rovnost nastává např. pro R Noetherovský

Zjevně $fd(A) \leq pd(A) \Rightarrow \text{Tor-dim}(R) \leq gl.\dim(R)$

$gl.\dim(R) = 0 \iff \forall A: pd(A) = 0 \iff A$ projektivní

$\iff$ každý modul je injektivní $\iff R$ polojednoduchý

$gl.\dim(R) = 1 \iff$ každý podmodul projektivního je projektivní

$\iff$ každý kvocient injektivního je injektivní $(0 \to M_1 \to P_0 \to P_0/M_1 \to 0)$, $\overset{A}{=}$

$\Leftarrow R$ je PID (obor hlavních ideálů) (jako pro $\mathbb{Z}$)

$\boxed{\text{VĚTA}}$ (Hilbertova o szyzygyích) $gl.\dim \Bbbk[x_1, \dots, x_r] = r$. ($\Bbbk$ těleso)

Dk: Stačí R kom., $gl.\dim R < \infty \Rightarrow gl.\dim R[x] = 1 + gl.\dim R$

- „$\leq$": Nechť $A \in \text{Mod-}R[x]$, který chápeme zároveň jako R-modul

$$0 \longrightarrow A \otimes_R R[x] \xrightarrow{\beta} A \otimes_R R[x] \xrightarrow{\mu} A \longrightarrow 0$$

$$a \otimes p \longmapsto a \cdot p$$

$$a \otimes p \longmapsto (ax) \otimes p - a \otimes (xp)$$

~~tvrzení~~: tato posloupnost je exaktní. μ surj., $\mu\beta = 0$ $(\mu(a \otimes 1) = a)$

Jako R-modul je $R[x] = \underset{n \geq 0}{\bigoplus} R \cdot x^n$ (volný) $\Rightarrow A \otimes_R R[x] \cong \underset{n \geq 0}{\bigoplus} A$ ← prvky $a \otimes$

injektivita β: $0 = \beta(a_0 \otimes 1 + \dots + a_n \otimes x^n)$ (jednoznač. vyj.), $a_n \neq 0$ $= a_0 x \otimes 1 - a_0 \otimes x + a_1 x \otimes x - a_1 \otimes x^2 + \dots + a_n x \otimes x^n - a_n \otimes x^{n+1}$

$a_n = 0$, SPOR $\Leftarrow$

Nechť $\mu(\underbrace{a_0 \otimes 1 + \dots + a_n \otimes x^n}_{z}) = 0$ ($\in \ker\mu$) $\quad z + \underbrace{\beta(a_n \otimes x^{n-1})}_{a_n x \otimes x^{n-1} - a_n \otimes x^n} = a_0 \otimes 1 + \dots + a_{n-2} \otimes x^{n-2} + (a_{n-1} + a_n x)$

má menší „stupeň" a lze pokračovat indukcí

$(z = a_0 \otimes 1 \in \ker\mu \Rightarrow a_0 = \mu(z) = 0)$

Z exaktnosti $0 \to A \otimes_R R[x] \xrightarrow{\beta} A \otimes_R R[x] \xrightarrow{\mu} A \to 0$

dostaneme aplikací $\text{Ext}^*_{R[x]}(-, B)$:

$$\text{Ext}^{r+1}(A \otimes_R R[x], B) \leftarrow \text{Ext}^{r+1}(A, B) \leftarrow \text{Ext}^r(A \otimes_R R[x], B)$$

$$pd_{R[x]} A \leq 1 + pd_{R[x]} A \otimes_R R[x] \leq 1 + pd_R A$$

definované pomocí $\text{Ext}^i_{R[x]}(A \otimes_R R[x], B)$

$R^i \text{Hom}_{R[x]}(A \otimes_R R[x], B) = R^i \text{Hom}$

$R \subseteq S$
$\text{Hom}_S(A \otimes_R S, B) \cong \text{Hom}_R(A, B)$

- „$\geq$" dokážeme pro nenulový $R[x]$-modul A t.ž. $A \cdot x = 0$

$pd_{R[x]} A \geq 1 + pd_R A$ — zejména A lze vzít lib. nenulový R-modul a dodefinovat $ax = 0$.

indukcí vzhledem k $d = pd_R A$ — $\boxed{d=0}$ tj. A je projektivní nad $R \Rightarrow$ není projekt. nad $R[x]$, protože násobení x je neinjektivní.

tj. $pd_{R[x]} A \geq 1 = 1+0 = 1 + pd_R A$

— $\boxed{d=1}$ $pd_{R[x]} A \leq 2$ podle první části

předp. < 2, uvažíme v Mod-$R[x]$ $\quad 0 \to P_1 \to P_0 \to A \to 0$ proj. rez. délky 1 (ex. $pd_{R[x]} A \leq 1$)

$-\otimes_{R[x]} R = R[x]/xR[x]$

$$\boxed{\begin{array}{l} 0 \to R[x] \xrightarrow{x} R[x] \xrightarrow{ev_0} R \to 0 \\ 0 \to A \xrightarrow{x} A \to 0 \end{array}}$$

P_0 projektivní

$$0 \longrightarrow Tor_1^{R[x]}(A,R) \longrightarrow P_1/xP_1 \to P_0/xP_0 \to A/xA \to 0$$

$Tor_1^{R[x]}(A,R) = \{a \mid ax = 0\} = A$, $\quad P_1/xP_1$, P_0/xP_0 proj. nad R, $\quad A/xA = A$

Protože $pd_R A = 1$, je A projektivní nad R — SPOR s $pd_R A = 0$.

