$$\begin{tikzcd} A\otimes A \arrow[r, "m"] \arrow[d, "d"'] & A \arrow[d, "d"] \\ A\otimes A \arrow[r, "m"'] & A \end{tikzcd}$$

$d(x\cdot y) = (dx)y + (-1)^{|x|} x\cdot(dy)$

gradovaná derivace = Leibnizovo pr.

Def. $\varphi: C \to C$ je gradovaná koderivace, jestliže komutuje

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\varphi"'] & C\otimes C \arrow[d, "\varphi\otimes id + id\otimes\varphi"] \\ C \arrow[r, "\Delta"'] & C\otimes C \end{tikzcd}$$

$(id\otimes\varphi)(x\otimes y) = (-1)^{|\varphi||x|} x\otimes\varphi(y)$

pomoc průkaz: Nechť $\Delta(x) = \sum x_i \otimes x^i$. Potom $\Delta(\varphi(x)) = \sum(\varphi(x_i)\otimes x^i + (-1)^{|\varphi|\cdot|x_i|} x_i\otimes\varphi(x^i))$

Bar konstrukce

Nechť A je augmentovaná dga, $\bar{A} = \ker\varepsilon$ augmentační ideál

Definujeme $BA = T^c(\bar{A}[1])$ (bar konstrukce A), $\bar{A}$ posunuté o $+1$, $(\bar{A}[1])_{n+1} = \bar{A}_n$

jako grad. koalg. $= \bigoplus_{n\geq} \bar{A}[1]^{\otimes n}$

$A_1\otimes A_2\otimes A_3$

A_2

A_1

0

k

Na BA definujeme diferenciál, který bude koderivací, takže BA bude dgc. Pišme pro $x\in A_n$ $sx \in (A[1])_{n+1}$ (ten samý jako x, jen posunutý)

$d(sx_1\otimes\cdots\otimes sx_n) = d^{\otimes}(\quad) + d^{alg}(\quad) =$

$d^{\otimes}$: diferenciál na $\bigoplus(\bar{A}[1])^{\otimes n}$

$= \sum \pm sx_1\otimes\cdots\otimes sx_{i-1}\otimes sdx_i\otimes sx_{i+1}\otimes\cdots\otimes sx_n$
$+ \sum sx_1\otimes\cdots\otimes sx_{i-1}\otimes s(x_i x_{i+1})\otimes sx_{i+1}\otimes\cdots\otimes sx_n$

Let us study coderivations on T^cV more concretely

$$\begin{tikzcd} x\in T^cV \arrow[r, "\Delta^{(n)}"] \arrow[d, "d"'] & T^cV^{\otimes} \arrow[d, "d"] \arrow[r, "q^{\otimes n}"] & V^{\otimes n} \\ T^cV \arrow[r, "\Delta^{(n)}"] & T^cV^{\otimes n} \arrow[ur, "q^{\otimes n}"'] & \end{tikzcd}$$

commutes; $= \sum id^{\otimes(i-1)}\otimes d\otimes id^{\otimes(n-i)}$

$y = dx$, $y = \sum y_n$, y_n the homogenous part of length n

$q: T^cV \to V$ projection onto the summand $V^{\otimes 1}$

To compute dx, we study $\boxed{x = x_1\otimes\cdots\otimes x_k}$ $\quad q^{\otimes n} d\Delta^{(n)}(x_1\otimes\cdots\otimes x_k) =$ ($= (dx)_n$)

$= \sum q^{\otimes n}(id^{\otimes(i-1)}\otimes d\otimes id^{\otimes(n-i)})\Delta^{(n)}(x_1\otimes\cdots\otimes x_k) =$ ($(T^cV)^{\otimes n}$)

$= \sum q^{\otimes n}(\quad)((x_1)\otimes\cdots\otimes(x_{i-1})(x_i\otimes\cdots\otimes x_j)\otimes(x_{j+1})\otimes\cdots\otimes(x_k))$

