Consider a projective representation of a group G on a vector space, i.e. a gp homo $G \to PGL_n(k)$

Q: When does it lift to an ordinary representation?

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{k}^\times \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & \mathbb{k}^\times \arrow[r] & GL_n(k) \arrow[r] & PGL_n(k) \arrow[r] & 1
\end{tikzcd}$$

A lift $G \to GL_n(\mathbb{k})$ is equivalent to a section of $E \to G$:

$$\begin{tikzcd}
G \arrow[rr, "id"] \arrow[dr, dashed] \arrow[ddr, dashed, bend right] & & G \arrow[dd] \\
& E \arrow[ur] \arrow[d] & \\
& GL_n(\mathbb{k}) \arrow[r] & PGL_n(\mathbb{k})
\end{tikzcd}$$

Now, the classification of extensions gives an answer: proj. repr. lifts iff the respective elt of $H^2(G; \mathbb{k}^\times)$ is zero.

Simplicial resolutions

B_G & B_G^u as simplicial res. $\mathbb{Z}G\text{-Mod} \longrightarrow \mathbb{Z}G\text{-Mod}$

$$M \longmapsto \mathbb{Z}G \otimes_{\mathbb{Z}} M$$

This is a "comonad" the str. of a $\mathbb{Z}G$-module is: $\mathbb{Z}G \otimes_{\mathbb{Z}} M \to M$, $(g \otimes x) \mapsto g x$

$$\eta: \mathbb{Z}G \otimes_{\mathbb{Z}} M \longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} M$$
$$g \otimes x \longmapsto g \otimes 1 \otimes x$$

The differential in B_G^u is

$$\mathbb{Z}G \otimes \cdots \otimes \underline{\mathbb{Z}G \otimes \mathbb{Z}} \longrightarrow \mathbb{Z}G \otimes \cdots \otimes \mathbb{Z}G \otimes \mathbb{Z}$$

with trivial $\mathbb{Z}G$-action

$$(g_1 \otimes \cdots \otimes g_n) \longmapsto g_1(g_2 \otimes \cdots \otimes g_n) = \varepsilon \quad = id \otimes \cdots \otimes id \otimes \varepsilon$$

simplicial object $\left\{ + \sum (-1)^i (g_1 \otimes \cdots \otimes g_{i-1} \otimes g_i g_{i+1} \otimes \cdots \otimes g_n) + (-1)^n (g_1 \otimes \cdots \otimes g_{n-1}) \right.$

$= id \otimes \cdots \otimes id \otimes \varepsilon$

$T = \mathbb{Z}G \otimes -$

$$T \circ \cdots \circ T \circ \boxed{T} \circ T \circ \cdots \circ T$$
$$\Downarrow \varepsilon$$
$$T \circ \cdots \circ T \circ \boxed{Id} \circ T \circ \cdots \circ T$$

$$\mathbb{Z}G \otimes \cdots \otimes \mathbb{Z}G \otimes \underline{(\overbrace{\mathbb{Z}G \otimes \cdots \otimes \mathbb{Z}G}^{n-i-1} \otimes \mathbb{Z})}$$

$\downarrow \varepsilon$ for the module $\underbrace{\mathbb{Z}G \otimes \cdots \otimes \mathbb{Z}G}_{n-i-1} \otimes \mathbb{Z}$

$$\mathbb{Z}G \otimes \cdots \otimes \mathbb{Z}G \otimes \underbrace{\mathbb{Z}G \otimes \cdots \otimes \mathbb{Z}G}_{n-i-1} \otimes \mathbb{Z}$$

$\perp: \text{Mod-}R \to \text{Mod-}R$ functor together with natural transformation

$\delta: \perp \to \perp^2$

~~[illegible]~~

$\varepsilon: \perp \to Id$

satisfying

$$\begin{tikzcd}
\perp \arrow[r, "\delta"] \arrow[d, "\delta"] & \perp^2 \arrow[d, "\delta \circ Id"] \\
\perp^2 \arrow[r, "Id \circ \delta"'] & \perp^3
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
\perp \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, "id"] & \perp^2 \arrow[d, "\varepsilon \circ Id"] \\
\perp^2 \arrow[r, "Id \circ \varepsilon"'] & \perp
\end{tikzcd}$$

associativitor?