— $\boxed{d \geq 2}$, v R-modulech uvažíme $\quad 0 \to M \to \overset{\text{proj.}}{P} \to A \to 0 \quad$ Mod-R

$pd_{R[x]} P = 1$

$$2 \leq d = pd_R A = 1 + pd_R M \overset{IP}{=} pd_{R[x]} M = -1 + pd_{R[x]} A \quad \blacksquare$$

VĚTA (Künneth) Nechť Tor-dim$(R) \leq 1$. Nechť C je řet. komplex t.ž. C_n je plochý (resp. projektivní, gl. dim $(R) \leq 1$). Potom existuje krátká exaktní posloupnost

"homologie C s koef. A"

$$0 \to H_n C \otimes_R A \longrightarrow H_n(C \otimes_R A) \longrightarrow Tor_1^R(H_{n-1} C, A) \longrightarrow 0$$

"homologie C s koef. A"

Pokud gl. dim$(R) \leq 1$, C_n projektivní, $0 \to Ext_R^1(H_{n-1} C, A) \to H^n(Hom(C,A)) \to$
$\to Hom(H_n C, A) \to 0$.

Dk: Máme kep: $\quad 0 \to Z_n \to C_n \to B_{n-1} \to 0 \quad$ aplikujeme $-\otimes A$

$$\underbrace{0 \to B_n \to Z_n \to H_n C \to 0}_{\text{plochá rezolventa (resp. projektivní)}}$$

homologie $0 \to B_n \otimes A \to Z_n \otimes A \to 0$ jsou $Tor_*^R(H_n C, A)$

$$Tor_1^R(B_{n-1}, A) \to Z_n \otimes A \to C_n \otimes A \to B_{n-1} \otimes A \to 0$$

$Tor_1^R(B_{n-1}, A) = 0 \leftarrow$ protože C_{n-1} je plochý modul, $B_{n-1} \subseteq C_{n-1}$ podmodul $\Big\}$ Tor-dim $\leq 1 \Rightarrow B_{n-1}$ je plochý

$$\boxed{\begin{array}{ccccc} B_{n-1} \to & C_{n-1} & \to & C_{n-1}/B_{n-1} & \to 0 \\ \text{pl.} & \Leftarrow \text{pl.} & & \text{lib.} & \end{array}}$$

Tuto krátkou ex. posl. budeme chápat jako KEP řet. komplexů

$$0 \to \underset{\text{nulový diferenciál}}{Z \otimes A} \longrightarrow C \otimes A \to \underset{\text{nulový diferenciál}}{B[1] \otimes A} \to 0$$

Dostáváme dl. ex. posl. homologických grup

$$B_n \otimes A \xrightarrow{incl_n \otimes id} Z_n \otimes A \longrightarrow H_n(C \otimes A) \longrightarrow B_{n-1} \otimes A \xrightarrow{incl_n \otimes id} Z_{n-1} \otimes A$$

$$0 \to \underbrace{coker(incl_n \otimes id)}_{= H_n C \otimes_R A} \longrightarrow H_n(C \otimes A) \to \underbrace{ker(incl_{n-1} \otimes id)}_{= Tor_1^R(H_{n-1} C, A)} \to 0$$

$\blacksquare$

pullback: $\begin{pmatrix} B \\ \downarrow \\ A \\ \uparrow \\ C \end{pmatrix} \longmapsto B\times_A C$

A, B, C abelovské grupy

homomorfismus takovýchto diagramů:

$$\begin{tikzcd} B' \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & A \\ C' \arrow[r] \arrow[u] & C \arrow[u] \end{tikzcd}$$

takže tento diag. komutuje

pullback je zleva exaktní

jakožto $\underset{\lim D}{\underset{\|}{B\times_A C}} \cong \mathrm{Hom}\left(\begin{matrix} \mathbb{Z} \\ \downarrow \mathrm{id} \\ \mathbb{Z} \\ \uparrow \mathrm{id} \\ \mathbb{Z}\end{matrix}, \begin{matrix} B \\ \downarrow \\ A \\ \uparrow \\ C\end{matrix}\right)$

$\underline{\mathbb{Z}}$ $\quad$ D ← rezolventy?