$= \sum x_1\otimes\cdots\otimes x_{i-1}\otimes \underbrace{(d(x_i\otimes\cdots\otimes x_j))_1}_{\text{the part in length 1}}\otimes x_{j+1}\otimes\cdots\otimes x_k$

Denote $(d(\ ))_1$ by d_ℓ if $\ell = j-i+1$ $\qquad (\ell = k-n+1)$

$d_\ell: V^{\otimes\ell} \subseteq T^cV \xrightarrow{d} T^cV \xrightarrow{q} V$

$= \sum_{i,j,\ell \text{ s.t. } \ell=j-i+1} x_1\otimes\cdots\otimes x_{i-1}\otimes d_\ell(x_i\otimes\cdots\otimes x_j)\otimes x_{j+1}\otimes\cdots\otimes x_k$

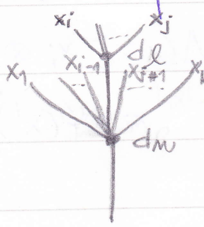
The point: $d_\ell : V^{\otimes \ell} \to V$ can be arbitrary!

$A = \bar{A} \oplus k[0]$ graded k-module

Def. A_n (weak) **A_∞-algebra** A is a dg-coalgebra structure on $T^c \bar{A}[1]$ 'cause it's differential

More concretely, it consists of maps $d_\ell : \bar{A}[1]^{\otimes \ell} \to \bar{A}[1]$ deg -1

satisfying $d^2 = 0$.



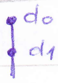
$\sum \pm d_\ell (x_1 \otimes \dots \otimes x_{i-1} \otimes d_m (x_i \otimes \dots \otimes x_j) \otimes$
$\otimes x_{j+1} \otimes \dots \otimes x_k) = 0$

k fixed $\quad$ $\ell = j - i + 1$, $m = k - \ell + 1$

Ex for low ℓ:

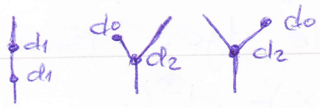
- $\ell = 0$: $d_0 : k \to \bar{A}[1]$ of deg -1, $1 \mapsto \Omega \in A_{-2}$
- $\ell = 1$: $d_1 : \bar{A}[1] \to \bar{A}[1]$ of deg -1 $\quad$ $\bar{A} \to \bar{A}$ "differential", but not quite

$d^2 = 0$ condition: $k = 0$



$\dots\ d_1(\Omega) = 0$

$k = 1$



$\pm d_1 d_1 (x) \pm d_2 (\Omega \otimes x)$
$\pm d_2 (x \otimes \Omega) = 0$

$d_1 d_1 (x) \pm [\Omega, x]$

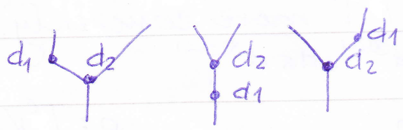
$D^2 \varphi = [\Omega \cdot \varphi]$

- $\ell = 2$: $d_2 : \bar{A}[1] \otimes \bar{A}[1] \to \bar{A}[1]$ deg -1

$m_2 : \bar{A} \otimes \bar{A} \to \bar{A} \longrightarrow \bar{A}$ deg 0
multiplication "A-bar"?

From now on, assume $d_0 = 0$. Then $d_1 d_1 = 0$. $\to$ weak A_∞-algebra

$k = 2$:

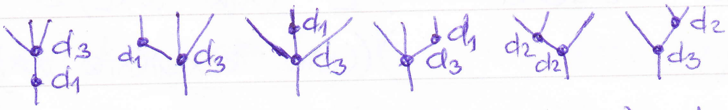


$(d^2 (x \otimes y)) \overset{!}{=} \big(d(\pm d_2(x \otimes y) \pm d_1 x \otimes y$
$\pm x \otimes d_1 y)\big) \overset{!}{=}$

$d_1 d_1 x \otimes y = 0$
$x \otimes d_1 d_1 y = 0$ $\Big\}$ by $k = 1$
$d_1 x \otimes d_1 y$
$d_1 x \otimes d_1 y$ $\Big\}$ cancel out

