A few general remarks on Cartmell's generalized algebraic theories (which capture Martin-Löf type theories with all sorts of logical rules) and contextual categories:

- $\mathrm{ContCat} \simeq \mathrm{Mod}(\mathcal{E}_{cc})$ for an essentially algebraic theory $\mathcal{E}_{cc}$ with sorts indexed by $\mathbb{N} + \mathbb{N}\times\mathbb{N}$.

  tree structure on objects

  accordingly indexed morphism-sets

- In fact, Cartmell states that

  Set-valued models

  1. $\forall\, T \in GAT\ \exists\, \mathcal{E} \in EAT:\quad \mathrm{Mod}_{GAT}(T) \simeq \mathrm{Mod}_{EAT}(\mathcal{E})$ (This appears to be left somewhat unproven though)
  2. $\forall\, \mathcal{E} \in EAT\ \exists\, T \in GAT^{\Sigma,=}:\quad \mathcal{C}(\mathcal{E}) \simeq \mathcal{C}(T)$

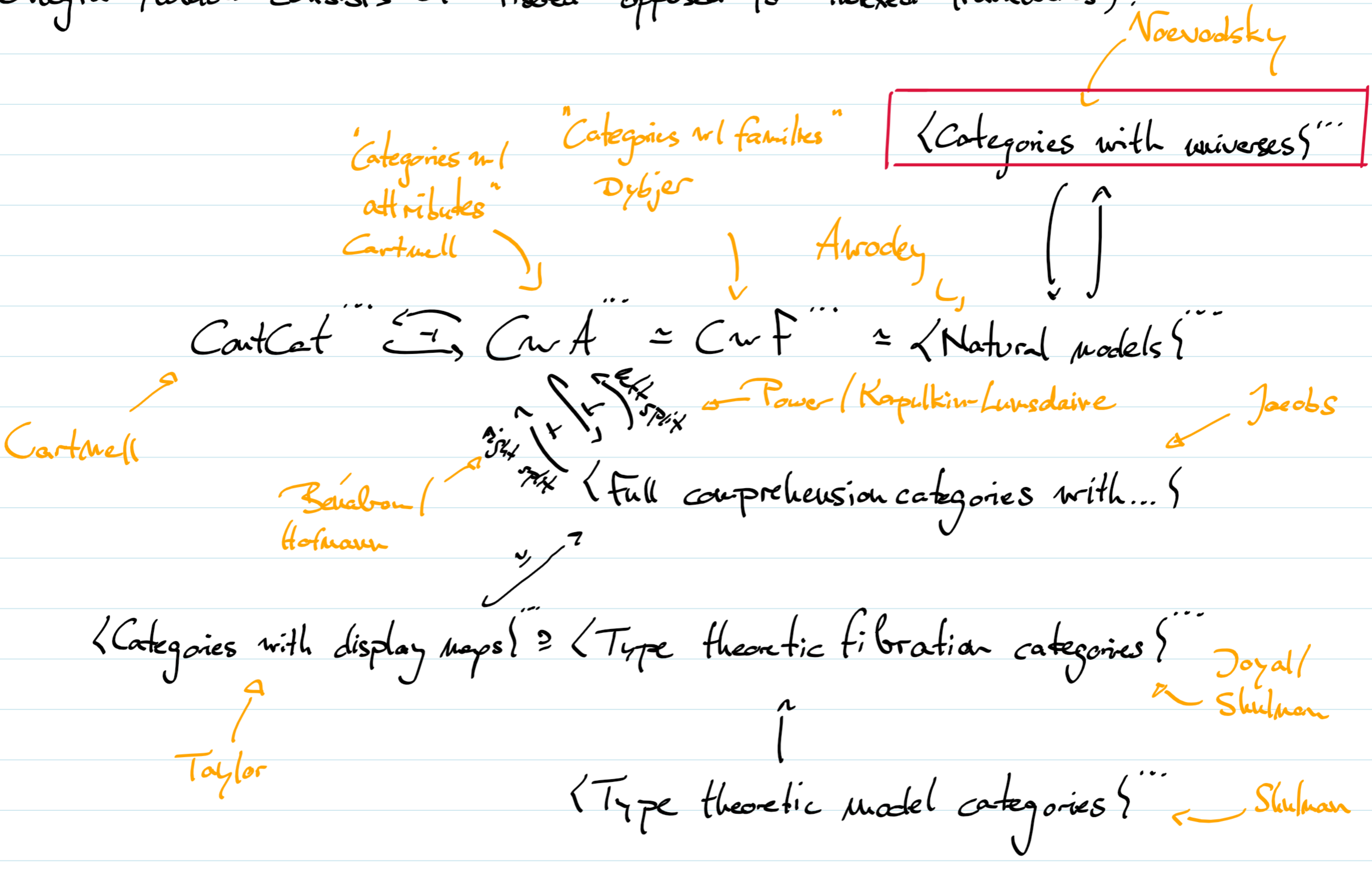
     as categories.