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "b"] \arrow[d, "\mathrm{id}"'] & B \arrow[d, "f"] \\ \mathbb{Z} \arrow[r, "a"] & A \\ \mathbb{Z} \arrow[r, "c"] \arrow[u, "\mathrm{id}"] & C \arrow[u, "g"'] \end{tikzcd}$$

$f(b)=a$

$g(c)=a$

Derivované funktory jsou $R^n \lim D = \underbrace{\lim^n D}_{\text{označení}} = \mathrm{Ext}(\underline{\mathbb{Z}}, D)$

$\mathbb{Z}_p$ p-adická čísla

$\cdots \to A_2 \to A_1 \to A_0$ limita se nazývá věž $\quad \mathbb{Z}/p^2 \to \mathbb{Z}/p$

Opět platí balancování, takže lze $\lim^n$ spočítat z projektivní rezolventy $\underline{\mathbb{Z}}$.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "1"] \arrow[d, "\binom{1}{0}"] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r, "\binom{1}{-1}"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "(1\ 1)"] & \mathbb{Z} \\ 0 \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r, "1"] \arrow[u, "\binom{0}{1}"] & \mathbb{Z} \arrow[u] \end{tikzcd}$$

proj. rez.

Lemma: $\mathrm{Hom}\left(\begin{matrix} \mathbb{Z} \\ \downarrow \\ \mathbb{Z}\oplus\mathbb{Z} \\ \uparrow \\ \mathbb{Z}\end{matrix}, \begin{matrix} B \\ \downarrow \\ A \\ \uparrow \\ C\end{matrix}\right) \cong B\times C \Longrightarrow \begin{matrix} \mathbb{Z} \\ \downarrow \\ \mathbb{Z}\oplus\mathbb{Z} \\ \downarrow \\ \mathbb{Z}\end{matrix}$ proj

$\mathrm{Hom}\left(\begin{matrix} 0 \\ \downarrow \\ \mathbb{Z} \\ \uparrow \\ 0\end{matrix}, \begin{matrix} B \\ \downarrow \\ A \\ \uparrow \\ C\end{matrix}\right) \cong A \Longrightarrow \begin{matrix} 0 \\ \downarrow \\ \mathbb{Z} \\ \uparrow \\ 0\end{matrix}$ projektivní

Důsledek: $\lim^n$ jsou kohomologie

$0 \leftarrow A \leftarrow B \times C \leftarrow 0$

$f(b)-g(c) \xleftarrow{(f,-g)} (b,c)$

$\lim^0 = B\times_A C$

$\lim^1 = A/\operatorname{im}((f,-g): B\times C \to A)$

Aplikace: $0 \to D'' \to D \to D' \to 0$ indukuje $0 \to \lim D' \to \lim D \to \lim D'' \to$

$\to \lim^1 D' \to \lim^1 D \to \lim^1 D'' \to 0$

Podobný př. $\begin{pmatrix} A \\ \downarrow f \\ B\end{pmatrix} \longmapsto \ker f \cong \mathrm{Hom}\left(\begin{matrix}\mathbb{Z} \\ \downarrow \\ 0\end{matrix}, \begin{matrix} A \\ \downarrow f \\ B\end{matrix}\right)$

$\lim^0 = \ker \quad \lim^1 = \operatorname{coker} \Longrightarrow$ snake lemma

grupa G má akci na A

Př. $A \circlearrowleft G \longmapsto A^G = \{a\in A \mid \forall g\in G: ga=a\} \cong \mathrm{Hom}(\underline{\mathbb{Z}}, A)$ ..... atd.

Kohomologie grup

G grupa, $\mathbb{Z}G$ grupová algebra (okruh) = množina lineárních kombinací prvků G s koeficienty v $\mathbb{Z}$.

$\mathbb{Z}G$-modul = abelovská grupa společně s akcí G pomocí homomorfismů grup, tj. $G \to \mathrm{Aut}(A)$ ← jako grupy

Def. $A \in \text{Mod-}\mathbb{Z}G$. Definujeme $A^G = \{a \in A \mid ga = a\} \subseteq A$

$A_G = A/(ga - a)$

$A^{\otimes m}_{\Sigma_n} = S^m A$

Lemma: $A^G \cong \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$ ← $g \cdot x = x$ triviální akce $\mathbb{Z}G$ — tato algebra je augmentovaná

$A_G \cong A \otimes_{\mathbb{Z}G} \mathbb{Z}$

augmentace $\mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z}$, $g \mapsto 1$

Důkaz: $\mathbb{Z}$ je gen. 1 jako $\mathbb{Z}$-modul, takže homo $\varphi : \mathbb{Z} \to A$ je jedn. dán obrazem 1, $\mathbb{Z}G$-linearita je ekviv. $g \cdot \varphi(1) = \varphi(g \cdot 1) = \varphi(1)$

Lépe $\mathbb{Z} = \mathbb{Z}G/(g-1)$ $1 \in G$ neutr. prvek jako $\mathbb{Z}G$-moduly