$= \pm d_1 d_2 (x \otimes y) \pm d_2 (d_1 x \otimes y) \pm d_2 (x \otimes d_1 y)$
$= \pm d_1 (x \cdot y) \pm (d_1 x) \cdot y \pm x \cdot d_1 y = 0$
$\to$ multiplication on A is a chain map

$k = 3$:



$\pm [d_1 d_3](x \otimes y \otimes z) = \pm d_1 d_3 (x \otimes y \otimes z) \pm d_3 (d_1 x \otimes y \otimes z) \pm$
$\pm d_3 (x \otimes d_1 y \otimes z) \pm d_3 (x \otimes y \otimes d_1 z) =$
$= \pm d_2 (d_2 (x \otimes y) \otimes z) \pm d_2 (x \otimes d_2 (y \otimes z)) = \pm (xy) z \pm x (yz)$

etc.

In general, $\cancel{d_l}$
$$\begin{tikzcd} \bar A[1]^{\otimes l} \arrow[r, "d_l"] & \bar A[1] \\ \bar A^{\otimes l} \arrow[u, "s^{\otimes l}", "\cong"'] \arrow[r, "m_l"] & \bar A \arrow[u, "s"', "\cong"] \end{tikzcd}$$

$s: \bar A \to \bar A[1]$ $\boxed{\deg 1}$
$x \mapsto sx = x$

$m_l : \bar A^{\otimes l} \to \bar A$ has degree $l + (-1) + (-1) = \underline{\underline{l-2}}$

$m_1 : \bar A \to \bar A$ diff

$m_2 : \bar A \otimes \bar A \to \bar A$ multip.

$m_3 : \bar A \otimes \bar A \otimes \bar A \to A$ assoc. htpy

In particular, any augmented dga A is an A_∞-algebra by setting: $m_1 = d \leftarrow$ differential in A, $m_2 = \mu \leftarrow$ mult. in A, $m_3 = 0, m_4 = 0, \dots$

– $d^2 = 0$ condition is $[d_1, d_4] = 0 \dots$

$$\begin{tikzcd} (-1)^{|x|} sx \otimes sy & \bar A[1] \otimes \bar A[1] \arrow[r, "d_2"] & \bar A[1] & s(xy) \\ & \bar A \otimes \bar A \arrow[u, "s \otimes s"] \arrow[r, "m_2"] & \bar A \arrow[u, "s"'] & \\ & x \otimes y & x \cdot y & \end{tikzcd}$$

& also $d_1(sx) = -s\,dx$

$d_2(sx \otimes sy) = (-1)^{|x|} s(x \cdot y)$

$d(sx_1 \otimes \dots \otimes sx_k) =$

$= \sum \pm sx_1 \otimes \dots \otimes sx_{i-1} \otimes \underset{\overset{\|}{\pm d_1(sx_i)}}{s(dx_i)} \otimes sx_{i+1} \otimes \dots \otimes sx_k$

$= \sum \pm sx_1 \otimes \dots \otimes sx_{i-1} \otimes s(x_i x_{i+1}) \otimes sx_{i+2} \otimes \dots \otimes sx_k$

$\to T^c \bar A$ with the differential is called **bar** of A, denoted **BA**.

Dual situation:

an **A_∞-coalgebra** $C = \bar C \oplus k[0]$ (a graded k-module)

is a dga-structure on $T\bar C[-1]$ tensor algebra

Again $d : T\bar C[-1] \to T\bar C[-1]$ is determined by maps $d_l : \bar C[-1] \to \bar C[-1]^{\otimes l}$,

namely $d(x_1 \otimes \dots \otimes x_k) = \sum \pm x_1 \otimes \dots \otimes x_{i-1} \otimes d_l(x_i) \otimes x_{i+1} \otimes \dots \otimes x_k$

require: $d_0 = 0$

$d_l = \bar C[-1] \to \bar C[-1]^{\otimes l}$ deg -1

$w_l = \bar C \to \bar C^{\otimes l}$ deg $l-2$

$d^2 = 0$ also translates into a condition on w_l's.