is called a **comonad**.

dually $\quad T^2 \to T$ "multipl." $\quad$ monad $\quad$ 1st square: "associativity"
$\qquad\quad \mathrm{Id} \to T$ "unit" $\qquad\qquad\qquad$ 2nd square: "unit law"

Ex $\perp : \mathbb{Z}G\text{-Mod} \to \mathbb{Z}G\text{-Mod}$

$$M \longmapsto \mathbb{Z}G \otimes_{\mathbb{Z}} M$$

$\varepsilon : \mathbb{Z}G \otimes_{\mathbb{Z}} M \to M$ the mult. by elts of $\mathbb{Z}G$

$\delta : \mathbb{Z}G \otimes_{\mathbb{Z}} M \to \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} M$

$\qquad g \otimes x \longmapsto g \otimes 1 \otimes x$

An augmented chain complex associated with $\perp$ and $M \in \text{Mod-}R$:

$$\underbrace{\overset{C_2}{\cdots \perp^3 M} \to \overset{C_1}{\perp^2 M} \to \overset{C_0}{\perp M}} \xrightarrow{\varepsilon} M$$

many components of the differential

$C_n = \perp^{n+1} M \xrightarrow{d_i} \perp^n M = C_{n-1}$

$\perp^i \perp \perp^{n-i} M \xrightarrow{\mathrm{Id} \circ \varepsilon \circ \mathrm{Id}} \perp^n M$

$i = 0, \dots, n$

$\mathbb{Z}G \otimes \mathbb{Z}G \otimes M \to \mathbb{Z}G \otimes M$

$g \otimes h \otimes x \xmapsto{d_0} gh \otimes x$

$\qquad\qquad \xmapsto{d_1} g \otimes hx$

the differential is $d = \sum_{i=0}^{n} (-1)^i d_i$ ... unnormalised version

normalised version: $\quad s_i : \underset{\perp^{n+1} M}{\overset{\shortparallel}{C_n}} \to \underset{\perp^{n+2} M}{\overset{\shortparallel}{C_{n+1}}}$ $\qquad \perp^i \perp \perp^n M \to \perp^{n+2} M$

$\qquad\qquad \mathrm{Id} \circ \delta \circ \mathrm{Id} \qquad i = 0, \dots, n$

MEWE POOJTATIVE

quotient out the subcomplex whose part in dim $\underline{n}$ is

$\sum_{i=0}^{n-1} \operatorname{im} s_i \subseteq C_n$

$\qquad s_i : C_{n-1} \to C_n$

Important Ex. Let $\Bbbk$ be a ground ring, commutative with a unit (think: a field)

Let R be a $\Bbbk$-algebra

i.e.
- R is a ring, associative with unit
- R is a (bi) module over $\Bbbk$

the additions agree:
$+ : R \times R \to R$
$- : R \times R \to R$
$\cdot : k \times R \to R$

- the ring multiplication $R \times R \to R$ is $\Bbbk$-bilinear
$\qquad R \otimes_{\Bbbk} R \to R$

Alternatively: $k \to R$ ring homo whose image lies in the centre of R
$k \mapsto k\cdot 1$

$$\begin{array}{ccc} k \otimes_k R & \xrightarrow{u \otimes id} & R \otimes_k R \\ r & \longmapsto & 1 \otimes r \end{array}$$

$$\begin{tikzcd} R \otimes_k R \otimes_k R \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"'] & R \otimes_k R \arrow[d, "m"] \\ R \otimes_k R \arrow[r, "m"] & R \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r] \arrow[d, "id \otimes u"'] \arrow[dr, "id"] & R \otimes_k R \arrow[d, "m"] \\ R \otimes_k R \arrow[r, "m"'] & R \end{tikzcd}$$

R is a R-R-bimodule $\quad (rx)s = r(xs)$
associativity = bimodule axiom

$R\text{-Mod}\cancel{-R} \to R\text{-Mod}\cancel{-R}$
$M \mapsto R \otimes_k M$
again a comonad
$\varepsilon : R \otimes_k M \to M \quad r \otimes x \mapsto rx$
$\delta : R \otimes_k M \to R \otimes_k R \otimes_k M \quad r \otimes x \mapsto r \otimes 1 \otimes x$