$A \otimes_{\mathbb{Z}G} \mathbb{Z} \cong A \otimes_{\mathbb{Z}G} \mathbb{Z}G/(g-1) \cong A/(g-1)$

Def. Nechť $A \in \text{Mod-}\mathbb{Z}G$. Definujeme $H_n(G;A) = L_n(-)_G(A) \cong \operatorname{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, A)$

$H^n(G;A) = R^n(-)^G(A) \cong \operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$

Př. $G = \mathbb{Z}$, $\mathbb{Z}G = \mathbb{Z}[t, t^{-1}]$ $\qquad \mathbb{Z}[t,t^{-1}] \xrightarrow{ev_1} \mathbb{Z}$

$1 \longmapsto 1$

$t \longmapsto t \cdot 1 = 1$ $\qquad t^{-1} \longmapsto t^{-1} \cdot 1 = 1$

$\sum_{n \in \mathbb{Z}} a_n t^n \longmapsto \sum a_n$

co je jádro? Laurentovy polynomy, které mají součet koeficientů 0?

↳ ideál $(t-1) \cong \mathbb{Z}[t, t^{-1}]$

$t - 1 \longleftarrow 1$ homo modulů, surjektivní, injektivita $\iff$ $t-1$ není dělitel nuly

$0 \to \mathbb{Z}[t,t^{-1}] \xrightarrow{(t-1)x} \mathbb{Z}[t,t^{-1}] \xrightarrow{ev_1} \mathbb{Z} \to 0$

$A \in \text{Mod-}\mathbb{Z}G$ $\quad H_n(\mathbb{Z}; A) = H_n(0 \to \overset{1}{A} \xrightarrow{(t-1)x} \overset{0}{A} \to 0) = \begin{cases} A_{\mathbb{Z}} & n = 0 \\ A^{\mathbb{Z}} & n = 1 \\ 0 & \text{jinak} \end{cases}$

$H^n(\mathbb{Z}; A) = H^n(0 \leftarrow \overset{1}{A} \xleftarrow{(t-1)x} \overset{0}{A} \leftarrow 0) = \begin{cases} A^{\mathbb{Z}} & n = 0 \\ A_{\mathbb{Z}} & n = 1 \\ 0 & \text{jinak} \end{cases}$

$0 \to A \to B \to C \to 0$ indukuje $0 \to A^{\mathbb{Z}} \to B^{\mathbb{Z}} \to C^{\mathbb{Z}} \to A_{\mathbb{Z}} \to B_{\mathbb{Z}} \to C_{\mathbb{Z}} \to 0$

Př. $G = \mathbb{Z}/m = C_m$ — cyklická grupa o m prvcích $1, t, \dots, t^{m-1}$

$\mathbb{Z}C_m \xrightarrow{\varepsilon} \mathbb{Z}$

$t^k \longmapsto 1$

$\mathbb{Z}C_m \xleftarrow{\underbrace{(1+t+\dots+t^{m-1})}_{=N}} \mathbb{Z}C_m \xleftarrow{(t-1)x} \mathbb{Z}C_m \longleftarrow \cdots$

$(t-1)x$

$N \longleftarrow t^k$

$\ker(Nx) = \ker \varepsilon = (t-1)$

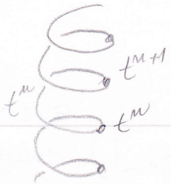
$H_n(C_m, \mathbb{Z}) = H_n(\cdots \xrightarrow{m} \overset{3}{\mathbb{Z}} \xrightarrow{0} \overset{2}{\mathbb{Z}} \xrightarrow{m} \overset{1}{\mathbb{Z}} \xrightarrow{0} \overset{0}{\mathbb{Z}} \to 0)$

$= \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/m & n \text{ liché} \\ 0 & n \text{ sudé}, n > 0 \end{cases}$

$H_n(G, \mathbb{Z}) \cong H_n(BG, \mathbb{Z})$

$\quad\quad BG \cong K(G,1)$

klasifikační prostor
$$BG = \begin{cases} S^1 & G=\mathbb{Z} \\ \mathbb{R}P^\infty & G=C_2 \\ \text{lens space} & G=C_n \\ \vdots & \end{cases}$$



Bar rezolventa

Definujme $B_m^{U=\text{unreduced}}$ volný $\overset{\text{levý}}{\mathbb{Z}G}$-modul na množině $\overbrace{G\times\dots\times G}^{n\text{-krát}}$

generátory budeme zapisovat $[g_1\otimes\dots\otimes g_n]$

B_n .... volný $\overset{\text{levý}}{\mathbb{Z}G}$-modul na $(G\setminus\{1\})\times\dots\times(G\setminus\{1\})$, gen. $[g_1|\dots|g_n]$