Independently of the status of 1., it generally may not be enough to characterize the Set-valued models of a GAT to characterize the GAT itself. Generic examples of models include the global sections functor ($\Gamma \in \mathrm{ContCat}(\mathcal{C}(T), \mathrm{Fam})$), and the constant terminal object functor.

An according higher categorical version of 2. has been proven Kapulkin and Szumiło, in the sense that they construct an equivalence between $GAT^{\Sigma,=}$ and the $(\infty,1)$-category of lex $(\infty,1)$-categories.

The issue with the construction of examples of contextual categories from real-world data is the notorious coherence problem: The requirement that pullback-action, modelling substitution, as well as the structures modelling the logical rules all commute *strictly* with pullback-action. The practice of category theory rather provides examples equipped with all necessary data but commuting with plb's only *up to isomorphism* (or even weaker notions of equivalence).

⤳ Strictification results have been developed, and hence categorical frameworks of dependent type-theoretical semantics which are less rigid and entail natural examples directly. They all come together with functorial strictification procedures which produce honest internal languages (see bottom half of the following diagram, which consists of "fibered" opposed to "indexed" frameworks).



The benefits of categories with universes are

- they solve coherence by design and hence don't require a further splitting procedure to be applied, and yet include important examples of real-world mathematical models; the framework in practice usually applies to categories which model MLTT with multiple universes.
- the framework can be extended to all type-formers of MLTT, including Id-types (which can be problematic in the "fibred" frameworks by virtue of the due splitting procedures).

Definition 10: Let $\mathcal{C}$ be a category. A universe in $\mathcal{C}$ is an object $\mathcal{U} \in \mathcal{C}$ together with a morphism $p : \hat{\mathcal{U}} \to \mathcal{U}$ in $\mathcal{C}$, and for each map $f : C \to \mathcal{U}$ a choice of pullback square

$$\begin{tikzcd}
(C;f) \arrow[r, "Q f"] \arrow[d, "P_{(f)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \hat{\mathcal{U}} \arrow[d, "p"] \\
C \arrow[r, "f"'] & \mathcal{U}
\end{tikzcd}$$

Idea: Like in Example 8, we'll use that composition of arrows is strictly functorial, and compose those as "names" for pullback-representations, rather than concatenate pullback-squares themselves. A universe $p : \hat{\mathcal{U}} \to \mathcal{U}$ here plays the role of an external Tarski-universe which represents the syntax modelled in $\mathcal{C}$, much like the reflexive object in a CCwR does for the associated untyped λ-calculus.

Notation: For a sequence of maps $f_1: C \to \mathcal{U}$, $f_2: (C; f_1) \to \mathcal{U}, \dots$, write $(C; f_1, \dots, f_n)$ for $((\dots(C; f_1); \dots), f_n)$. For $n=0$, that yields $(C;) = C$.