The chain complex associated with $\perp$ and $R \in R\text{-Mod-}R$ has

$$\begin{array}{cccccccc} & C_2 & & C_1 & & C_0 & & \\ \cdots \to & R \otimes_k R \otimes_k R \otimes_k R & \to & R \otimes_k R \otimes_k R & \to & R \otimes_k R & \xrightarrow{\varepsilon} & R \end{array}$$

augmentation

$r \otimes s \otimes t \otimes u \longmapsto rs \otimes t \otimes u - r \otimes st \otimes u - r \otimes s \otimes tu$
$r \otimes s \longmapsto rs$
$r \otimes s \otimes t \longmapsto rs \otimes t - r \otimes st$

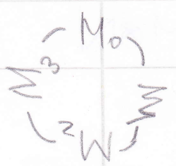
Denote this chain complex by B_R

Hochschild cohomology of a k-algebra R with coefficients in a bimodule M is $\quad H^n(R,M) = H^n(\mathrm{Hom}_{R\text{-}R}(B_R, M))$

= vector space equipped with a bilinear product
násobení, sčítání, násobení skalárem

The Hochschild homology groups are defined "dually using a tensor product" $\quad H_n(R,M) = H_n(B_R \otimes_{R^e} M)$

when M,N are R-R-bimodules define $M \otimes_{R^e} N$ as a quotient $(M \otimes_k N)/\sim$
by $xr \otimes y \sim x \otimes ry$
$rx \otimes y \sim x \otimes yr$



$R^e = R \otimes_k R^{op}$
R-R-bimodule $=$
$= R^e$-module left & rig

$\otimes = \otimes_k$

Ex. $H^0(R,M)$:

$$\mathrm{Hom}_{R\text{-}R}(R\otimes R\otimes R, M) \xleftarrow{d^*} \mathrm{Hom}_{R\text{-}R}(R\otimes R, M) \longleftarrow 0$$

$1\otimes 1 \mapsto x$

$r(1\otimes 1)x = r\otimes s \mapsto rxs$ } is a cocycle $\iff sx = xs$ $\forall s\in R$

$1\otimes s\otimes 1 \mapsto s\otimes 1 - 1\otimes s$

$\mapsto sx - xs$

$r(\,)t$ $\rightsquigarrow$ $r\otimes s\otimes t \mapsto rs\otimes t - r\otimes st \mapsto rsxt - rxst$

$1\otimes s\otimes 1\otimes x \mapsto s\otimes 1\otimes x - 1\otimes s\ldots$

$H^0(R,M) = \{x\in M \mid rx = xr \ \forall r\in R\}$

$1\otimes s\otimes 1\otimes x \mapsto 1\otimes 1\otimes x$

$(R\otimes R\otimes R)\otimes_{R^e} M$ $\quad (R\otimes R)\otimes_{R^e} M$

$H_0(R,M) = M/\sim$ with $xs = sx$

$R\otimes M \longrightarrow M$

$s\otimes x \qquad xs - sx$

Ex. $H^m(\mathbb{Z}G, M) \cong H^m(G,M)$ if a left $\mathbb{Z}G$-module is made into a $\mathbb{Z}G$-$\mathbb{Z}G$-bimodule via $x\cdot g = x$ (trivial right G-action)

$H_m(\mathbb{Z}G, M) \cong H_m(G,M)$

What is $H^1(R,M)$?

$$\mathrm{Hom}(R\otimes R\otimes R\otimes R, M) \xleftarrow{d^*} \mathrm{Hom}_{R\text{-}R}(R\otimes R\otimes R, M) \xleftarrow{d^*} \mathrm{Hom}_{R\text{-}R}(R\otimes R, M)$$

image: $1\otimes s\otimes 1 \mapsto sx - xs$

kernel: $1\otimes s\otimes 1 \mapsto f(s)$

$r\otimes s\otimes t \mapsto rf(s)t$

$1\otimes r\otimes s\otimes 1$

$\xmapsto{d} r\otimes s\otimes 1 - 1\otimes rs\otimes 1 + 1\otimes r\otimes s$

$\xmapsto{f} rf(s) - f(rs) + f(r)s$ $\rightsquigarrow$ f lies in the kernel iff $f(rs) = f(r)s + rf(s)$ i.e. f is a derivation + image = principal derivation $f(s) = sx - xs$