Definujeme $d_0[g_1\otimes\dots\otimes g_n] = g_1[g_2\otimes\dots\otimes g_n]$

$d_i[g_1\otimes\dots\otimes g_n] = [g_1\otimes\dots\otimes g_{i-1}\otimes g_i g_{i+1}\otimes g_{i+2}\otimes\dots\otimes g_n]$ $i=1,\dots,n-1$

$d_n[g_1\otimes\dots\otimes g_n] = [g_1\otimes\dots\otimes g_{n-1}]$

$d = \sum_{i=0}^{n}(-1)^i d_i : B_n^u \to B_{n-1}^u$, $\varepsilon: B_0^u \to \mathbb{Z}$, $[\,] \mapsto 1$

$\boxed{\text{VĚTA}}$ $\dots\to B_n^u \to B_{n-1}^u \to \cancel{B}\dots\to B_0^u \to \mathbb{Z}\to 0$ je proj. rezolventa — nered. bar. re

B je kvocient B^u podle $[g_1\otimes\dots\otimes g_{i-1}\otimes 1\otimes g_{i+1}\otimes\dots\otimes g_n]$ bar. rez.

a je to taky rezolventa $\mathbb{Z}$.

$B_G^{u=\text{unnormalised}}$ -- volný $\mathbb{Z}G$-modul na množině $[g_1\otimes\dots\otimes g_n]$, kde $g_i\in G$

$d[g_1\otimes\dots\otimes g_n] = g_1[g_2\otimes\dots\otimes g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1\otimes\dots\otimes g_{i-1}\otimes g_i g_{i+1}\otimes g_{i+2}\otimes\dots\otimes g_n]$

$+(-1)^n[g_1\otimes\dots\otimes g_{n-1}]$

B_G a quotient complex — (normalised) bar resolution — $B_G = B_G^u / \{[g_1\otimes\dots\otimes \underset{\text{the unit of }G}{1}\otimes\dots\otimes g_n]\}$

the class of $[g_1\otimes\dots\otimes g_n]$ in B_G is denoted by $[g_1|\dots|g_n]$

augmentation: $\varepsilon: (B_G^u)_0 \to \mathbb{Z}$

$[\,] \longmapsto 1$ $\quad g[\,] \longmapsto 1$

$\boxed{\text{Theorem}}$ $B_G^u \xrightarrow{\varepsilon} \mathbb{Z}$ is a free resolution and so is $B_G \xrightarrow{\varepsilon} \mathbb{Z}$

Proof: $B_G^u \xrightarrow{\varepsilon} \mathbb{Z}$ is a chain map between chain cx's of $\mathbb{Z}G$-modules,

$d^2=0$ easy

Acyclicity of $B_G^u \to \mathbb{Z}$ will follow from a contraction

that is only $\mathbb{Z}$-linear: $(B_G^u)_n$ is a free $\mathbb{Z}$-module

with basis $g[g_1\otimes\dots\otimes g_n]$

$h(g[g_1\otimes\dots\otimes g_n]) = [g\otimes g_1\otimes\dots\otimes g_n]$

Computing $(dh+hd)(g[g_1 \otimes \dots \otimes g_n])$ $\overset{\text{everything cancelles}}{\text{down except}}$ $= g[g_1 \otimes \dots \otimes g_n]$,

i.e. $dh + hd = id$

for $n=0$: $(dh+hd)(g[\,]) = d[g] + h(1) = g[\,] - [\,] + [\,] = g[\,]$

define $h(1) = [\,]$

$$\begin{tikzcd}
(B^u_G)_1 \arrow[d, "d"'] \arrow[ur] & (B^u_G)_1 \arrow[d, "d"] \\
(B^u_G)_0 \arrow[d, "\varepsilon"'] \arrow[ur, "h"] & (B^u_G)_0 \arrow[d, "\varepsilon"] \\
\mathbb{Z} \arrow[ur, "h"] & \mathbb{Z}
\end{tikzcd}$$

Therefore $id \sim 0$ via h

$\Rightarrow H(B^u_G \xrightarrow{\varepsilon} \mathbb{Z}) = 0$

$\Rightarrow B^u_G \xrightarrow{\varepsilon} \mathbb{Z}$ is a quasi iso

$A \longmapsto \mathbb{Z}G \otimes_{\mathbb{Z}} A$

$\text{Mod-}\mathbb{Z} \longrightarrow \text{Mod-}\mathbb{Z}G$

$\longleftarrow$

$M \longleftarrow\!\shortmid M$

Normalised B_G: $d[g_1 \otimes \dots \otimes \overset{i}{1} \otimes \dots \otimes g_n] =$ many terms with 1 somewhere

$+ (-1)^{i-1}[g_1 \otimes \dots \otimes g_{i-1} 1 \otimes g_{i+1} \otimes \dots \otimes g_n] + (-1)^i [g_1 \otimes \dots \otimes g_{i-1} \otimes 1 g_{i+1} \otimes \dots \otimes g_n] = 0$

$\Rightarrow$ generators of B^u_G with some $g_i = 1$ form a subcomplex and this makes the quotient B_G into a chain cx. The rest is analogous $\blacksquare$

