§2 Categorical semantics of dependent type theories

In the beginning of §1 Syntax, we have discussed various choices of meta-theoretical presentation. Each such presentation of syntax determines - or at least suggests - an accordingly structured presentation of the associated semantics.

① In Chapter III we presented the simply typed λ-calculus in the Church version. Here, types are absolute notions. We interpreted types as objects and closed terms as morphisms. General terms were not directly interpreted - instead we bounded away all their free variables by λ-Abstraction and interpreted the resulting closed term.

A similar semantics is definable even for dependent type theories (when presented in the according Church version). Indeed, we have seen that most structural theory can be reduced to the empty context by way of the logical rules for Π- and Σ-types - much in the same way we reduced open terms to closed terms in S.T.T.

A corresponding categorical semantics of $MLTT^{\Sigma,\Pi,1,=}$ with judgemental Σ-uniqueness and extensional identity-types (i.e. with Equality-Reflection) has been worked out by Seely in "Locally cartesian closed categories and type theory".

Here, closed types are once again interpreted as objects of a category, closed terms of function type are interpreted as its morphisms.

- → A terminal object is given by the terminal type 1.
- → Pullbacks of spans $A \xrightarrow{a} C \xleftarrow{b} B$ are computed via

$$\begin{tikzcd}
\sum\limits_{x:A} \sum\limits_{y:B} \mathrm{App}(a,x) =_C \mathrm{App}(b,y) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "b"] \\
A \arrow[r, "a"'] & C
\end{tikzcd}$$

- → Exponentials in the slice over a closed type C are given for

$$\begin{tikzcd}
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

via identification with

$$\begin{tikzcd}
\sum\limits_{c:C} \underbrace{\sum\limits_{a:A} f(a)=c}_{f^{-1}(c)} \arrow[r, "\pi_1"] & C & \sum\limits_{c:C} \underbrace{\sum\limits_{b:B} g(b)=c}_{g^{-1}(c)} \arrow[l, "\pi_1"']
\end{tikzcd}$$

by the term

$$\pi_1 : \Big(\sum_{c:C} \big(f^{-1}(c) \to g^{-1}(c)\big)\Big) \longrightarrow C$$

(via Lemma 3.2.3 in Seely's paper).

Caveat: There is a coherence issue with Seely's original construction concerning the interpretation of substitution of terms in type families. It has been rectified by Curien, Hofmann, and ultimately by Clairambault and Dybjer. They show that this interpretation gives rise to a biequivalence of 2-categories between a 2-category of $\mathrm{MLTT}^{\Sigma,\Pi,1,=}$-type theories, and the 2-category of locally cartesian closed categories.

② In the Curry-de Bruijn version we chose, we have a third fundamental notion: The contexts. A categorical interpretation of the structural dependency of contexts, types and terms is given by Cartmell's contextual categories. They constitute today's standard for categorical expressions of type theoretical syntax. Indeed, among all categorical frameworks associated to MLTT, it is the closest to the syntax, and hence a common reference point.

Motivation: Given Σ-types, a type family $x:A \vdash B$ type gives rise to a "fibered" type $\emptyset \vdash \pi_1 : \sum_{a:A} B \twoheadrightarrow A$ with fibers $B[a/x]$.

Given furthermore universes, we equivalently obtain an 'indexed' type $\beta : A \to \mathcal{U}_i$ s.th.

$$\begin{tikzcd} \sum_{a:A} B \arrow[rr, "\simeq"] \arrow[dr, two heads, "\pi_1"'] & & \sum_{a:A} El(App(\beta, a)) \arrow[dl, two heads, "\pi_1"] \\ & A & \end{tikzcd}$$

Thus, some form of comprehension (reminding of the Grothendieck construction) is implicitly baked into MLTT.