Construction 11: Given a category $\mathcal{C}$ with a universe $\mathcal{U}$ and a terminal object 1, we obtain a contextual category $\mathcal{C}_\mathcal{U}$ as follows:

* $Ob_n(\mathcal{C}_\mathcal{U}) := \{(f_1, \dots, f_n) \in (\mathcal{C}_1)^n \mid \forall 1 \leq i \leq n: f_i : (1; f_1, \dots, f_n) \to \mathcal{U}\}$
* $\mathcal{C}_\mathcal{U}((f_1, \dots, f_n), (g_1, \dots, g_m)) := \mathcal{C}((1; f_1, \dots, f_n), (1; g_1, \dots, g_m))$
* $1_{\mathcal{C}_\mathcal{U}} := \emptyset$
* $ft(f_1, \dots, f_{n+1}) := (f_1, \dots, f_n)$
* $p_{(f_1, \dots, f_{n+1})} := p_{((1; f_1, \dots, f_n); f_{n+1})} : (f_1, \dots, f_{n+1}) \to (f_1, \dots, f_n)$
* Given a map $\alpha : (g_1, \dots, g_m) \to (f_1, \dots, f_n)$ in $\mathcal{C}_\mathcal{U}$ and an extension $(f_1, \dots, f_{n+1})$, the canonical pullback $\alpha^*(f_1, \dots, f_{n+1})$ is given by

$$\begin{tikzcd}
(1; g_1, \dots, g_m, f_{n+1} \circ \alpha) \arrow[r, dashed] \arrow[d, "p_{(\dots, f_{n+1} \circ \alpha)}"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, "Q(f_{n+1} \circ \alpha)"] & (1; f_1, \dots, f_{n+1}) \arrow[r, "Q(f_{n+1})"] \arrow[d, "p_{(\dots, f_{n+1})}"] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{U}} \arrow[d] \\
(1; g_1, \dots, g_m) \arrow[r, "\alpha"'] & (1; f_1, \dots, f_n) \arrow[r, "f_{n+1}"'] & \mathcal{U}
\end{tikzcd}$$

Exercise 12: * The data in Construction 11 defines a contextual category $\mathcal{C}_\mathcal{U}$.

* The contextual cat. $\mathcal{C}_\mathcal{U}$ is well-defined up to isomorphism given just $\mathcal{C}$ and $p: \tilde{\mathcal{U}} \to \mathcal{U}$ (independently of the explicit choice of pb's and 1).

Proposition 13: Every small contextual category is isomorphic (in CxtCat) to $\mathcal{C}_{\mathcal{U}}$ for some category $\mathcal{C}$ with a universe $\mathcal{U}$ and a terminal object.

Proof: Given a contextual category $\mathcal{D}$, consider $\mathcal{C} := [\mathcal{D}^{op}, Set]$ and $p : \hat{\mathcal{U}} \to \mathcal{U}$ given by

$$\begin{array}{cl} & \hat{\mathcal{U}}(B) := \{ (E, e) \mid ft\, E = B,\ E \underset{p_E}{\overset{e}{\rightleftarrows}} B \text{ is a section} \} \\ (E,e) & \\ \downarrow & \Big\downarrow p_B \\ E & \mathcal{U}(B) := \{ E \mid ft\, E = B \} \end{array}$$

Exercise: Then any choice of pullbacks yields an isomorphism $\mathcal{C}_{\mathcal{U}} \cong \mathcal{D}$ in CxtCat.

E.g.

$$\overset{\in Ob_1(\mathcal{C})}{\begin{tikzcd} yC \arrow[r, "{(C, id_C)}"] \arrow[d, "yp_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & \hat{\mathcal{U}} \arrow[d, "p"] \\ y1 \arrow[r, "\ulcorner C \urcorner"'] & \mathcal{U} \end{tikzcd}}, \quad \overset{\in Ob_2(\mathcal{C})}{\begin{tikzcd} yE \arrow[r] \arrow[d, "yp_E"'] \arrow[dr, phantom, "\lrcorner", very near start] & \hat{\mathcal{U}} \arrow[d, "p"] \\ yC \arrow[r, "\ulcorner E \urcorner"'] & \mathcal{U} \end{tikzcd}}, \ldots \text{ etc.}$$

$\square$

Remark 14: The universe $\hat{\mathcal{U}} \xrightarrow{p} \mathcal{U}$ in $[\mathcal{D}^{op}, Set]$ in the proof of Proposition 13 is the prototype of a "natural model" in the sense of Awodey.