Proposion: $H^1(R,M) = \mathrm{Der}(R,M)/\mathrm{PDer}(R,M)$

Products on Hochschild cohomology

multiplication

$\tilde{f}: R^{\otimes(m+2)} \to M$ $\qquad f: R^{\otimes m} \to M$

$\tilde{g}: R^{\otimes(n+2)} \to N$ $\qquad g: R^{\otimes n} \to N$

$\widetilde{f\cdot g}: R^{\otimes(m+n+2)} \to M\otimes N$ $\qquad f\cdot g: R^{\otimes(m+n)} \to M\otimes N$

$H^*(R,R)$ admits a structure of a Gerstenhaber algebra

$f \in Hom_{R-R}(R^{\otimes(n+2)}, R) \cong Hom_K(R^{\otimes n}, R)$

$$f(x_0 \otimes \dots \otimes x_{n+1}) \mapsto f(1 \otimes x_1 \otimes \dots \otimes x_n \otimes 1)$$

$$= x_0 f(1 \otimes x_1 \otimes \dots \otimes x_n \otimes 1) x_{n+1}$$

cup product: $|f| = n \quad |g| = m \Rightarrow |f \cup g| = n+m$

$$(f \cup g)(x_1 \otimes \dots \otimes x_{n+m}) = f(x_1 \otimes \dots \otimes x_n) g(x_{n+1} \otimes \dots \otimes x_{n+m})$$

$$\delta(f \cup g) = \delta f \cup g + f \cup \delta g - (-1)^{m}(f(x_1 \otimes \dots \otimes x_n) x_{n+1}) g(x_{n+2} \otimes \dots \otimes x_{n+m+1}) + (-1)^{m} f(x_1 \otimes \dots \otimes x_n)(x_{n+1} g(x_{n+2} \otimes \dots \otimes x_{n+m+1}))$$

Corollary: f, g cocycles $\Rightarrow f \cup g$ cocycle

in addition: one of f, g coboundary $\Rightarrow f \cup g$ coboundary

$f = \delta \tilde{f} \Rightarrow f \cup g = \delta(\tilde{f} \cup g)$

$\Rightarrow$ there is a well defined operation $\cup : H^n(R;R) \otimes H^m(R;R) \to H^{n+m}(R;R)$

$$[f] \otimes [g] \longmapsto [f \cup g]$$

If R is commutative, then the cup-product on $H^*(R;R)$ is also commu

$|f \circ_i g| = n+m-1$

$(f \circ_i g)(x_1 \otimes \dots \otimes x_{n+m-1}) = \pm f(x_1 \otimes \dots \otimes x_{i-1} \otimes g(x_i \otimes \dots \otimes x_{m-i-1}) \otimes x_{m-1} \otimes \dots \otimes x_{n+m-1}$

$[f, g] = \sum \pm (f \circ_i g - (-1)^{nm} g \circ_i f)$

$\delta[f, g] = [\delta f, g] \pm [f, \delta g] \quad (-1)^{|f||g|}$

$[f, [g, h]] = [[f, g], h] \pm [g, [f, h]]$ Jacobi identity?

$[f, gh] = [f, g]h \pm g[f, h] \quad (-1)^{|f||g|}$

$[f, -]$ is a graded derivation w.r.t. $[,]$

Deligne conjecture: Gerstenhaber algebra structure on $H^*(R;R)$ comes from some particular structure on $C^*(R,R)$

Hochschild cochains

Last time: $H^0(R,M)=\{x\in M \mid rx=xr\}$ $\qquad$ $H_0(R,M)=M/(rx-xr)$

$H^1(R,M)=\mathrm{Der}(R,M)/\mathrm{PDer}(R,M)$

$H_1(R,M)$ in the case that R is commutative & $rx=xr$ (i.e. M is a module made into a bimodule)