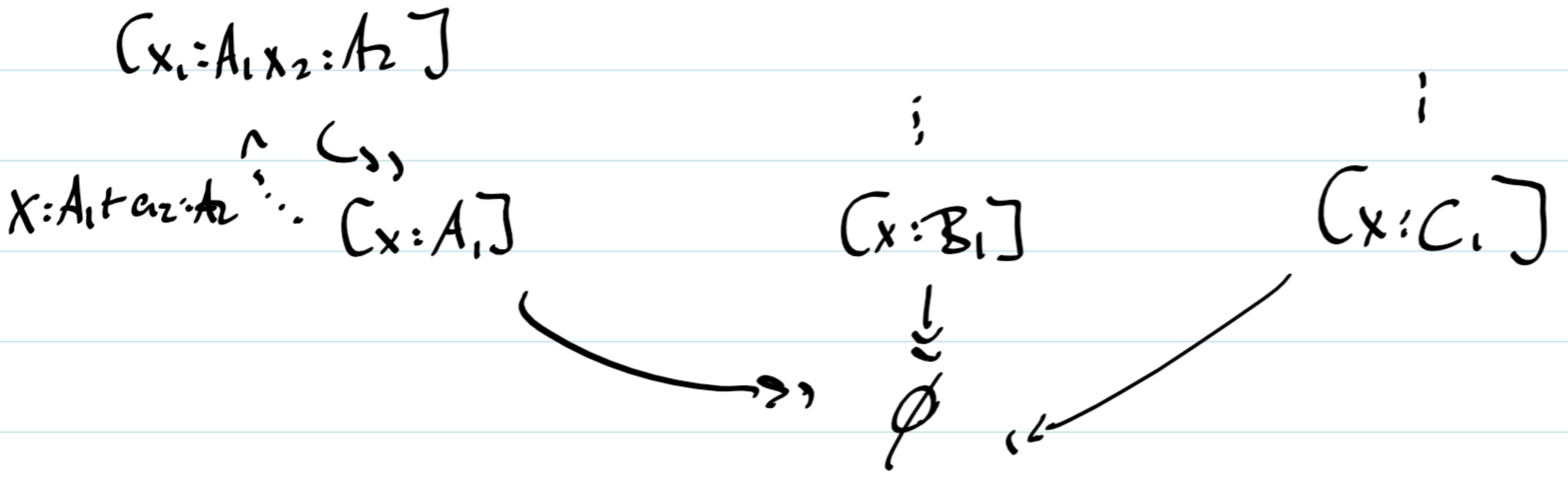
Idea: We may interpret type families as fibrations right away (independently of the specific logical rules at hand).

In this abstract framework, the "fibrations" are often referred to as dependent projections or display maps).

Remark: Recall that we interpreted in the untyped and simply typed cases contexts of free variables as (plain) projections as well.

We will understand

* (Congruence classes of) contexts as objects
* (Congruence classes of) context extensions $\Gamma \vdash A\ \mathrm{type}$ and their finite iterations as distinguished arrows $(\Gamma, x:A) \twoheadrightarrow \Gamma$ called dependent projections.
* Sequences of (congruence classes of) terms as morphisms, s.th. single terms $\Gamma \vdash a : A$ correspond to sections $\Gamma \overset{a}{\dashrightarrow} (\Gamma, x:A)$.



A contextual category comes with a tree structure.

We don't interpret typability judgements, but only the ass. ctx. formations.

Yet, we interpret type-assignment directly.

Definition 1: A contextual category is a category $\mathcal{C}$ with

1. a terminal object 1;
2. a tree structure "$\leq$" on the set $Ob(\mathcal{C})$ with 1 as root. We obtain a partition $Ob(\mathcal{C}) = \coprod_{n \geq 0} Ob_n(\mathcal{C})$ together with maps $ft : Ob_{n+1}(\mathcal{C}) \to Ob_n(\mathcal{C})$ such that $Ob_0(\mathcal{C}) = \{1\}$;
3. for every $n \geq 0$, $C \in Ob_{n+1}(\mathcal{C})$, a distinguished morphism $p_C : C \twoheadrightarrow ft(C)$,

called the dependent projection from C and denoted by $C \twoheadrightarrow ft(C)$;