Example: $H_n(G; \overset{\in \text{Mod-}\mathbb{Z}G}{A}) = H_n(A \otimes_{\mathbb{Z}G} B^{(u)}_G)$ $\qquad \overset{A/(x \sim xg)}{A_G} = A \otimes_{\mathbb{Z}G} \mathbb{Z}$

$H_0(G; \mathbb{Z}) = H_0(\mathbb{Z} \otimes_{\mathbb{Z}G} B^u_G)$

$\underbrace{\mathbb{Z} \otimes_{\mathbb{Z}G} (B^u_G)_1}_{\text{gener. by } [g]} \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \overset{\mathbb{Z}G \text{ gen. by } [\,]}{(B^u_G)_0}$

$\mathbb{Z}\{[g] \mid g \in G\} \longrightarrow \mathbb{Z}[\,]$

$[g] \longmapsto \text{"}g[\,] - [\,]\text{"} = 0$ $\qquad H_0(G, \mathbb{Z}) = \mathbb{Z}$

$\overset{\text{gen. by } [g \otimes h]}{\mathbb{Z} \otimes_{\mathbb{Z}G} (B^u_G)_2}$

$= \mathbb{Z}\{[g \otimes h] \mid g, h \in G\}$

$H_1(G, \mathbb{Z}) = H_1(\mathbb{Z} \otimes_{\mathbb{Z}G} B^u_G) = \overset{\text{everything}}{\ker} / \mathrm{im} = \mathbb{Z}\{[g] \mid g \in G\} / ([h] - [gh] + [g])$

$[g \otimes h] \longmapsto \text{"}g[h] - [gh] + [g]\text{"}$ $= [h] - [gh] + [g]$

$=$ free abelian group generated by ~~elts~~ elements $g \in G$ subject to $[gh] = [g] + [h]$ $(= [hg])$

Thus, $H_1(G; \mathbb{Z})$ is the abelianization of G

$= G / [G, G]$

commutator subgroup

$H_1(\overset{K(G,1)}{BG}; \mathbb{Z}) = G / [G, G]$

$H^1(G;M)$ and derivations

A derivation of $\mathbb{Z}G$ with values in a $(\mathbb{Z}G,\mathbb{Z}G)$-bimodule M is a group homeomorphism ($\mathbb{Z}$-linear) $\varphi: \mathbb{Z}G \to M$ such that $\varphi(g\cdot h) = \underbrace{\varphi(g)}_{M}\cdot \underbrace{h}_{G} + \underbrace{g}_{G}\cdot \varphi($

When M is a left $\mathbb{Z}G$-module, we make it into a bimodule with trivial right action, i.e. $x.g \overset{def.}{=} x$ for $g \in G$

Then the condition becomes $\varphi(g\cdot h) = g\varphi(h) + \varphi(g)$.

A principal derivation $D_x: \mathbb{Z}G \to M$, $x \in M$

$D_x(g) = gx - xg$ | trivial right action $Dx(g) = gx - x$

This is always a derivation (easy)

Theorem $\underset{\text{first cohomology group of } G \text{ with coefficients in } M}{H^1(G;M)} \cong \mathrm{Der}(G,M)/\mathrm{PDer}(G,M)$

colim $D = D$ cle $D \ast_D^l W$

Proof: $H^1(G;M) = H^1(\mathrm{Hom}(B^u_G; M))$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\{[g\otimes h]\}, M) \xleftarrow{d^*} \mathrm{Hom}_{\mathbb{Z}}(\underset{=\mathbb{Z}G}{\mathbb{Z}\{[g]\}}, M) \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M)$

$\mathrm{Set}(\{g\}, M)$ $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G\{[g]\}, M)$

$([g\otimes h] \mapsto \varphi(g[h] - [gh] + [g])) \longleftarrow\!\!| \ \varphi$

$= g\,\varphi[h] - \varphi[gh] + \varphi[g]$ ... therefore,

$\varphi \in \ker d^* \iff \varphi[gh] = g\,\varphi[h] + \varphi[g]$ i.e. φ is a derivation

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, M) \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M)$

$([g] \mapsto gx - x) \longleftarrow\!\!| \ x$ $\quad \varphi \in \mathrm{im}\, d^* \iff \varphi$ is a principal derivatio

$H^2(G;M)$ and extensions

We will be interested in extensions $0 \to \underset{\text{abelian group}}{M} \to \underset{\text{group}}{X} \to \underset{\text{group}}{G} \to 1$

M is a normal subgroup of X

again up to isomorphism of ext

$$\begin{array}{ccccccccc} 0 & \to & M & \to & \cancel{X} & \to & G & \to & 1 \\ & & \| & & \downarrow\cong & & \| & & \\ 0 & \to & M & \to & Y & \to & G & \to & 1 \end{array}$$