Ex. C a dgc $\Rightarrow$ C an A_∞-coalg. By setting $w_1 = d$, $w_2 = \Delta$, $w_3 = 0$, $w_4 = 0, \dots$

$d(s^{-1}x_1 \otimes \dots \otimes s^{-1}x_k) = \sum \pm s^{-1}x_1 \otimes \dots \otimes s^{-1}x_{i-1} \otimes s^{-1}dx_i \otimes s^{-1}x_{i+1} \otimes \dots \otimes s^{-1}x_k$

$+ \sum \pm s^{-1}x_1 \otimes \dots \otimes s^{-1}x_{i-1} \otimes (s^{-1} \otimes s^{-1}) \Delta x_i \otimes s^{-1}x_{i+1} \otimes \dots \otimes s^{-1}x_k$

This dga is denoted by **ΩC** .... **cobar** of C.

$$\begin{tikzcd} DGA \arrow[r, shift left, "B"] & DGC \arrow[l, shift left, "\Omega"] \end{tikzcd}$$

$\boxed{\textbf{Theorem:}}$ Let k be a field. Then $\exists$ $\Omega BA \xrightarrow{\text{dga-map}} A$ a q-iso.

$\exists$ $C \xrightarrow{\text{dgc-map}} B\Omega C$ a q-iso.

Pf: ΩBA as an algebra is generated by $s^{-1}(\underbrace{sx_1 \otimes \dots \otimes sx_n}_{\in BA}) =: (x_1|\dots|x_n)$

The map $\Omega BA \to A$ sends $(x) \mapsto x$

$(x_1|\dots|x_n) \mapsto 0 \quad n>1$

There are subcomplexes of ΩBA formed by products $(x_1|\dots|x_n)\dots(z_1|\dots|z_m)$ with $n+\dots+m \leq \ell$

each A_k is free (k field) restrict x_i's to basis elts

$$\begin{tikzcd}
C_1 \arrow[r, "\simeq"] & A \\
\cap & \\
C_2 & \\
\cap & \\
\vdots & \\
\cap & \\
\bigcup_\ell C_\ell = \Omega BA \arrow[uuuuur] &
\end{tikzcd}$$

show: $C_\ell / C_{\ell-1}$ is contractible

contraction: $(x)(x_1|\dots|x_n)\varphi \overset{h}{\mapsto} (x|x_1|\dots|x_n)\varphi$ (φ = the rest)

any other product $\mapsto 0$

$$\mathrm{DGC} \underset{B}{\overset{\Omega}{\rightleftarrows}} \mathrm{DGA} \quad \text{it is an adjunction}$$

$$\underline{\mathrm{Hom}_{DGA}(\Omega C, A)} \cong TC(C,A) \cong \mathrm{Hom}_{DGC}(C, BA) \qquad (*)$$

$\Omega C = T(s^{-1}C)$ with a suitable differential

"twisting cochains"

uniquely determined by $s^{-1}C \xrightarrow{t} A$ $\quad ((s^{-1}C)^{\otimes n} \xrightarrow{t^{\otimes n}} A^{\otimes n} \xrightarrow{\text{mult.}} A)$

subject to the condition that the algebra map induced by t commutes with the differential:

$$dt(x\cdot y) = d(t(x)\cdot t(y)) = \underbrace{dt(x)\cdot t(y) \pm t(x)\cdot dt(y)}_{\text{suppose that } td=-dt \text{ on } x,y} = -td(x)t(y) \mp t(x)td(y) =$$

$$= t(-d(x)\cdot y \mp x\cdot dy) = -td(x\cdot y)$$

$\Rightarrow$ if $td+dt=0$ on $s^{-1}C$ then $td+dt=0$ on ΩC

$$\begin{tikzcd}
s^{-1}x \arrow[d, mapsto] & s^{-1}C \arrow[r, "t"] \arrow[d, "d"] & A \arrow[d, "d"] \\
-s^{-1}(dx) \pm (s^{-1}\otimes s^{-1})\Delta(x) & \Omega C \arrow[r, "t"] & A
\end{tikzcd}$$

in terms of $t: C \to A$

$\underline{tdx \pm m(t\otimes t)\Delta(x) = 0}$ twisting cochain condition

$+dtx$

$[d,t](x)$

$$\begin{tikzcd}
C \arrow[d, "\Delta"] \arrow[r] & A \\
C\otimes C \arrow[r, "t\otimes t"] & A\otimes A \arrow[u, "m"]
\end{tikzcd}$$