$\Omega_{R|k}$ ---. "differential 1-forms" $\qquad$ $f_1 dg_1 + \dots + f_r dg_r$

$\Omega_{R|k}$ is an R-module generated by $\{dr \mid r\in R\}$

subject to relations $d(r_0+r_1)=dr_0+dr_1$, $d(r_0r_1)=r_0dr_1+r_1dr_0$

$\boxed{\text{Theorem}}$ R commutative, $rx=xr$. Then $H_1(R,M)=M\otimes_R \Omega_{R|k}$

In particular, $H_1(R,R)=\Omega_{R|k}$, $1\otimes r\otimes s\otimes 1 \mapsto r\otimes s\otimes 1 - 1\otimes rs\otimes 1 + 1\otimes r\otimes s$.

Pf. $M\otimes R\otimes R \longrightarrow M\otimes R \xrightarrow{0} M$

$x\otimes r \longmapsto rx-xr=0$

$x\otimes r\otimes s \longmapsto xr\otimes s - x\otimes rs + sx\otimes r$

$\Rightarrow H_1(R,M)=M\otimes R/(xr\otimes s - x\otimes rs + sx\otimes r)$

$H_1(R,M)\xrightarrow{\varphi} M\otimes_R\Omega_{R|k}$ $\qquad$ $M\otimes_R \Omega_{R|k}\xrightarrow{\psi} H_1(R,M)$

$x\otimes r\longmapsto x\otimes dr$ $\qquad$ $x\otimes s\,dr\longmapsto xs\otimes r$

Clearly $\varphi\psi=\mathrm{id}$, $\psi\varphi=\mathrm{id}$, once we show that they're well defined

for φ: $xr\otimes s - x\otimes rs + sx\otimes r \longmapsto xr\otimes ds - x\otimes d(rs) + sx\otimes dr$

$= x\otimes(\underbrace{r\,ds - d(rs) + s\,dr}_{0})$

for ψ:

$x\otimes(d(r_0r_1)-r_0dr_1-r_1dr_0)\longmapsto x\otimes r_0r_1 - xr_0\otimes r_1 - xr_1\otimes r_0 =$

$=-d(x\otimes r_0\otimes r_1)=0$ in $H_1(R,M)$

$x\otimes(d(r_0+r_1)-dr_0-dr_1)= x\otimes(r_0+r_1)-x\otimes r_0 - x\otimes r_1=0$ ■

$\boxed{\text{Theorem}}$ (Hochschild-Kostant-Rosenberg)

Let R be a commut. alg., essentially of finite type over a field k. If R is smooth over (k a field ... equivalent to separable), then there is an iso $\Lambda^*\Omega_{R|k}\xrightarrow{\cong} H_*(R,R)$

exterior algebra over R ... $\Lambda^n\Omega_{R|k}$ gen. by $r_0dr_1\wedge\dots\wedge dr_n$

H^2 and extensions

extensions : $0 \to M \to E \xrightarrow{\varepsilon} R \to 0$

- E an algebra, ε an algebra map
- $M = \ker \varepsilon \subseteq E$ a square zero ideal $M \cdot M = 0$
- ε splits as a map of k-modules (is k-split) (always the case when k is a field)

Given a splitting $\sigma : R \to E$ gives a decomposition $E \cong R \oplus M$

$\sigma(r) + \iota(x) \leftarrow\!\shortmid (r, x)$

$e \longmapsto (\varepsilon(e), \iota^{-1}(e - \sigma\varepsilon(e)))$

The square zero condition amounts to $(r_1, x_1)(r_2, x_2) =$

$(\sigma(r_1) + \iota(x_1))(\sigma(r_2) + \iota(r_2)) = \sigma(r_1)\sigma(r_2) + \iota(x_1)\sigma(r_2) +$
$+ \sigma(r_1)\iota(x_2) + \iota(x_1)\iota(x_2) = 0$ $\quad$ 〃 $(r_1 r_2 ?)$ $\quad$ 〃 $(0, x_1 r_2)$

$= (r_1 r_2, x_1 r_2 + r_1 x_2 + f(r_1, r_2))$,

because ε is an algebra map

Where M becomes an R-R-bimodule via $rxs = \sigma(r)x\sigma(s)$

bimodule structure

multiplication in E; does not depend on σ because any other choice differs by an elt of M and M is square zero.