The conjugation action of X on M

$x \in X$, $a \in M$ ... $xax^{-1} = {}^x a \in M$ factors through $X/M = G$

--- for $x \in M$, the conjugation $^x a = a$

Thus, there is a conj. action of G on M and M is a $\mathbb{Z}G$-module

Consider a mapping $\sigma: G \to X$ s.t. $p\sigma = id$ and $\sigma(1)=1$
– call σ a based section. factor set $[\,,\,]: G \times G \longrightarrow M$
Define for $g,h \in G$ an element of M $[g,h] = \sigma(g)\sigma(h)\sigma(gh)^{-1} \in \ker p = M$
Reformulation of the conjugation action: $^{g}a = \sigma(g)\, a\, \sigma(g)^{-1}$

Lemma: Suppose that two extensions X, Y could be equipped with based sections in such a way that the factor sets I, J are equal. Then X, Y are isomorphic.

Proof: Consider bijections $M \times G \cong X$ $\quad M \times G \cong Y$
$(a,g) \longmapsto a.\sigma(g)$ $\quad (a,g) \longmapsto a.\tau(g)$

Claim: the composition $X \cong M \times G \cong Y$ is an isomorphism.
Transporting the group structures from X to $M \times G$
& from Y to $M \times G$.

this is the same as to say that these group str's are the same.

Transport from X $\quad (a,g)(b,h) \longmapsto (a.\sigma(g))(b.\sigma(h)) = a\,\sigma(g)\, b\, \sigma(g)^{-1}\sigma(g)\sigma(h)\,\sigma(gh)^{-1}\sigma(gh)$
$= \underbrace{a \cdot {}^{g}b \cdot [g,h]}_{\text{lie in } M} \cdot \sigma(gh)$

$\Rightarrow$ we have $(a,g)(b,h) = (a + {}^{g}b + [g,h],\ gh)$. The same for Y ■

Theorem There is a bijection $H^2(G;M) \cong \left\{ \text{extensions } \begin{array}{c} 0 \to M \to X \to G \to 1 \\ \parallel \quad\ \ \downarrow \quad\ \parallel \\ 0 \to M - Y \to G \to 1 \end{array} \right\} / \text{iso}$

$\mathrm{Hom}_{\mathbb{Z}G}((B_G)_3, M) \xleftarrow{\delta} \mathrm{Hom}_{\mathbb{Z}G}((B_G)_2, M) \xleftarrow{\delta} \mathrm{Hom}_{\mathbb{Z}G}((B_G)_1, M)$

$G \times G \times G \to M$ $\qquad [\,,\,]: G \times G \to M$
$\left.\begin{array}{l}(1,h) \mapsto 0 \\ (g,1) \mapsto 0\end{array}\right\}$ normalised

cocycle condition: $(f|g|h) \xmapsto{\partial} f(g|h) - (fg|h) + (f|gh) - (f|g) \xmapsto{[\,,\,]}$
$\xmapsto{[\,,\,]} \boxed{f[g,h] - [fg,h] + [f,gh] - [f,g] = 0}$
2-cocycle

Lemma: The factor set $[g,h] = \sigma(g)\sigma(h)\sigma(gh)^{-1}$ satisfies the normalised cocycle condition.

Proof: $\sigma(f)\sigma(g)\sigma(h)\sigma(gh)^{-1}\sigma(f)^{-1} - \sigma(fg)\sigma(h)\sigma(fgh)^{-1} + \sigma(f)\sigma(gh)\sigma(fgh)^{-1} -$
$- \sigma(f)\sigma(g)\sigma(fg)^{-1} = \sigma(f)\sigma(g)\sigma(h)\sigma(fgh)^{-1} - \sigma(f)\sigma(g)\sigma(h)\sigma(fgh)^{-1} = 0$ ■

What happens if we change σ to another based section σ'?
$\sigma'(g) = \beta(g)\sigma(g)$ where $\beta : G \to M$ s.t. $\beta(1)=0$

Lemma: In this case, $[\,,\,]$ and $[\,,\,]'$ differ by a coboundary.

Proof: $[g,h]' = \sigma'(g)\sigma'(h)\sigma'(gh)^{-1} = \beta(g)\sigma(g)\beta(h)\sigma(h)\sigma(gh)^{-1}\beta(gh)^{-1}$

$$= \beta(g) + \sigma(g)\beta(h)\sigma(g)^{-1} + \sigma(g)\sigma(h)\sigma(gh)^{-1} - \beta(gh) = \beta(g) + g\cdot\beta(h) + [g,h] - \beta(gh)$$

Therefore $[g,h]' - [g,h] = g\cdot\beta(h) - \beta(gh) + \beta(g) = \delta\beta(g|h)$. ∎

$\boxed{\text{Theorem}}$ The association $\{\text{extensions } 0\to \overset{\text{abelian}}{M}\to X\to G\to 1\}/\text{iso} \xrightarrow{\cong} H^2(G;M)$

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\cong"] & G \arrow[r] \arrow[d, equal] & 1 \\ 0 \arrow[r] & M \arrow[r] & Y \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

$$(0\to M\to X\to G\to 1) \longmapsto [\,,\,]$$

So far, we have proved that this is well-defined.