M.-Cartan equation $\quad [d,t] + t^2 = 0$

The "dual version" is the same / proving $(*)$

counit ... the DGA-map that corresponds to the identity

$\mathrm{id}: BA \to BA \quad (C = BA)$

$$\begin{array}{c} T^c(SA) \\ \| \\ BA \longrightarrow A \\ sx_1\otimes\cdots\otimes sx_n \longmapsto \begin{cases} x_1 & n=1 \\ 0 & \text{else} \end{cases} \end{array}$$

$\Omega BA \to A$ canonical twisting cochain

$\uparrow$ write $(x_1|\dots|x_n)$ for $s^{-1}(sx_1\otimes\dots\otimes sx_n)$

$(x_1|\dots|x_n) \longmapsto \begin{cases} x_1 & n=1 \\ 0 & \text{else} \end{cases}$

$\boxed{\text{Theorem}}$ This map $\Omega BA \to A$ is a htpy equivalence if all k-modules A_n are free (projective) – in particular, this is the case for k a field.

Pf. F_ℓ = subcomplex of ΩBA formed by elts of length $\leq \ell$

where length $(x_1|\dots|x_n) = n$ and length$(x\cdot y)$ = length(x) + length(y)

$d(x|y) = (dx|y) \pm (x|dy) \pm (x)(y) \pm (xy)$

$\underset{\{(x) \mid x\in A\}}{F_1} \subseteq \Omega BA \to A$ is an iso

$F_1 \subseteq F_2 \subseteq \dots$ and $\Omega BA = \bigcup_\ell F_\ell$

$H_*(\Omega BA) \cong \operatorname{colim}(H_*F_1 \to H_*F_2 \to \dots)$ (directed colimits are exact)

Therefore, ETS (= enough to show): $F_{\ell-1} \hookrightarrow F_\ell$ q-iso

The LES associated with for $0 \to F_{\ell-1} \to F_\ell \to F_\ell/F_{\ell-1} \to 0$ shows that it is enough that $H_*(F_\ell/F_{\ell-1}) = 0$. (a htpy $\mathrm{id} \simeq 0$)

A contraction of $F_\ell/F_{\ell-1}$ is given by $(x)(x_1|\dots|x_n)\cdot\{ \;\longmapsto \pm(x|x_1|\dots|x_n)\cdot\{$

else $\longmapsto 0$

the generators of ΩBA $\quad \underbrace{(\cdot|\dots|\cdot)\dots\dots(\cdot|\dots|\cdot)}_{\text{basis elts of } A}$

the diff on $F_\ell/F_{\ell-1}$ is

$$d(x_1,\dots,x_n) = \sum \pm (x_1|\dots|x_{i-1}|dx_i|x_{i+1}|\dots|x_n) \pm \sum \pm (x_1|\dots|x_i)(x_{i+1}|\dots|x_n)$$

$n>1$

$$\begin{tikzcd} (x_1|\dots|x_n)\{ \arrow[r, "h"] \arrow[d, "d"'] & 0 \arrow[d, "d"] \\ \pm(x_1)(x_2|\dots|x_n)\{ + \text{other terms} \arrow[r, "h"] & 0 \quad (x_1|x_2|\dots|x_n)\{ \end{tikzcd}$$

$\blacksquare$

If q-iso's are thought of as iso's then we get an equiv.