4. for every $C \in Ob_{n+1}(\mathcal{C})$ and every $f : B \to ft(C)$, a distinguished representative $(f^*C, q(f,C))$ of pullback along p_C s.th.

$$\begin{tikzcd} f^*C \arrow[r, "q(f{,}C)"] \arrow[d, two heads, "p_{f^*C}"'] & C \arrow[d, two heads, "p_C"] \\ B \arrow[r, "f"'] & ft(C) \end{tikzcd}$$

is a pullback square in $\mathcal{C}$, and such that these representatives are strictly functorial:

- $(id_{ft(C)}^* C, q(id_{ft(C)}, C)) = (C, id_C)$
- for $f : B \to ft(C)$, $g : A \to B$,
  $((f \circ g)^* C, q(f \circ g, C)) = (g^* f^* C, q(f,C) \circ q(g, f^*C))$.

Proposition 2: Every type theory T satisfying the structural rules from §1 gives rise to a contextual category $\mathcal{C}(T)$.

Proof: Let $(Ob(\mathcal{C}(T)), \leq)$ be the tree of congruence classes of valid context-extensions, i.e. $1 = \emptyset$, and $\Gamma \leq \Delta$ iff $\Delta = (\Gamma, \theta)$ for some precontext θ. So $ft(\Gamma, x : A) = \Gamma$ whenever $\Gamma \vdash A$ type.

Given contexts $\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$, $\Delta = (y_1 : B_1, \ldots, y_m : B_m)$, let

$$Hom(\Gamma, \Delta) := \{ (f_1, \ldots, f_m) \in \text{Preterm} \mid \Gamma \vdash f_1 : B_1, \Gamma \vdash f_2 : B_2[f_1/y_1], \ldots, \Gamma \vdash f_m : B_m[f_1/y_1] \ldots [f_{m-1}/y_{m-1}] \} / \Gamma \vdash \_ \equiv \_ : \Delta$$

i.e. $\Gamma \vdash (f_1, \ldots, f_m) : \sum_{y_1 : B_1} \ldots \sum_{y_{m-1} : B_{m-1}} B_m$ whenever Σ-types exist.

where $\Gamma \vdash \vec{f} \equiv \vec{g} : \Delta$ iff $\Gamma \vdash f_1 \equiv g_1 : B_1, \ldots, \Gamma \vdash f_m \equiv g_m : B_m[\vec{f}/\vec{y}]$.

Let

$$id_\Gamma := (x_1, \ldots, x_n) \in Hom(\Gamma, \Gamma),$$

and whenever $\Gamma \vdash A\ type$, let

$$p_{(\Gamma, x:A)} := (x_1, \ldots, x_n) \in Hom((\Gamma, x:A), \Gamma).$$

In particular, whenever $\Gamma \vdash a : A$, we get

$$f_a := (x_1, \ldots, x_n, a) : \Gamma \longrightarrow (\Gamma, x:A),$$

and we have $Hom(\Gamma, \phi) = \{\phi\}$.

Given $f = (f_1, \ldots, f_m) : \Gamma \longrightarrow \Delta$, $g = (g_1, \ldots, g_k) : \Delta \longrightarrow \Theta$, let

$$g \circ f := (g_1[\vec{f}/\Delta], \ldots, g_k[\vec{f}/\Delta]).$$

Substitution of terms for free variables in terms is modelled by composition of arrows and hence always strictly functorial.

Exercise 3: 1. The composition is well-defined on congruence classes, and associative.

2. $g \circ f \in Hom(\Delta, \Theta)$.

3. $id_\Delta \circ f = f = f \circ id_\Gamma \in Hom(\Gamma, \Delta)$.