Surjectivity: Given a cocycle $[\,,\,]: G\times G\to M$, define a multiplication on $M\times G$ by $(a,g)(b,h) = (a+gb+[g,h], gh)$

Then this really makes $M\times G$ into a group with inverse $(a,g)^{-1} = (-g^{-1}a - g^{-1}[g,g^{-1}], g^{-1})$. (Associativity follows from the cocyle condition by yet another direct computation)

There is an obvious based section $\sigma: G\to M\times G$, $\sigma(g) = (0,g)$

and $[g,h] = (0,g)(0,h)(0,gh)^{-1} = ([g,h], gh)(-h^{-1}g^{-1}[gh, h^{-1}g^{-1}], h^{-1}g^{-1})$

$$= ([g,h] + gh(-h^{-1}g^{-1}[gh, h^{-1}g^{-1}]) + [gh, h^{-1}g^{-1}], ghh^{-1}g^{-1}) = ([g,h],1)$$

corresponds to $[g,h] \in M \subseteq M\times G$

Injectivity: Suppose that two extensions X, Y have cohomologous factor sets $\leftarrow$ they yield the same cohomology class i.e. they differ by a cohomology.

$[g,h]_\tau - [g,h]_\sigma = \delta\beta(g|h)$ Then change σ to $\sigma' = \beta\cdot\sigma$ and get $[g,h]_\tau = [g,h]_{\sigma'}$. Then $X \cong Y$ by one of the lemmas. ∎

$H^2(G;M) = 0$ if $\gcd(|G|,|M|)=1 \Rightarrow$ any ext splits

Theorem If G is finite of order m then multiplication by m is zero on $H_n(G;M)$, $H^n(G;M)$, $n>0$

Proof: Consider the following two chain maps $B_G \to B_G$

$$\begin{tikzcd}
(B^u_G)_1 \arrow[r, "0"] \arrow[d] & (B^u_G)_1 \arrow[d] \\
\mathbb{Z}G = (B^u_G)_0 \arrow[r, "N\cdot"] & (B^u_G)_0 = \mathbb{Z}G
\end{tikzcd}
\qquad \longleftarrow \qquad
\begin{tikzcd}
{[g]} \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\
g[\,]-[\,] \arrow[r, mapsto] & N\cdot[\,]-N\cdot[\,] \equiv 0
\end{tikzcd}$$

$N = \sum_{g\in G} g$ the norm elt $N\cdot g = N = g\cdot N$

and $B^u_G \xrightarrow{m\cdot} B^u_G$

The idea is that those two are chain homotopic - then they induce htpic maps $\mathrm{Hom}_{\mathbb{Z}G}(B^u_G, M)$ and thus the same maps on cohomology groups i.e. $0 = m\cdot$ as maps $H^n(G;M) \to H^n(G;M)$

The chain htpy is given by $\nu : (B^u_G)_n \longrightarrow (B^u_G)_{n+1}$

$$(g_1\otimes\dots\otimes g_n) \longmapsto (-1)^{n+1}\sum_{g\in G}(g_1\otimes\dots\otimes g_n\otimes g)$$

$$(d\nu+\nu d)(g_1\otimes\dots\otimes g_n) = (-1)^{n+1}\sum_{g\in G}(-1)^n(g_1\otimes\dots\otimes g_{n-1}\otimes g_n g) + (-1)^{n+1}(g_1\otimes\dots\otimes g_n)$$
$$+(-1)^n\sum_{g\in G}(-1)^n(g_1\otimes\dots\otimes g_{n-1}\otimes g) =$$
$$= \sum_{g\in G}(g_1\otimes\dots\otimes g_n) = m\cdot(g_1\otimes\dots\otimes g_n) = (\beta-\alpha)(g_1\otimes\dots\otimes g_n)$$

for $n=0$: $d\nu[\,] = -\sum_{g\in G} d[g] =$
$$= \sum_{g\in G} -g[\,]+[\,] = m[\,]-N[\,] = (\beta-\alpha)[\,] \quad \blacksquare$$

Corollary: If both G, M are finite and $\gcd(|G|,|M|)=1$ then $H^n(G;M)=0$, $n>0$.

Proof: In $H^n(G;M)$, multiplication by $|M|$ has an inverse (it has an inverse in $\mathbb{Z}/d$) and on the other hand, multiplication by $|M|$ is zero on $M^{|G|} \Rightarrow H^n(G;M) = 0$ $\blacksquare$

Corollary: If both G, M are finite and $\gcd(|G|,|M|)=1$, then any extension $0\to M\to X\to G\to 1$ splits. ($X \cong M \rtimes G$ the action of G on M)

The same works for M non-commutative (Schur-Zassenhaus)