccc} 0 & \to & P & \to & R & \to & Q & \to & 0 \\ & & \downarrow \wr & & \downarrow & & \downarrow \wr & & \\ 0 & \to & A[0] & \to & B[0] & \to & C[0] & \to & 0 \end{array}$$

By 5-lemma applied to the resulting long exact sequence of homology groups.

$$\begin{array}{ccccccccccc} \to & H_{n+1} Q & \to & H_n P & \to & H_n R & \to & H_n Q & \to & H_{n-1} P & \to \dots \\ & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\ \to & H_{n+1} C[0] & \to & H_n A[0] & \to & H_n B[0] & \to & H_n C[0] & \to & H_{n-1} A[0] & \to \dots \end{array}$$

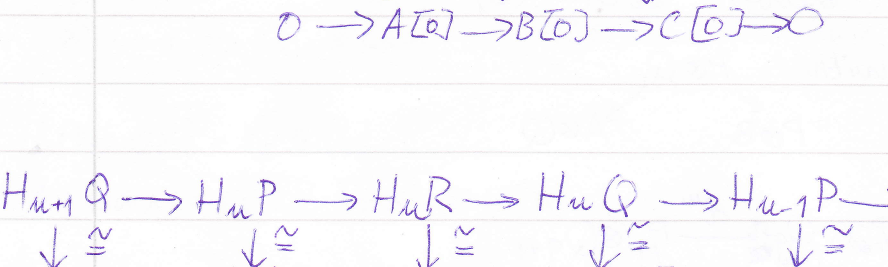
(Ex) $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B)$

- compute it from $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$

$$\dots \to \underset{\overset{\|}{0}}{\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Q}, B)} \to \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B) \to \mathbb{Z} \otimes_{\mathbb{Z}} B \to \mathbb{Q} \otimes_{\mathbb{Z}} B \to \mathbb{Q}/\mathbb{Z} \otimes B \to 0$$

$\Rightarrow \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B) = \ker(B \longrightarrow \mathbb{Q} \otimes B)$ $\qquad \mathbb{Q} = \operatorname{colim} \mathbb{Z}$

$\searrow \{b \in B \mid \exists m \in \mathbb{N}: mb = 0\} = TB$



<u>Preexample:</u> $\operatorname{Tor}_1^{\mathbb{Z}}(A, B) = 0$ if A is flat

↑ a derived functor of $A \otimes_R -$, which is exact (next time)

Chceme ukázat, že 2 různé rezolventy dají izomorfní derivované funktory

Balancing of Tor and Ext

Let $P \xrightarrow{\sim} A[0]$, $Q \xrightarrow{\sim} B[0]$ be projective resol.

Then $H_n(P \otimes_R B) \cong H_n(A \otimes_R Q)$

$\quad\; \| \qquad\qquad\qquad\qquad \|$

$\mathrm{Tor}_m^R(A,B) \qquad \mathrm{Tor}_m^R(A,B)$

$P_1 \to P_0 \to A$

$\mathrm{Hom}(P_1,B) \leftarrow \mathrm{Hom}(P_0,B) \leftarrow \mathrm{Hom}(A,B)$

Let $P \xrightarrow{\sim} A[0]$ be a proj. res.

and $B[0] \xrightarrow{\sim} E$ be an inj. res. — nemá tu být třeba Q? nebo E?

$H^n \mathrm{Hom}(P,B) \cong H^n \mathrm{Hom}(A,B) = \mathrm{Ext}^n_R(A,B)$

$\mathrm{Ext}^1_R(A,B) = \{0 \to B \to ? \to A \to 0\}/\cong$

Main goal: A_R (right R-module) ${}_R B$ (left R-module)

take $P \xrightarrow{\sim} A$, $Q \xrightarrow{\sim} B$ projective resolutions

Then $H_n(P \otimes B) \cong H_n(A \otimes Q)$ (over R)

$\qquad\; \| \qquad\qquad\qquad \|$

$L_n(- \otimes B)(A) \quad L_n(A \otimes -)(B)$

Main idea: compare with

$$\begin{array}{ccc} & P\otimes Q & \\ \swarrow & & \searrow \\ P\otimes B & & A\otimes Q \end{array}$$

What is $P \otimes Q$?

$$\begin{array}{ccccc}
P_{p-1}\otimes Q_{q+1} & \xleftarrow{d^h} & P_p \otimes Q_{q+1} & \xleftarrow{d^h} & P_{p+1}\otimes Q_{q+1} \\
\downarrow d^v & & \downarrow d^v & & \downarrow d^v \\
P_{p-1}\otimes Q_q & \xleftarrow{d^h} & P_p \otimes Q_q & \xleftarrow{d^h} & P_{p+1}\otimes Q_{q+1} \\
 & & \downarrow d^v & & \downarrow d^v \\
 & & P_{p-1}\otimes Q_q & \xleftarrow{d^h} & P_{p+1}\otimes Q_{q-1}
\end{array}$$