4. For any $a \in$ Preterm, if $\Gamma \vdash a : A$, then $p_{(\Gamma, x:A)} \circ f_t = id_\Gamma$.

5. If $g = (g_1, \ldots, g_{n+1}) : \Gamma \longrightarrow (\Gamma, x:A)$ is a section to $p_{(\Gamma, x:A)}$, then $g = f_{g_{n+1}} \in Hom(\Gamma, (\Gamma, x:A))$.

We are left to construct split pullback presentations as required in Def. 1.4.

Thus, given $f : \Gamma \to \Delta$, $\Delta \vdash A$ type, let

$$f^*(\Delta, x : A) := (\Gamma, y : A[\vec{f}/\Delta])$$
$$q(f, (\Delta, y : A)) := (\vec{f}, y) : (\Gamma, y : A[\vec{f}/\Delta]) \longrightarrow (\Delta, y : A).$$

Exercise 4:
$$\begin{tikzcd} (\Gamma, y : A[\vec{f}/\Delta]) \arrow[r, "(\vec{f}, y)"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner"] & (\Delta, y : A) \arrow[d, "p"] \\ \Gamma \arrow[r] & \Delta \end{tikzcd}$$
is a pb.-square in $\mathcal{C}(T)$.

Substitution of terms for free variables in types is modelled by pb. action $\rightsquigarrow$ therefore the split pres.!

The fact that this cleavage of pb. presentations is split corresponds exactly to the fact that substitution (of terms in types) in T is strictly associative and reflexive. □

The category ContCat of contextual categories is given by an according notion of contextual functors which preserve all contextual structure on the nose. A category of barebone dependent type theories is given by Cartmell's category GAT of generalized algebraic theories and interpretations (up to equivalence wrt. "intended identity of denotation"), see Section 12 in Cartmell's paper.

Theorem 5 (Cartmell): The term-model construction gives rise to an equivalence

$\mathcal{C} : \mathrm{GAT} \longrightarrow \mathrm{ContCat}$ of categories. □

In principle – via the "Initiality Conjecture / Theorem" – for any set S of logical rules, there is a set $\bar{S}$ of corresponding categorical structures on contextual categories s.th. $C : GAT \to CxtCat$ extends to an equivalence

$$C : GAT^{(s)s \in S} \longrightarrow CxtCat^{(s)s \in \bar{S}} .$$

Example 6: A Π-type structure on a contextual category $\mathcal{C}$ is a triple of assignments (Π, λ, app) of the following form.

1. (Π-formation) $\Pi : Ob_{n+2}(\mathcal{C}) \to Ob_{n+1}(\mathcal{C})$ is a function such that $ft(\Pi(C)) = ft(ft(C))$ f.a. $C \in Ob_{n+2}(\mathcal{C})$, $n \geq 0$;

2. (Π-Introduction) $$\lambda : \underbrace{\left\{ \begin{tikzcd} C \arrow[d, two heads, "p_C"'] \\ ft(C) \arrow[u, dashed, bend right, "s"'] \end{tikzcd} \right\}}_{=: Sec(p_C)} \longrightarrow \underbrace{\left\{ \begin{tikzcd} \Pi(C) \arrow[d, two heads] \\ ft(ft(C)) \arrow[u, dashed, bend right] \end{tikzcd} \right\}}_{=: Sec(p_{\Pi(C)})}$$ for all $C \in Ob_{n+2}(\mathcal{C})$;

3. (Π-Elimination) For each pair of sections $f \in Sec(p_{\Pi(C)})$, $t \in Sec(p_{ft(C)})$, a section $app(f,t) \in Sec(p_{t^*C})$,

$$\begin{tikzcd}
 & & C \arrow[d, two heads] & \\
t^*C \arrow[urr] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & ft(C) \arrow[d, two heads] & \Pi(C) \arrow[dl, two heads] \\
ft(ft(C)) \arrow[u, dashed, bend left, "app(f,t)"] \arrow[urr, dashed, "t"] & = & ft(ft(C)) \arrow[ur, dashed, bend right, "f"'] &
\end{tikzcd}$$

such that

(a) (Π-Computation) For all sections $s \in Sec(p_C)$, $t \in Sec(p_{ft(C)})$:

$$app(\lambda s, t) = t^* s \quad ;$$

(b) (Stability / Substitution) The assignments are stable under pullback:

For $f : D \to ft(ft(C))$,

* $f^*(\Pi(C)) = \Pi(f^*C)$,
* $\lambda(f^*s) = f^*(\lambda_s)$,
* $f^*\mathrm{app}(s,t) = \mathrm{app}(f^*s, f^*t)$;

(c) (Π-Uniqueness / η-Congruence) For any section $s \in \mathrm{Sec}(p_C)$, the section $\lambda s \in \mathrm{Sec}(p_{\Pi(C)})$ is the unique section of $p_{\Pi(C)}$ which satisfies (a): i.e. whenever $f \in \mathrm{Sec}(p_{\Pi(C)})$ is a section s.th. $\mathrm{app}(f,t) = t^*s$ f.a. sections $t \in \mathrm{Sec}(p_{ft(C)})$, then $f = \lambda s$.

There are no counterparts to Equality and Congruence rules as the categorical semantics is defined up to judgemental equality.

Theorem 7 (Cartmell): $C : GAT^{\Pi} \xrightarrow{\simeq} ContCat^{\Pi}$, where the respective notions of Π-preserving interpretations and Π-structure preserving contextual functors are defined accordingly.

Further logical rules including $\Sigma, 0, 1, \mathbb{N}, =, +, \mathcal{U}_i$ for $i < \omega$ have been proven to lead to according results in Streicher's "Semantics of type theory".

Example 8: The category Fam of families of sets is a contextual category.

Its objects at level $n \geq 1$ are sequences of functions of sets

$\vec{X} = (X_n \xrightarrow{f_n} X_{n-1} \to \dots \xrightarrow{f_1} X_1)$ presented by a specific iterated indexed set under

the equivalences $\int(X, -) : \mathrm{Fun}(X, \mathrm{Set}) \xrightarrow{\simeq} \mathrm{Set}/X$ for $X \in \mathrm{Set}$.

i.e. for a sequence of functions $X \xrightarrow{f_1} \mathrm{Set}$, $\int(X, f_1)_0 \xrightarrow{f_2} \mathrm{Set}, \dots,$

$\int(\dots\int(\int(X, f_1), f_2), \dots), f_{n-1})_0 \xrightarrow{f_n} \mathrm{Set}$, let

$$\int(X, f_1, \dots, f_n) := \int(\dots\int(\int(X, f_1), f_2), \dots), f_n)_0.$$

Let $\mathrm{Ob}_0(\mathrm{Fam}) = \{\emptyset\}$. For $n \geq 1$, let

$$\mathrm{Ob}_n(\mathrm{Fam}) = \{ (f_1, \dots, f_n) \in \mathrm{Set}^{\to} \mid \forall 1 \leq i \leq n : \; f_i : \int(\{\emptyset\}, f_1, \dots, f_{i-1}) \to \mathrm{Set} \}$$

Morphisms $(f_1, \dots, f_n) \to (g_1, \dots, g_m)$ are dependent families of "operators" in the sense of Cartmell, or equivalently, simply functions

$\int(\{\emptyset\}, f_1, \dots, f_n) \to \int(\{\emptyset\}, g_1, \dots, g_m)$.