$\mathbb{D}(\mathcal{D}GC) \simeq \mathbb{D}(\mathcal{D}GA)$

"derived category"

(later)

Improvement: $\mathcal{D}GC_{nilt} \underset{B}{\overset{\longrightarrow}{\longleftarrow}} A_\infty\text{-Alg}$

$BA = T^c(sA)$ with a correct differential $= A_\infty$-str.

$\Omega_\infty C$ ... add freely $m_2, m_3, \dots$ to $\bar{s}^1 C$ with a suitable diff

$m_3(m_2(x,y),z,w)$

$$d(\bar{s}^1 x) = \pm \bar{s}^1(dx) \pm m_2(\bar{s}^1 \otimes \bar{s}^1)\Delta^{(2)} x \longleftarrow \Omega C$$
$$\pm m_3(\bar{s}^1 \otimes \bar{s}^1 \otimes \bar{s}^1)\Delta^{(3)} x$$
$$\pm \dots$$

Ex $\Omega_\infty BA$ $\quad d(x_1|x_2|x_3) =$ the same as in ΩBA $+ m_3((x_1),(x_2),(x_3))$

The same works (with a lot of work, HOPEFULLY) $\mathcal{D}(\mathcal{D}GC_{nilt}) \simeq \mathcal{D}(A_\infty\text{-Alg})$

Consequence: For an A_∞-alg. A, we have $\Omega_\infty BA \xrightarrow{q\text{-iso}} A$

$\Omega_\infty BA \downarrow \Omega BA$ — Send all m_3's, m_4's to zero

$\boxed{\text{Theorem:}}$ is q-iso

Rectification: Any A_∞-alg is q-iso to a dga.

Our goal now: $\quad A \dots A_\infty$-Alg, $f: A \simeq A'$ chain htpy equiv

$\Longrightarrow A'$ is also an A_∞-Alg and f is an "A_∞-map"

Together: A_∞-alg $=$ something htpy equiv. to a dga

Homotopy equiv's and strong deformation retractions (SDR)

SDR: $(f,g,h): C \rightrightarrows D$ s.t. • $fg = id$

where $f: C \to D$ ch. map deg 0 • $id - gf = [d,h]$

$g: D \to C$ ch. map deg 0 • $fh = 0$, $hg = 0$, $hh = 0$

$h: C \to C$ deg 1

Proposition: Any htpy equiv "can be replaced" by a span of SDR

$$\begin{tikzcd} & C \arrow[dl, Rightarrow, "{(f,g,h)}"'] \arrow[dr, Rightarrow, "{(f',g',h')}"] & \\ D \arrow[rr, bend right, "\varphi = f'g"'] & & D' \end{tikzcd}$$

$C = cyl(\varphi)$

$\varphi = f'g$ ← the original htpy equiv

Suppose A is an A_∞-alg.

$A \xrightarrow[f]{\simeq} A'$ — f is a part of a SDR (f,g,h)

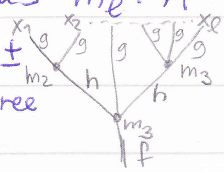
$BA \dashrightarrow BA'$? what is the differential?

$BA = T^c(sA)$ with a certain differential

$BA' = T^c(sA')$

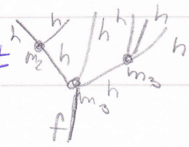
---

differential $\equiv$ operations $m_\ell : A'^{\otimes \ell} \to A'$ deg $\ell-2$

$m_\ell(x_1 \otimes \dots \otimes x_\ell) = \sum_{T \text{ tree}} \pm$  $\pm f m_3(h\, m_2(g(x_1), g(x_2)), g x_3, h\, m_3(g x_{\ell-2}, g x_{\ell-1}, g x_\ell))$

what about coalg. ~~[illegible]~~ $BA \to BA'$?

tw. c. $BA \to A'$

i.e. $s^{-1}BA \to A'$ $\qquad (x_1|\dots|x_\ell) \mapsto \sum_{T \text{ tree}} \pm$ 

$\oplus\, s^{-1}(sA)^{\otimes \ell} \to A'$

Def. A_∞-map $A \to A'$ = coalg. map $BA \to BA'$

8.-15.6. away