Def. A **double (chain) complex** is a collection of modules $C_{p,q}$ $p,q \in \mathbb{Z}$ together with homomorphisms $d^h: C_{p,q} \to C_{p-1,q}$, $d^v: C_{p,q} \to C_{p,q-1}$ (horizontal, vertical) such that $d = d^h + d^v$ is of order two, i.e. $0 = d^2 = (d^h + d^v)(d^h + d^v) =$
$= \underbrace{d^h d^h}_{0} + \underbrace{(d^h d^v + d^v d^h)}_{0} + \underbrace{d^v d^v}_{0}$.

Now $P \otimes Q$ is a double complex with $(P\otimes Q)_{p,q} = P_p \otimes Q_q$

and $d^h(x \otimes y) = (dx) \otimes y$

$d^v(x \otimes y) = (-1)^p x \otimes (dy)$ … idea: we swap d & x (d has degree -1, x has degree $|x| = p$ $(x \in P_p)$)

the formula $d^2 = 0$ really works and suggests a way of forming a chain complex out of a double complex C

$(\mathrm{Tot}^{\oplus} C)_n = \bigoplus_{p+q=n} C_{p,q}$

... then $d = d^h + d^v : (\mathrm{Tot}^{\oplus} C)_n \longrightarrow (\mathrm{Tot}^{\oplus} C)_{n-1}$ and $d^2 = 0$. ("total" written above Tot)

Another total complex: $(\mathrm{Tot}^{\times} C)_n = \prod_{p+q=n} C_{p,q}$. For $P \otimes Q$, they are the same since both P, Q are non-neg. graded.

Global analysis: $d(\omega \wedge \theta) = (d\omega) \wedge \ldots + (-1)^{|\omega|} \omega \wedge (d \ldots$

$\mathrm{Tot}(P \otimes Q)_n = \bigoplus_{p+q=n} P_p \otimes Q_q$

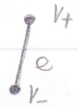
$d(x \otimes y) = (dx) \otimes y + (-1)^p x \otimes (dy)$

Reinterpret $cyl(C)$ (C: chain complex) in terms of the tensor product

$cyl(R[0]) = \quad 0 \to R \xrightarrow{\binom{-1}{1}} R \oplus R \to 0$ (degrees 1, 0)

"interval" chain complex

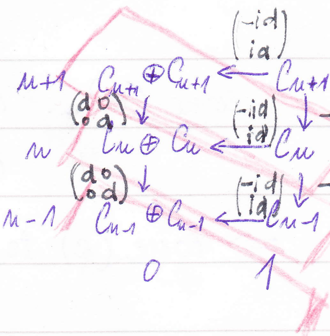
1 dim; 0 dim × 2



$de = v_+ - v_-$

Lemma: $cyl(C) \cong \mathrm{Tot}(cyl(R[0]) \otimes C)$

$$\begin{array}{ccc} (R\oplus R) \otimes C_n & \cong & C_n \oplus C_n \\ \downarrow id \otimes d & & \downarrow \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \\ (R \oplus R) \otimes C_{n-1} & \cong & C_{n-1} \oplus C_{n-1} \end{array} \qquad \begin{array}{ccc} R \otimes C_n & \cong & C_n \\ \downarrow -id \otimes d & & \downarrow -d \\ R \otimes C_{n-1} & \cong & C_{n-1} \end{array}$$

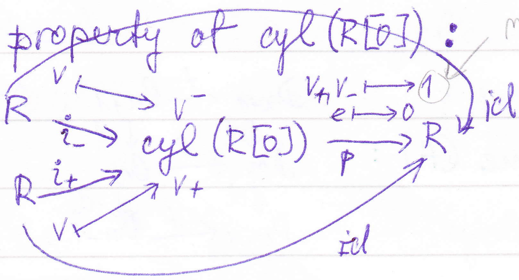


or:

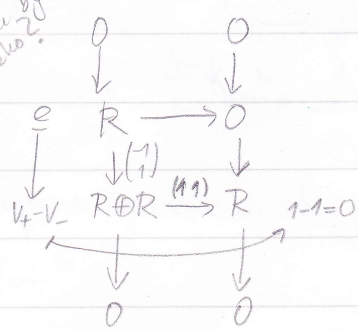
$d(v_\pm \otimes x) = v_\pm \otimes (dx)$

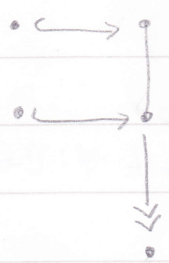
$d(e \otimes x) = (v_+ - v_-) \otimes x - e \otimes (dx)$

An important property of $cyl(R[0])$:



(nemá tu být něco jiného?)