The mapping $f : R \times R \to M$ induces a 2-cocycle $f : R \otimes R \to M$ and thus an elt of $H^2(R, M)$

$\boxed{\text{Theorem}}$ Given a k-algebra R and an R-R bimodule M, the equivalence classes of square zero, k-split extensions of R by M are in bijective correspondence with $H^2(M; R)$

$(0 \to M \to E \to R \to 0) \mapsto f \in H^2(R, M)$ $\quad$ $f(r_1, r_2) = \sigma(r_1)\sigma(r_2) - \sigma(r_1 r_2)$ $\quad$ 〃 $(r_1 r_2, 0)$

the independence of f on the choice of σ:

any other σ' satisfies : $\sigma'(r) = \sigma(r) + g(r)$, where $g : R \to M$

Hence, $f'(r_1 r_2) = \sigma'(r_1)\sigma'(r_2) - \sigma'(r_1 r_2)$

$= (\sigma(r_1) + g(r_1))(\sigma(r_2) + g(r_2)) - \sigma(r_1 r_2) - g(r_1 r_2)$

$= f(r_1 r_2) + (\delta g)(r_1, r_2)$,

i.e. $f' - f = \delta g$ and they yield the same elt in $H^2(R; M)$

$0 \to M \to R \oplus M \to R \to 0 \quad \leftarrow\!\shortmid f$

with multiplication : $(r_1, x_1)(r_2, x_2) = (r_1 r_2, r_1 x_2 + x_1 r_2 + f(r_1, r_2))$

What do elements of $H^n(R, M)$ classify?

$n=2$:
$$\begin{array}{l} 0 \\ \downarrow \\ M \\ \downarrow^{2} \\ E \xrightarrow{\varepsilon} R \\ \downarrow \\ 0 = A \end{array}$$

... equivalent to exactness: $A \xrightarrow[\sim]{\varepsilon} R$ ... quasi-iso

What is the structure on A?

it is a differential graded algebra $\underbrace{\quad}_{A_n \cdot A_m \subseteq A_{n+m}}$

in particular: $EE \subseteq E$, E algebra

$EM \subseteq M$, $ME \subseteq M$, M ideal

$MM = 0$ square zero

diferenciálně gradovaná algebra

A **differential graded algebra** (over $\Bbbk$) or **dga** is a chain complex A of k-modules together with a multiplication maps (unital, assoc.) $A_n \times A_m \to A_{n+m}$, that are k-bilinear and satisfy the graded Leibniz rule $d(a \cdot b) = (d \cdot a) \cdot b + (-1)^{|a|} a \cdot (db)$

Condensed version: multiplication is a chain map $A \otimes A \to A$

$A \in Ch_k$ + mult $A \otimes A \mapsto A$ } řetězcové homy

unit $\underbrace{k[0]}_{\text{unit for } \otimes} \mapsto A$

in dim k: $\bigoplus_{n+m=k} A_n \otimes A_m \to A$

the chain cond.:
$$\begin{array}{ccc} a \otimes b & \longmapsto & ab \\ \downarrow & & \downarrow \\ da \otimes b + (-1)^{|a|} a \otimes db & \longmapsto & d(ab) \end{array}$$

+ asociativita

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes id} & A \otimes A \\ {\scriptstyle id \otimes m} \downarrow & & \downarrow {\scriptstyle m} \\ A \otimes A & \xrightarrow[m]{} & A \end{array}$$

($|1| = 0$
$|xy| = |x| + |y|$)

speciální případ: A je obyčejná algebra, pak $A[0]$ je dga.