The dependent projection of $(f_1, \dots, f_{n+1})$ is given by

$$\int(\{\emptyset\}, f_1, \dots, f_n)(f_{n+1}) : \int(\{\emptyset\}, f_1, \dots, f_{n+1}) \twoheadrightarrow \underbrace{\int(\{\emptyset\}, f_1, \dots, f_n)}_{=: \mathrm{ft}(\int(\{\emptyset\}, f_1, \dots, f_{n+1}))}.$$

Lastly, given $(f_1, \dots, f_{n+1}) \in \mathrm{Fam}$ and a map $\alpha : (g_1, \dots, g_m) \to (f_1, \dots, f_n)$, the canonical pullback $\alpha^*(f_1, \dots, f_{n+1}) \in \mathrm{Fam}$ is given by

$$\int\left((\{\emptyset\}, g_1, \dots, g_m), \int(\{\emptyset\}, g_1, \dots, g_m) \xrightarrow[\alpha]{} \int(\{\emptyset\}, f_1, \dots, f_n) \xrightarrow[f_{n+1}]{} \mathrm{Set}\right),$$

using that "unstraightening" $\int_-$ takes precomposition to pullback-action.

This is strictly functorial, because both precomposition and unstraightening are.

The contextual category Fam plays the same role for dependent type theories as the category Set does for 1st order theories:

For any $\mathbb{T} \in \mathrm{GAT}$, one may define the category $\mathrm{Mod}(\mathbb{T})$ of Set-valued models. Then $\mathrm{Mod}(\mathbb{T}) \simeq \mathrm{ContCat}(\mathcal{C}(\mathbb{T}), \mathrm{Fam})$.

Example 9: An identity-type structure on a contextual category $\mathcal{C}$ consists of

1.+2. For every object $C \in \mathrm{Ob}_n(\mathcal{C})$, $n \geq 1$, an object $\mathrm{Id}(C) \twoheadrightarrow p_C^* C$ together with a section $\mathrm{refl}_C : C \to \mathrm{Id}(C)$ s.th. $p_{\mathrm{Id}(C)} \circ \mathrm{refl}_C = (1_C, 1_C)$.

3.+4. For each $D \twoheadrightarrow \mathrm{Id}(C)$, $d \in \mathrm{Sec}(p_{\mathrm{Id}(C)} \circ p_D)$ s.th. $p_D \circ d = \mathrm{refl}_C$, a section $J_{D,d} \in \mathrm{Sec}(p_D)$ s.th. $J_{D,d} \circ \mathrm{refl}_C = d$, and

5. s.th. all notions are strictly pullback-stable.

$$\begin{tikzcd}
 & & D \arrow[d, two heads] \\
 & & \mathrm{Id}(C) \arrow[d, two heads] \arrow[u, bend right, dotted, "J_{D,d}"'] \\
C \arrow[rr, "\Delta"'] \arrow[urr, dashed, "\mathrm{refl}_C"] \arrow[uurr, dashed, bend left, "d"] & & C \times_{ft(C)} C
\end{tikzcd}$$

Thus, identity-type structure on a contextual category $\mathcal{C}$ yields f.a. $C \in \mathrm{Ob}_n(\mathcal{C})$, $n \geq 1$, a factorization of the diagonal $C \to C \times_{ft(C)} C$ through a dependent projection $C \xrightarrow{\mathrm{refl}_C} \mathrm{Id}(C) \twoheadrightarrow C \times_{ft(C)} C$ s.th. for all (sdid) squares of the form

$$\begin{tikzcd}
C \arrow[r, "d"] \arrow[d, "\mathrm{refl}_C"'] & D \arrow[d, two heads, "p_D"] \\
\mathrm{Id}(C) \arrow[r, equal] \arrow[ur, dashed, "J_{D,d}"] & \mathrm{Id}(C)
\end{tikzcd}$$

there is a (dotted) lift. Thus, identity-type structure equips $\mathcal{C}(\mathbb{T})$

with path-objects for a notion of fibrations generated by the dependent projections. Indeed, identity-type structures induce a stable weak factorization system on $\mathcal{C}$ generated by the dependent projections. See Gambino and Garner's "The identity type weak factorisation system".

The observation that identity types have semantic interpretation as path-objects in suitable model categories so that the J-Eliminator yields the characteristic lifts, was first noted in Awodey, Warren- "Homotopy theoretic models of identity types".