Lemma: Both $i_\pm$ are htpy inverses of p. Better: $cyl(R[0])$ deforms onto $Rv_\pm$

Proof: Take $-$

$cyl(R[0]) \xrightarrow{p} R[0] \xrightarrow{i_-} cyl(R[0])$

$e \longmapsto 0$

$v_+ \longmapsto v_-$

$v_- \longmapsto v_+$

$h : cyl(R[0])_0 \longrightarrow cyl(R[0])_1$

$e \longmapsto 0$

$v_+ \longmapsto e$

$v_- \longmapsto 0$

$(dh + hd)(e) = d\underset{0}{he} + hde$

$= h(v_+ - v_-) = e$

$(dh + hd)(v_+) = dhv_+ + h\underset{0}{dv_+} = de = v_+ - v_-$

$(dh + hd)(v_-) = \underset{0}{dhv} + h\underset{0}{dv} = 0$

<u>Remark</u>: i_+, i_- are quasi-isos by a direct computation of $H_*(cyl(R[0]))$ but htpy equiv > q-iso. (co?)

Tensor all i_+, i_-, p, h with C to get (From now on $P \otimes Q = Tot(P \otimes Q)$)

$x \mapsto$
$C \cong R[0] \otimes C \to V_- \otimes X$
$\to cyl(C) \longrightarrow R[0] \otimes C \cong C$
$C \cong R[0] \otimes \to V_+ \otimes X$
$x \mapsto$

h also induces a htpy $h \otimes id$ between i_p, id.

Another such construction: **cone (C)** = cone$(R[0]) \otimes C$

cone $(R[0])$ =
e: R, 1
$\downarrow 1$
v: R, 0
$\downarrow$
0

$de = v$ } contractible: $hv = e$, $dh + hd = id$

cone (C)... $d(e \otimes x) = v \otimes x - e \otimes (dx)$
$d(v \otimes x) = v \otimes (dx)$

$\Rightarrow$ cone (C) is contractible (contraction $h \otimes id$)

contraction = htpy between 0, id
C contractible - alternatively C htpy equivalent to 0

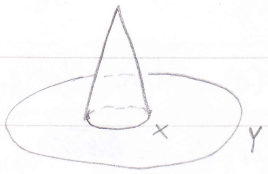
cone(C):
$C_{n+1} \xleftarrow{1} C_{n+1}$
$d \downarrow \quad \downarrow -d$
$C_n \xleftarrow{1} C_n$
$d \downarrow \quad \downarrow -d$
$C_{n-1} \xleftarrow{1} C_{n-1}$

$0 \to C \to cone(C) \to C[1] \to 0$ (quotient, but shifted by 1)

Let $f: C \to D$ be a chain map. **Cone (f)**:
$D_{n+1} \xleftarrow{f} C_{n+1}$
$\downarrow d \quad \downarrow -d$
$D_n \xleftarrow{f} C_n$
$\downarrow d \quad \downarrow -d$
$D_{n-1} \xleftarrow{f} C_{n-1}$

$0 \to D \to cone(f) \to C[1] \to 0$

Pushout square:
$C \longrightarrow cone(C)$
$f \downarrow \qquad \downarrow$
$D \longrightarrow cone(f)$



What is pushout?
$A \xrightarrow{f} B$
$g \downarrow \quad \downarrow$
$C \longrightarrow P$

$P = (B \oplus C) / \underbrace{(f(a) - g(a))}_{M}$

$A \to B$
$\downarrow \quad \downarrow$
$C \to B \oplus C \to B \oplus C / M$

<u>universal property</u>:
$A \xrightarrow{f} B$
$g \downarrow \quad \downarrow$
$C \to B \oplus C \to B \oplus C / M$
h
$\to D$

The exact sequence $0 \to D \to \mathrm{cone}(f) \to C[1] \to 0$ induces a long exact sequence of homology groups

$$H_{n+1}\,\mathrm{cone}(f) \longrightarrow H_n C \xrightarrow{f_*} H_n D \longrightarrow H_n\,\mathrm{Cone}(f) \longrightarrow H_{n-1} C \to H_{n-1}\,\mathrm{cone}\ldots$$

$0 \to D \to \mathrm{cone}(f) \to C[1] \to 0$

$(0,x) \longmapsto x$

$\downarrow d$

$f(x) \longmapsto (f(x),0)$

in a long exact sequence every third map is izo only if homology is zero

(ne u? $* = \forall n$)

Proposition: f is a quasi-iso $\iff H_*\,\mathrm{cone}(f) = 0$ ... $\mathrm{cone}(f)$ is acyclic = e...