V je k-modul

$T(V[d])$ ... teuzorová algebra $T(V[k]) = \bigoplus_{n \geq 0} V[d]^{\otimes n}$ ... n-tá tenzorová mocnina

– v dimenzi nd je $V^{\otimes n}$

chcx:
$$\begin{array}{l} 0 \\ \downarrow \\ V \text{ dimenze } d \\ \downarrow \\ 0 \end{array}$$
jinak samé nuly

$d=1$
$$\begin{array}{c} V \otimes V \\ \downarrow 0 \\ V \\ \downarrow \sigma \\ k \end{array}$$

augmentace: $\varepsilon: A \to k[0]$ homomorfismus dga – zachovává 1, násobení a diferenciál

$$\begin{array}{ccc} A_1 & \to & 0 \\ d\downarrow & \square & \downarrow \\ A_0 & \xrightarrow{\varepsilon} & k \\ \downarrow & & \downarrow \\ A_{-1} & \to & 0 \end{array} \qquad \varepsilon d(x) = 0$$

typický příklad: $A_n = 0$ pro $n < 0$

$A_0 = k \cdot 1$

$d: A_1 \xrightarrow{0} A_0$ $\qquad \begin{array}{ccc} A_1 & \to & 0 \\ \downarrow 0 & & \downarrow \\ \varepsilon: A_0 = k & \xrightarrow{id} & k \\ \downarrow & & \\ 0 & & \end{array}$

- A je **souvislá** ... máme kanonickou augmentaci
- $\ker \varepsilon$ je **augmentační ideál**, značíme $\overline{A}$

(pro souvislou algebru je $\overline{A}$ pozitivní část A). $\overline{A} = \bigoplus_{n>0} A_n$

Hlavní idea: A dga, $A \simeq B \leftarrow$ htpicky ekvivalentní jako řet. komplexy

Jakou má B strukturu? $\qquad fg - id = \partial h - h\partial$

položme $xy = g(f(x) f(y))$; $\quad x \otimes y \otimes z \xrightarrow{} ~~(x \otimes y) \otimes z~~ \ (xy) \cdot z$

$\xrightarrow{2} ~~x \otimes (y \otimes z)~~ \ x (y \cdot z)$

jsou dvě homotopická zobrazení $B \otimes B \otimes B \rightrightarrows B$

cesta $(x \cdot y) \cdot z \xrightarrow{a_{xyz}} x \cdot (y \cdot z)$ $\quad$ asociativita pro x, y, z

co pro 4 prvky? $((xy)z)w \xrightarrow{a_{xy,z,w}} (xy)(zw) \xrightarrow{a_{x,y,zw}} x \cdot (y(zw))$

$a_{x,y,z} \cdot w \downarrow \qquad A$

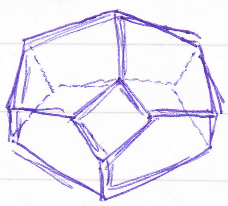
$(x \cdot (yz))w \xrightarrow{a_{x,yz,w}} x \cdot ((yz) \cdot w) \xrightarrow{x \cdot a_{y,z,w}} x \cdot (y(zw))$

 $\times B \times B \times B \times B \xrightarrow{A} B$

pro 5 prvků:

asociahedron - stěny jsou pětiúhelníky a čtverce $\quad K_5$

$K_5 \times B^5 \to B$



(Stasheff)

a pokračuje to dál...

K_n je konvexní polyedr dimenze $n-2$

$K_n \times B^n \to B$

umím sčítat = Algebraicky je vše o něco jednodušší v tom, že ∂K_n je součtem (až na znaménka) jednotlivých stěn a nemusíme řešit, jak se přesně tyto stěny potkávají.

Algebraicky konkrétně pro nízké dimenze:

$m_2: B \otimes B \to B$ násobení

$m_3: B \otimes B \otimes B \to B$

$[\partial, m_3](x,y,z) = \underbrace{m_2(x, m_2(y,z)) - m_2(m_2(x,y), z)}_{\text{stupeň roven } |x|+|y|+|z|}$

$\quad \| \quad$ ~~hranice~~ diferenciál v $Hom(B \otimes B \otimes B, B)$

$\partial m_3(x,y,z) + m_3(\partial x, y, z) + (-1)^{|x|} m_3(x, \partial y, z) + (-1)^{|x|+|y|} m_3(x, y, \partial z)$