Example 15: The category Set can be equipped with the universes

$$\begin{array}{l} Set_\bullet^{(\kappa)} := \{ (X, x) \mid X \in V_\kappa,\ x \in X \} \\ \Big\downarrow p_\kappa \\ Set^{(\kappa)} := V_\kappa \end{array}$$

for cardinals κ.

Then $\mathrm{Set}_{p\kappa}$ is isomorphic to the contextual category Fam_κ of κ-small families of sets:

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Set}_\bullet^{(\kappa)} \arrow[d, "p\kappa"] \\ 1 \arrow[r, "\ulcorner X_1 \urcorner"'] & \mathrm{Set}^{(\kappa)} \end{tikzcd} \quad , \quad \begin{tikzcd} \Sigma(X_1, f_1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Set}_\bullet^{(\kappa)} \arrow[d, "p\kappa"] \\ X_1 \arrow[r, "f_1"'] & \mathrm{Set}^{(\kappa)} \end{tikzcd} \quad , \ldots$$

Indeed, the construction of $\mathrm{Set}_{p\kappa}$ recovers exactly the definition of Fam in Example 8 up to the size restriction!

To define logical structure on the contextual category $\mathcal{C}_\mathcal{U}$, rather than to define its instances on all type families individually and assure their coherence, we may define it globally on the universe $p: \tilde{\mathcal{U}} \to \mathcal{U}$. The individual instances for each type family in $\mathcal{C}_\mathcal{U}$ are then obtained by pullback along their names in $\mathcal{U}$; coherence is virtually automatic by functoriality of precomposition.

Definition 16: Given a locally cartesian closed category $\mathcal{C}$ with a universe $\tilde{\mathcal{U}} \xrightarrow{p} \mathcal{U}$, let $\mathcal{U}^{(2)} := [\tilde{\mathcal{U}}, \mathcal{U}\times\mathcal{U}]_\mathcal{U} \longrightarrow \mathcal{U}$ be the internal hom-object of the pair $\begin{tikzcd}[column sep=small] \tilde{\mathcal{U}} \arrow[dr, "p"'] & & \mathcal{U}\times\mathcal{U} \arrow[dl, "\pi_1"] \\ & \mathcal{U} & \end{tikzcd}$ in the slice $\mathcal{C}/\mathcal{U}$. Let

$$\begin{tikzcd} A_{gen} \arrow[r] \arrow[d, "\alpha_{gen}"'] \arrow[dr, phantom, "\lrcorner"] & \tilde{\mathcal{U}} \arrow[d, "p"] \\ \mathcal{U}^{(2)} \arrow[r, "p"'] & \mathcal{U} \end{tikzcd} \quad , \text{ and } \quad \begin{tikzcd} B_{gen} \arrow[r] \arrow[d, "\beta_{gen}"'] \arrow[dr, phantom, "\lrcorner"] & \tilde{\mathcal{U}} \arrow[d, "p"] \\ \check{A}_{gen} \doteq [\tilde{\mathcal{U}}, \mathcal{U}] \times \mathcal{U} \arrow[r, "ev"'] & \mathcal{U} \end{tikzcd} .$$

Exercise 17: The sequence $(\alpha_{gen}, \beta_{gen})$ is universal in the following sense:

For every pair of maps $\Gamma \xrightarrow{\ulcorner A \urcorner} \mathcal{U}$ and $A := p^* \ulcorner A \urcorner \xrightarrow[\ulcorner B \urcorner]{} \mathcal{U}$, there is a unique map $\ulcorner (A,B) \urcorner : \Gamma \longrightarrow \mathcal{U}^{(2)}$ such that

$$\begin{tikzcd}
B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B_{gen} \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & & \tilde{\mathcal{U}} \arrow[d, "p"] \\
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] \arrow[rrr, bend right=15, "\ulcorner B \urcorner"] & A_{gen} \arrow[r] \arrow[d] \arrow[rr, bend left, "ev"] \arrow[dr, phantom, "\lrcorner"] & \hat{\mathcal{U}} \arrow[d, "p"] & \mathcal{U} \\
\Gamma \arrow[r, "\ulcorner (A,B) \urcorner"'] \arrow[rr, bend right, "\ulcorner A \urcorner"'] & \mathcal{U}^{(2)} \arrow[r] & \mathcal{U} &
\end{tikzcd}$$

Thus, $\mathcal{U}^{(2)}$ represents context extensions of length 2 (over all and hence not necessarily "syntactic" bases wrt. $\mathcal{C}_{\mathcal{U}}$).

In particular, for contexts $(f_1, \dots, f_n) \in Ob_n(\mathcal{C}_{\mathcal{U}})$ we obtain a 1-1 correspondence between context extensions $(f_1, \dots, f_n, \ulcorner A \urcorner, \ulcorner B \urcorner) \in Ob_{n+2}(\mathcal{C}_{\mathcal{U}})$ and maps

$\ulcorner (A,B) \urcorner : (1; f_1, \dots, f_n) \longrightarrow \mathcal{U}^{(2)}$.

Definition 18: A Π-structure on a universe $\mathcal{U}$ in a loc. cart. closed category $\mathcal{C}$ is a map $\Pi : \mathcal{U}^{(2)} \longrightarrow \mathcal{U}$ which represents the dependent product $\prod_{\alpha_{gen}} B_{gen}$:

$$\begin{tikzcd}
\prod_{\alpha_{gen}} B_{gen} \arrow[r] \arrow[d, "\prod_{\alpha_{gen}} \beta_{gen}"'] \arrow[dr, phantom, "\lrcorner"] & \tilde{\mathcal{U}} \arrow[d, "p"] \\
\mathcal{U}^{(2)} \arrow[r, "\Pi"'] & \mathcal{U}
\end{tikzcd}$$

Definition 19: A **Σ-structure** on a universe $\mathcal{U}$ in a l.c.c.c. $\mathcal{C}$ is a map $\Sigma: \mathcal{U}^{(2)} \longrightarrow \mathcal{U}$ which represents the dependent sum $\Sigma_{\alpha_{gen}} B_{gen}$:

$$\begin{tikzcd} \Sigma_{\alpha_{gen}} B_{gen} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \hat{\mathcal{U}} \arrow[d, "p"] \\ \mathcal{U}^{(2)} \arrow[r, "\Sigma"'] & \mathcal{U} \end{tikzcd}$$

Remark 20: Π- and Σ-structure on a universe $\mathcal{U}$ in a l.c.c.c. $\mathcal{C}$ make use of the interpretation of Π- and Σ-types in the extensional Martin-Löf type theory given by the l.c.c.c. $\mathcal{C}$ (see beginning of this section) as dependent products and dependent sums respectively. Thus, both Definition 18 and 19 don't require new structure on $\mathcal{C}$, but rather "$\mathcal{U}$-smallness" of existing structure when applied to "$\mathcal{U}$-small" inputs.

The same holds for e.g. the 0 and 1 types and a few others, but does not does not apply to all type-formers, such as identity-types. Recall that identity-type structure on a contextual category $\mathcal{C}$ equips $\mathcal{C}$ with path-objects

$$\begin{tikzcd} (\Gamma, A) \arrow[rr, "\Delta"] \arrow[dr, bend right, "\mathrm{refl}_A"'] & & (\Gamma, A) \times_p (\Gamma, A) \\ & (\Gamma, \mathrm{Id}(A)) \arrow[ur, "p_{\mathrm{Id}(A)}"'] & \end{tikzcd} \qquad (*)$$

such that the left maps refl_A are weakly left orthogonal to the class of dependent projections. Since the refl_A-terms remain $\mathrm{refl}_{A[-]}$-terms under substitution of terms in types, the left map refl_A in $(*)$ is in fact <u>stably</u> weakly left orthogonal to the class of dep. projections: here, stability means that the class of left maps is stable under pullback along right maps.