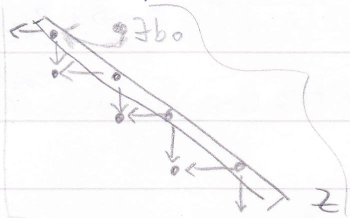
(Ex) $P \xrightarrow{\varepsilon} A$ is a resolution $\iff \ldots P_1 \xrightarrow{-d} P_0 \xrightarrow{\varepsilon} A$ is exact.

$$A \otimes Q \xleftarrow{\varepsilon \otimes id} \mathrm{Tot}^{\oplus}(P \otimes Q)$$

$A \otimes Q_1$	$P_0 \otimes Q_1 \longleftarrow P_1 \otimes Q_1$	$A \otimes Q_1 \longleftarrow P_0 \otimes Q_1 \longleftarrow P_1 \otimes Q_1$
$\downarrow$	$\downarrow \qquad \downarrow$	$\downarrow \qquad \downarrow \qquad \downarrow$
$A \otimes Q_0$	$P_0 \otimes Q_0 \longleftarrow P_1 \otimes Q_0$	$A \otimes Q_1 \longleftarrow P_0 \otimes Q_0 \longleftarrow P_1 \otimes Q_0$

D :" double complex

claims:
- D has exact rows ($A \leftarrow P_0 \leftarrow P_0 \leftarrow \cdots$ ex & Q_q projective $\Rightarrow$ flat)
- therefore $\mathrm{Tot}^{\oplus} D$ is acyclic (next time)
- $\varepsilon \otimes id$ is a quasi-izo $\left(A \otimes Q \underset{q\text{-iso}}{\xleftarrow{\varepsilon \otimes id}} \mathrm{Tot}^{\oplus}(P \otimes Q) \underset{q\text{-iso}}{\xrightarrow{id \otimes \eta}} P \otimes B\right)$

$C =$ 

cycle in $\mathrm{Tot}^{\oplus} C$

replace z by $z - (d^h + d^v) b_0$ and repeat

C double complex (first quadrant) (upper half plane)
nonneg only if P&Q nonneg.

C_{*p} exact (acyclic) $\Rightarrow \mathrm{Tot}^{\oplus} C$ exact (acyclic)

$z \in (\mathrm{Tot}^{\oplus} C)_n = \bigoplus_{p+q=n} C_{pq}$ a cycle

$(z_{0n}, \ldots, z_{n0})$

$0 \xleftarrow{d^h} z_{0n} \longleftarrow c_{1n}$?

$d^h z_{0n} = 0 \Rightarrow \exists c_{1n} \in C_{1n} : d^h c_{1n} = z_{0n}$

then $z \sim z - dc_{1n} = (0, z_{1(n-1)}, \ldots, z_{n0})$ has fewer terms;

$\uparrow$ represents the same htpy class

repeat after n steps

$\leftarrow z_{n0}$

$z \sim z - dc_{1n} - dc_{2(n-1)} - \cdots - dc_{n+1,0} = 0$

i.e. $z = d(c_{1n} + \cdots + c_{n+1,0})$

Balancing Ext

$P \xrightarrow{\sim} A[0]$ (proj. res.), $B[0] \xrightarrow{\sim} I$ (inj. res.)

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\mathrm{Hom}(P_1,B) \arrow[r] \arrow[u] & \mathrm{Hom}(P_1,I_0) \arrow[r] \arrow[u] & \mathrm{Hom}(P_1,I_1) \arrow[r] \arrow[u] & {} \\
\mathrm{Hom}(P_0,B) \arrow[r] \arrow[u] & \mathrm{Hom}(P_0,I_0) \arrow[r] \arrow[u] & \mathrm{Hom}(P_0,I_1) \arrow[r] \arrow[u] & {} \\
 & \mathrm{Hom}(A,I_0) \arrow[r] \arrow[u] & \mathrm{Hom}(A,I_1) \arrow[r] \arrow[u] & {}
\end{tikzcd}$$

cone $(\mathrm{Hom}(P,B) \to \mathrm{Tot}^{\times}\mathrm{Hom}(P,I))$ acyclic $\Rightarrow$ $\mathrm{Hom}(P,B) \xrightarrow{\sim} \mathrm{Tot}^{\times}\mathrm{Hom}(P,I) \leftarrow \mathrm{Hom}(A,I)$

$\underbrace{R^n\mathrm{Hom}(-,B)(A) \cong R^n\mathrm{Hom}(A,-)(B)}_{\text{denoted } \mathrm{Ext}^n(A,B)}$

Lemma: $\mathrm{Ext}^n(\bigoplus A_i, B) \cong \prod \mathrm{Ext}^n(A_i,B)$

$\mathrm{Ext}^n(A, \prod B_i) \cong \prod \mathrm{Ext}^n(A,B_i)$

Proof: $P_i \xrightarrow{\sim} A_i[0] \Rightarrow \bigoplus P_i \xrightarrow{\sim} \bigoplus A_i[0]$

a direct sum is exact (preserves exact seq)