$d(x \otimes y \otimes z) = dx \otimes y \otimes z + (-1)^{|x|} x \otimes dy \otimes z + (-1)^{|x|+|y|} x \otimes y \otimes dz$

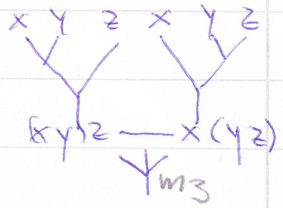
$|m_3| = 1$, protože $|[d, m_3]| = 0$

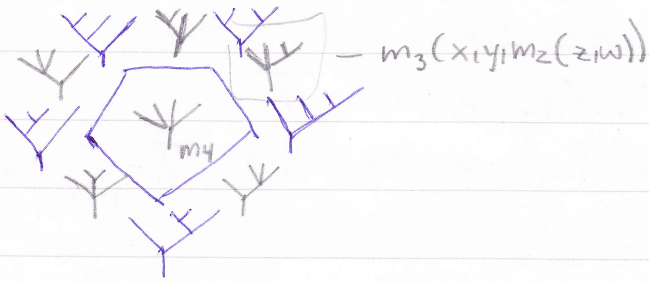
... homotopie mezi homotopiemi ⇒ stupeň 2

m_4: $B\otimes B\otimes B\otimes B \to B$ + strašná rovnice

$[d, m_4]$ = 5 členů odpovídajících 5 hranám K_4

uzávorkování lze reprezentovat binárním stromem





A_∞ algebras

Koalgebry a jejich koderivace

klasicky (ne-diferenciálně-gradované)

C... k-modul spolu s k-lin. zobr.

komnásobení $\Delta: C \to C\otimes C$, kojednotka $C \to k$

koasociativita

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C\otimes C \arrow[d, "\Delta\otimes id"] \\ C\otimes C \arrow[r, "id\otimes\Delta"'] & C\otimes C\otimes C \end{tikzcd}$$

unique $\Delta^{(n)}: C \to C^{\otimes n}$

kounitalita

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "id"] & C\otimes C \arrow[d, "\varepsilon\otimes id"] \\ C\otimes C \arrow[r, "id\otimes\varepsilon"'] & C \end{tikzcd}$$

Př. Tenzorová koalgebra

V... k-modul ... $T^cV = \bigoplus_{n\geq 0} V^{\otimes n}$

komponenty komnásobení jsou $\underset{T^cV}{V^{\otimes n}} \to \underset{T^cV\otimes T^cV}{V^{\otimes k}\otimes V^{\otimes l}}$ $\begin{cases} id & n=k+l \\ 0 & \text{jinak} \end{cases}$

$\Delta(u\otimes v) = \underset{k=V^{\otimes 0}}{1}\otimes \underset{V^{\otimes 2}}{(u\otimes v)} + \underset{V^{\otimes 1}}{u}\otimes \overset{\in V^{\otimes 1}}{v} + \underset{V^{\otimes 2}}{(u\otimes v)}\otimes 1_{\in V^{\otimes 0}}$

$\varepsilon: V^{\otimes n} \to k$ $\begin{cases} id & n=0 \\ 0 & \text{jinak} \end{cases}$

$(\varepsilon\otimes id)\Delta(u\otimes v) = u\otimes v + 0 + 0$

graded coalgebra: $\Delta(x_1\otimes\cdots\otimes x_n) = \sum_{i=0}^{n} \underset{T^cV}{(x_1\otimes\cdots\otimes x_i)} \otimes \underset{T^cV}{(x_{i+1}\otimes\cdots\otimes x_n)}$

Diferenciálně gradovaná koalgebra (dgc) C je $\begin{cases} \text{- řet. komplex} \\ \text{- koalgebra (koasoc., kounit.)} \end{cases}$

$\Delta: C \to C\otimes C$

$\varepsilon: C \to k[0]$

jsou řetězcové homomorfismy

$\varepsilon\, d(x) = 0$

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "d"'] & C\otimes C \arrow[d, "d = d\otimes id + id\otimes d"] \\ C \arrow[r, "\Delta"'] & C\otimes C \end{tikzcd}$$

$\equiv d$ je gradovaná koderivace

co Leibniz

$\boxed{d^2=0}$