Definition 21: An *Id-structure* on a universe $\mathcal{U}$ in a category $\mathcal{C}$ is a pair of maps $\mathrm{Id}\colon \hat{\mathcal{U}} \times_{\mathcal{U}} \hat{\mathcal{U}} \to \mathcal{U}$, $r\colon \hat{\mathcal{U}} \to \mathrm{Id}^*\hat{\mathcal{U}}$ which yield a factorization of the form

$$\begin{tikzcd}
\hat{\mathcal{U}} \arrow[r, "r"] \arrow[dr, "\Delta_{\hat{\mathcal{U}}}"'] & \mathrm{Id}^*\hat{\mathcal{U}} \arrow[d, "\mathrm{Id}^*p"] \\
& \hat{\mathcal{U}} \times_{\mathcal{U}} \hat{\mathcal{U}}
\end{tikzcd},$$

such that r is <u>stably</u> orthogonal to $\hat{\mathcal{U}} \xrightarrow{p} \mathcal{U}$ (and hence to all its pullbacks).

[In Kapulkin and Lumsdaine's paper, this is defined to be slightly stronger than what we defined above, but this additional strength is of no importance here.

Definition 22: A *0-structure* on a universe $\mathcal{U}$ is a map $\ulcorner 0 \urcorner\colon 1 \to \mathcal{U}$ s.th. $\ulcorner 0 \urcorner^*\hat{\mathcal{U}} \cong \emptyset$.

A *1-structure* on $\mathcal{U}$ is a map $\ulcorner 1 \urcorner\colon 1 \to \mathcal{U}$ s.th. $\ulcorner 1 \urcorner^*\hat{\mathcal{U}} \cong 1$.

Definition 23: A sequence of internal universes consists of arrows

$\ulcorner u_i \urcorner : 1 \to \mathcal{U}$, $\ulcorner El_i \urcorner : \mathcal{U}_i := u_i^* \tilde{\mathcal{U}} \to \mathcal{U}$, $el_i : \mathcal{U}_i \to \mathcal{U}_{i+1}$ f.a. $i \geq 0$

s.th. $\ulcorner El_{i+1} \urcorner \circ el_i = \ulcorner El_i \urcorner$ f.a. $i \geq 0$.

Given a sequence of internal universes, the pullback squares
$$\begin{tikzcd} \tilde{\mathcal{U}}_i \arrow[r] \arrow[d, "p_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{U}} \arrow[d, "p"] \\ \mathcal{U}_i \arrow[r, "\ulcorner El_i \urcorner"'] & \mathcal{U} \end{tikzcd}$$
induce a universe structure on the maps p_i from p.

The internal universes are closed under Π-types if each p_i has a Π-structure Π_i such that the squares

$$\begin{tikzcd} \mathcal{U}_i^{(2)} \arrow[r, "\ulcorner El_i \urcorner^{(2)}"] \arrow[d, "\Pi_i"'] & \mathcal{U}^{(2)} \arrow[d, "\Pi"] \\ \mathcal{U}_i \arrow[r, "\ulcorner El_i \urcorner"'] & \mathcal{U} \end{tikzcd}$$

commute.

Similarly for the other structures.

Theorem 24 (Voevodsky): A $\Pi/\Sigma/Id/0/1$-structure on a universe $\mathcal{U}$ in a (l.c.c.) category $\mathcal{C}$ induces an according type structure on $\mathcal{C}_\mathcal{U}$.

Moreover, a sequence of internal universes closed under any combination of $\Pi/\Sigma/0/1$-types induces a sequence of Tarski-Universes in $\mathcal{C}_\mathcal{U}$ closed under the corresponding type-formers.

Proof: See Theorem 1.4.15 in Kapulkin, Lumsdaine "The simplicial model of univalent foundations". $\square$