$\mathrm{Ext}^n(\bigoplus A_i,B) = H^n\mathrm{Hom}(\bigoplus P_i,B) \cong H^n\prod \mathrm{Hom}(P_i,B) \cong \prod H^n\mathrm{Hom}(P_i,B)$ (direct product is exact)

$= \prod \mathrm{Ext}^n(A_i,B)$ ∎

Lemma: If F is exact then $H_nFC \cong FH_nC$

Proof:

$0 \to Z_n \to C_n \xrightarrow{d} B_{n-1} \to 0$ $\quad\xrightarrow{F}\quad$ $0 \to FZ_n \to FC_n \xrightarrow{d_*} FB_{n-1} \to 0$

$0 \to B_n \to Z_n \to H_nC \to 0$ $\qquad\qquad$ $0 \to \underbrace{FB_n \to FZ_n}_{\text{coker } = H_nFC} \to FH_nC \to 0$

FZ_n are then cycles of FC and FB_n are the boundaries ∎

(Ex) $R = \mathbb{Z}$: $\mathrm{Ext}^n(A,B) = 0$ when $n > 1$ ... A has a projective resolution of length 1 $0 \to P_1 \to P_0 \to A$

- $\mathrm{Ext}^n(A,-) = 0$ for all $n \geq 1$ iff A is projective
- $\mathrm{Ext}^n(-,B) = 0$ for all $n \geq 1$ iff B is injective

- $\mathrm{Ext}^n(\mathbb{Z}/n, B)$ ... compute by resolving $\mathbb{Z}/n$

$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n$ apply $\mathrm{Hom}(-,B)$

$$\begin{tikzcd}
0 & \mathrm{Hom}(\mathbb{Z},B) \arrow[l] \arrow[d, phantom, "\cong"] & \mathrm{Hom}(\mathbb{Z},B) \arrow[l, "n^*"'] \arrow[d, phantom, "\cong"] \\
0 & B \arrow[l] & B \arrow[l, "n"'] & \cdots
\end{tikzcd}$$

$\mathrm{Ext}^0(\mathbb{Z}/n,B) = \mathrm{Hom}(\mathbb{Z}/n,B) = \ker(n \cdot B \to B) = B = \{b \in B \mid nb =$

$\mathrm{Ext}^1(\mathbb{Z}/n,B) = \mathrm{coker}(n \cdot B \to B) = B/nB$

Ex • A torsion; $\mathrm{Ext}^n(A,\mathbb{Z})$, injective resolution of $\mathbb{Z}$:

$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ apply $\mathrm{Hom}(A,-)$

$0 \to \mathrm{Hom}(A,\mathbb{Z}) \to \mathrm{Hom}(A,\mathbb{Q}) \to \mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \mathrm{Ext}^1(A,\mathbb{Z}) \to \mathrm{Ext}^1(A,\mathbb{Q}$

$\mathrm{Hom}(A,\mathbb{Z}) = 0$, $\mathrm{Hom}(A,\mathbb{Q}) = 0$ (A is torsion, Q has only 0 torsion), $\mathrm{Ext}^1(A,\mathbb{Q}) = 0$ (Q is injective)

$\Rightarrow \mathrm{Ext}^1(A,\mathbb{Z}) \cong A^*$ ... Pontryagin dual of A

In particular, if A is finite (abelian), then $A^\times = A$ $\quad \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \cong {}_n(\mathbb{Q}/\mathbb{Z})$
$= \{z \in \mathbb{Q}/\mathbb{Z} \mid z^n = 1\} = \mathbb{Z}/n$.

• $\mathrm{Ext}^1(\mathbb{Z}[1/p], \mathbb{Z}) \left(\cong \mathrm{Hom}(\mathbb{Z}[1/p]/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})\right) \neq 0$ (It's only flat, not projective)

Another formalism regarding Hom

$C \otimes D = \mathrm{Tot}^{\oplus}(C_p \otimes D_q)$
a tensor product of chain compl[exes]
$\mathrm{Hom}(C,D)$
ch. maps $(C\otimes D, E) \cong$ ch. maps $(C, \mathrm{Hom}$
(chain complexes are a monoid[al] closed cathegory
symmetric: $C \otimes D \cong D \otimes C$, $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$ (to ma[ke] a ch[ain map]))

C, D chain complexes ...

... make $\mathrm{Hom}(C,D)$ into a double ch. cplx

Define $\mathrm{Hom}(C,D)_{pq} = \mathrm{Hom}(C_p, D_q)$ ← total degree $q-p$

$\mathrm{Tot}^\times \mathrm{Hom}(C,D)_k = \prod_n \mathrm{Hom}(C_n, D_{n+k})$ ← in degree $(-n, n+k)$

The differential, or rather its signs, are chosen in such a [way]
that $ev: \mathrm{Hom}(C,D) \otimes C \to D$ is a chain map
(δ, d, d; $f \otimes x$)

$d(f(x)) = d(ev(f\otimes x)) = ev(\delta f \otimes x + (-1)^{|f|} f \otimes$ [dx])
$= \delta f(x) + (-1)^{|f|} f(dx)$

$\Rightarrow \delta f(x) = df(x) - (-1)^{|f|} f(dx)$

$\delta f = df - (-1)^{|f|} fd = [d,f]$ ... a graded commutator

(The following hold: $\delta(g\circ f) = [d, g\circ f] = [d,g]\circ f + (-1)^{|g|} g \circ [d,f]$)

What is a cycle in $\mathrm{Hom}(C,D)$?

$C_{n+1} \to D_{n+k+1}$
$C_n \to D_{n+k}$
$C_{n-1} \to D_{n+k-1}$

$f \in \prod_n \mathrm{Hom}(C_n, D_{n+k})$

$0 = [d,f] = df - (-1)^k fd$ i.e. $df = (-1)^k fd$ ... a chain map of degree k; $C \xrightarrow{\text{ch. map}} D[k]$

f, g are **homologous** (i.e. they determine the same homology class) iff they are chain homotopic

$g - f = [d,h] = dh - (-1)^{k+1} hd = dh + (-1)^k hd$ (← $|h| = |f|+1$) i.e.

Cartan, Eilenberg
MacLane
Hilton, Wu, Lie

Conclusion: $H_k \mathrm{Hom}(C,D)$ = chain homotopy classes of maps of degree k.

Ext and extensions

Let's try to prove the following: $\mathrm{Ext}^1(A,B)=0 \iff$ every $\xi: 0 \xrightarrow{i} B \xrightarrow{p} X \xrightarrow{} A \to 0$ splits.

Apply $\mathrm{Hom}(A,-)$ to ξ: $0 \to \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,X) \xrightarrow{p_*} \mathrm{Hom}(A,A) \to \mathrm{Ext}^1(A,B) \to \mathrm{Ext}^1(A,X) \to$

$j \longmapsto pj = id$

and note that ξ splits iff

id lies in the image of p_* iff it goes to 0 in $\mathrm{Ext}^1(A,B)$

Denote the image of id in $\mathrm{Ext}^1(A,B)$ by $\Theta(\xi)$.

$\Theta(\xi)$ a unique obstruction to the existence of a splitting.

This already proves $\Rightarrow$: if $\mathrm{Ext}^1(A,B)=0$ then $\Theta(\xi)=0$ and ξ split.

Important: naturality of the long exact sequence of derived functors

$$\begin{array}{ccccccccc} 0 \to & A & \to & B & \to & C & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & A' & \to & B' & \to & C' & \to 0 \end{array} \Rightarrow \begin{array}{ccccccccc} 0 \to & R^0FA & \to & R^0FB & \to & R^0FC & \to & R^1FA & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & \circlearrowleft & \downarrow & \\ 0 \to & R^0FA' & \to & R^0FB' & \to & R^0FC' & \to & R^1FA' & \to \cdots \end{array}$$

commutes

Will not prove it

Now we will produce a SES (short exact sequence) with any given obstruction $\alpha \in \mathrm{Ext}^1(A,B)$

$$\begin{array}{ccccccccc} 0 \to & B & \xrightarrow{g} & I & \to & C & \to 0 \\ & & & \uparrow \lrcorner & & \uparrow f & \\ 0 \to & B & \to & I \times_C A & \leftarrow & A & \to 0 \end{array}$$

(I injective, C = coker g; $I\times_C A =: X$)

$$\begin{array}{ccccccc} \mathrm{Hom}(A,I) \to & \mathrm{Hom}(A,C) & \xrightarrow{h_*} & \mathrm{Ext}^1(A,B) & \to & \mathrm{Ext}^1(A,I) = 0 & \to \\ \uparrow & \uparrow f_* & & \uparrow id & & & \\ \mathrm{Hom}(A,X) \to & \mathrm{Hom}(A,A) & \to & \mathrm{Ext}^1(A,B) & \to & \mathrm{Ext}^1(A,X) & \to \end{array}$$

$m_f \longmapsto \alpha_\epsilon$; I injective

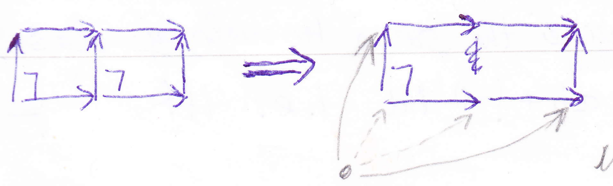
$id \longmapsto \Theta(\xi)=\alpha$

Aside: pullbacks

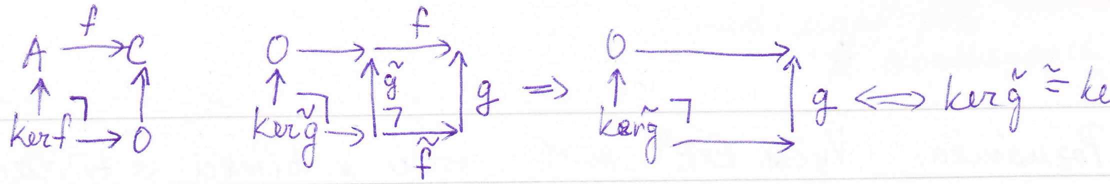
$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \tilde{g} \uparrow & \lrcorner & \uparrow g \\ A \times_C B & \xrightarrow{\tilde{f}} & B \end{array} = \{(a,b) \in A \times B \mid f(a) = g(b)\}$$

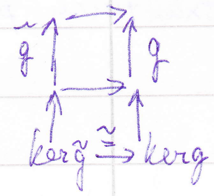
Lemma: (Universal property) $\mathrm{Hom}(D, A\times_C B) \xrightarrow{\cong}$ {commutative squares $\begin{array}{ccc} A & \xrightarrow{f} & C \\ \uparrow & & \uparrow g \\ D & \to & B \end{array}$ }

$$h \longmapsto \left(\begin{array}{ccc} A & \xrightarrow{f} & C \\ \tilde{g}h \uparrow & & \uparrow g \\ D & \xrightarrow{\tilde{f}h} & B \end{array}\right)$$



univ. property

Special case 



A little observation g surjective $\Rightarrow \tilde{g}$ surjective

VĚTA Existuje bijekce $\left\{ \text{rozšíření} \atop = \text{KEP} \; \xi: 0 \to B \to X \to A \to 0 \right\} /_{izo} \equiv Ext^1(A,B)$
$\forall A, B \in$ Mod-R $\qquad \xi \longmapsto \Theta(\xi)$ ... obstrukce

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izomorfismus rozšíření:

$$\begin{array}{ccccccccc} 0 & \to & B & \to & X & \to & A & \to & 0 \\ & & \downarrow id & & \downarrow & & \downarrow id & & \\ 0 & \to & B & \to & Y & \to & A & \to & 0 \end{array}$$

podle 5-lemmatu to uprostřed musí být izo

Dk: Na $0 \to B \xrightarrow{i} X \xrightarrow{p} A \to 0$ se použije $Hom(A,-)$

$$\dots \to Hom(A,A) \to Ext^1(A,B) \to \dots$$
$$id \longmapsto \Theta(\xi)$$

$\alpha \in Ext^1(A,B) \qquad 0 \to B \xrightarrow{j} I \xrightarrow{q} C \to 0$ (injektivní I)

$$\dots \to Hom(A,I) \xrightarrow{q^*} Hom(A,C) \to Ext^1(A,B) \to Ext^1(A,I)$$
$$f \longmapsto \alpha \qquad \qquad Ext^1(A,I) = 0$$
$$f + qg \longmapsto \alpha$$

f ... lib. vzor α, kde $g: A \to I$

$$\begin{array}{ccccccccc} 0 & \to & B & \xrightarrow{j} & I & \xrightarrow{q} & C & \to & 0 \\ & & & & \uparrow & \lrcorner & \uparrow f & & \\ 0 & \to & B & \to & X_f & \to & A & \to & 0 \end{array}$$

$b \longmapsto (j(b), 0)$

$X_f = \{(x,a) \in I \times A \mid q(x) = f(a)\}$

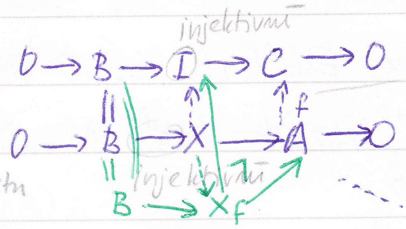
$X_{f+qg} = \{(x,a) \mid q(x) = f(a) + qg(a)\}$

$(x,a) \longmapsto (x + g(a), a)$

$b \longmapsto (j(b), 0)$

$$\begin{array}{ccccccccc} 0 & \to & B & \to & X_f & \to & A & \to & 0 \\ & & \downarrow id & & \downarrow & & \downarrow id & & \\ 0 & \to & B & \to & X_{f+qg} & \to & A & \to & 0 \end{array}$$

$b \longmapsto (j(b), 0)$

Zbývá 

z univ. vl. kojádra

složení dvou směrů dá identitu

... I je izo X_f, takže kompozice je identita

$$\begin{array}{ccc} Hom(A,C) & \to & Ext^1(A,B) \\ f_* \uparrow & & \uparrow id \\ Hom(A,A) & \to & Ext^1(A,B) \end{array}$$

f ... $id \mapsto \Theta(\xi)$

(Př) $Ext^1(\mathbb{Z}/m, \mathbb{Z}/n) = (\mathbb{Z}/n)/m(\mathbb{Z}/n) = \mathbb{Z}/gcd(m,n)$

zvláštní případy:
- $gcd(m,n) = 1 \Rightarrow$ všechna rozšíření se štěpí, $X \cong \mathbb{Z}_m \oplus \mathbb{Z}_n$
- $Ext^1(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p$ neštěpící se rozšíření $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to$; $1 \longmapsto 1$; redukce $1 \longmapsto k \cdot p, k \neq 0$
- $Ext^1(\mathbb{Z}/6, \mathbb{Z}/6) \cong \mathbb{Z}/6$: $\left(\mathbb{Z}/2 \oplus \mathbb{Z}/2 \text{ nebo } \mathbb{Z}_4 \right) \oplus \left(\mathbb{Z}/3 \oplus \mathbb{Z}/3 \text{ nebo } \mathbb{Z}/9 \right)$
- $Ext^1(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/3) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Poznámka: Vyšší Ext^n, $n>1$ jsou v bijekci s třídami exaktních posl.

$$0 \to B \to X_n \to \dots \to X_2 \to X_1 \to A \to 0$$
$$\| \qquad\qquad\qquad\qquad \|$$
$$0 \to B \to Y_n \to \dots \to Y_2 \to Y_1 \to A \to 0 \quad \leftarrow \text{ekvivalence}$$

Homologická dimenze

Nechť R je okruh (lépe komutativní), $A \in$ Mod-R

pd(A) (projektivní dimenze) = délka nejkratší projektivní rezolventy A (∞ pokud neex konečná)
$= \inf\{n \mid \exists$ exakt. posl. $0 \to P_n \to \dots \to P_0 \to A \to 0$ t.ž. P_i projektivní$\}$

fd(A) = —〃— ploché rezolventy A (P_i ploché)

id(B) = —〃— injektivní ($0 \to B \to I_0 \to \dots \to I_n \to 0$, I_i injektivní)

Lemma: Následující podmínky jsou ekvivalentní:

(i) $pd(A) \leq n$

(ii) v každé ex. posl. $0 \to M_n \to \underbrace{P_{n-1} \to \dots \to P_0}_{proj.} \to A \to 0$ je i M_n projektivní

(iii) $Ext^{n+1}(A,B) = 0$ pro $\forall B \in$ Mod-R

Dk: Jednoduché: (ii) ⇒ (i) ⇒ (iii), zbývá (iii) ⇒ (ii)

Rozštěpme ex. posl. na krátké: $0 \to M_n \to P_{n-1} \to M_{n-1} \to 0$ ($M_{n-1} = im(P_{n-1} \to P_{n-2})$)

$0 \to M_{n-1} \to P_{n-2} \to M_{n-2} \to 0$

$\vdots$

$0 \to M_1 \to P_0 \to M_0 \to 0$, $M_0 = A$

Aplikací $Ext^*(-,B)$ na $0 \to M_i \to P_{i-1} \to M_{i-1} \to 0$ dostaneme

$$\underset{0}{\underset{\|}{Ext^{j+1}(P_{i-1},B)}} \leftarrow Ext^{j+1}(M_{i-1},B) \xleftarrow{\cong} Ext^j(M_i,B) \leftarrow \underset{0}{\underset{\|}{Ext^j(P_{i-1},B)}} \quad \text{pro } j \geq 1$$

Proto $Ext^1(M_n,B) \cong Ext^2(M_{n-1},B) \cong \dots \cong Ext^n(M_1,B) \cong Ext^{n+1}(A,B) = 0$ pro $\forall B$ ($M_0 = A$)

⇒ M_n projektivní ■

Důsledek: $\sup\{pd(A) \mid A \in \text{Mod-R}\} \overset{\text{z Lemmatu}}{=} \inf\{n \mid Ext^{n+1}(A,B) = 0 \ \forall A,B\} \overset{\text{Lemma op}}{=}$
$= \sup\{id(B) \mid B \in \text{Mod-R}\}$

Toto číslo nazýváme (pravá) globální dimenze R. (gl.dim(R))
Podobná charakterizace funguje pro fd, $\sup\{fd(A) \mid A \in \text{Mod-R}\} =$
$= \inf\{n \mid Tor_{n+1}(A,B) = 0 \ \forall A,B\} \overset{def}{=}$ Tor-dim(R)