In order to construct $\mathrm{MLTT}^{\Pi,\Sigma,0,1,=,\mathcal{U}_i}$ - type theories from a category $\mathcal{C}$ (and hence, among others, obtain (relative) consistency results of various rules and axioms), by Theorem 24 it suffices to construct a universe $p: \tilde{\mathcal{U}} \to \mathcal{U}$ in $\mathcal{C}$ with the according logical structure.

Example 25: (Logical structure in the category of sets)
Let Set be the category of sets with universe $p_\kappa : \mathrm{Set}_\bullet^{(\kappa)} \to \mathrm{Set}^{(\kappa)}$ for a cardinal κ as in Example 15.

A "family" of sets $q: X \to Y$ is a type family wrt. p_κ (up to choice of explicit pullbacks) if there is a pb. square of the form

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Set}_\bullet^{(\kappa)} \arrow[d, "p_\kappa"] \\
Y \arrow[r, dashed, "\ulcorner q \urcorner"'] & \mathrm{Set}^{(\kappa)}
\end{tikzcd}$$

That is, if and only if the fibers $q^{-1}(y)$ f.a. $y \in Y$ are κ-small sets.

To define logical structure on p_κ, we don't have to impose choices of pullback squares first; instead, we may define the necessary pullback squares first, and make a choice for all other pullbacks of p_κ afterwards.

- $\forall \kappa \in \mathrm{Card}$: p_κ has a (unique) 0-structure:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Set}_\bullet^{(\kappa)} \arrow[d] \\
\{\emptyset\} \arrow[r, "\ulcorner \emptyset \urcorner"'] & \mathrm{Set}^{(\kappa)}
\end{tikzcd}$$

- $\forall \kappa > 0$: p_κ has 1-structure:
$$\begin{tikzcd} \{\emptyset\} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Set}_\bullet^{(\kappa)} \arrow[d] \\ \{\emptyset\} \arrow[r, "\ulcorner \{\emptyset\} \urcorner"'] & \mathrm{Set}^{(\kappa)} \end{tikzcd}$$

- Towards Π- and Σ-structure, we compute
$$\left(\mathrm{Set}^{(\kappa)}\right)^{(2)} := \left[\mathrm{Set}_\bullet^{(\kappa)}, \mathrm{Set}^{(\kappa)} \times \mathrm{Set}^{(\kappa)}\right]_{\mathrm{Set}^{(\kappa)}} = \coprod_{X \in \mathrm{Set}^{(\kappa)}} \mathrm{Fun}\big(\overbrace{p_\kappa^{-1}(X)}^{=X}, \mathrm{Set}^{(\kappa)}\big),$$

$$\begin{tikzcd} A_{gen} = \coprod\limits_{X \in \mathrm{Set}^{(\kappa)}} X \times \mathrm{Fun}(X, \mathrm{Set}^{(\kappa)}) \arrow[r] \arrow[d, "\alpha_{gen}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Set}_\bullet^{(\kappa)} \arrow[d, "p_\kappa"] \\ \coprod\limits_{X \in \mathrm{Set}^{(\kappa)}} \mathrm{Fun}(X, \mathrm{Set}^{(\kappa)}) \arrow[r, "\pi_1"'] & \mathrm{Set}^{(\kappa)} \end{tikzcd},$$

$$\begin{tikzcd} B_{gen} = \coprod\limits_{X \in \mathrm{Set}^{(\kappa)}} \coprod\limits_{f: X \to \mathrm{Set}^{(\kappa)}} \overbrace{\coprod\limits_{x \in X} f(x)}^{=\mathrm{graph}(f)} \arrow[r] \arrow[d, "\beta_{gen}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Set}_\bullet^{(\kappa)} \arrow[d, "p_\kappa"] \\ \coprod\limits_{X \in \mathrm{Set}^{(\kappa)}} X \times \mathrm{Fun}(X, \mathrm{Set}^{(\kappa)}) \arrow[r, "\coprod_X ev_X"'] & \mathrm{Set}^{(\kappa)} \end{tikzcd}.$$

$\rightsquigarrow$ p_κ has Σ-structure iff the fibers of the composition
$B_{gen} \xrightarrow{\beta_{gen}} A_{gen} \xrightarrow{\alpha_{gen}} \left(\mathrm{Set}^{(\kappa)}\right)^{(2)}$ are contained in $\mathrm{Set}^{(\kappa)}$.
That is, iff f.a. pairs $(X \in \mathrm{Set}^{(\kappa)}, f: X \to \mathrm{Set}^{(\kappa)})$, the graph $\coprod_{x \in X} f(x)$ of f is again contained in $\mathrm{Set}^{(\kappa)} = V_\kappa$. I.e. p_κ has Σ-structure iff κ is a regular cardinal.

– p_κ has Π-structure iff the fibers of $\Pi_{\alpha_{gen}} B_{gen} \longrightarrow (Set^{(\kappa)})^{(2)}$ are again contained in $Set^{(\kappa)}$. That is, iff f.a. pairs $(X \in Set^{(\kappa)}, f: X \to Set^{(\kappa)})$, the set $\prod_{x \in X} f(x) := \{ s : X \to Set \mid \forall x \in X : s(x) \in f(x) \}$ is again contained in $Set^{(\kappa)}$. Thus, whenever p_κ has Π-structure, it is a strong limit cardinal.

It follows that p_κ has both Σ- and Π-structure iff it is a strongly inaccessible cardinal.

– An Id-structure on p_κ is a pair of maps

$$\mathrm{Id} : \underbrace{Set_\bullet^{(\kappa)} \times_{Set^{(\kappa)}} Set_\bullet^{(\kappa)}}_{\cong \{(X,x,y) \mid X \in Set^{(\kappa)},\, x,y \in X\}} \longrightarrow Set^{(\kappa)} \quad , \quad r : Set_\bullet^{(\kappa)} \longrightarrow \underbrace{\mathrm{Id}^* Set_\bullet^{(\kappa)}}_{\cong \{(X,x,y,p) \mid X \in Set^{(\kappa)},\, x,y \in X,\, p \in \mathrm{Id}(X,x,y)\}}$$

which factors the diagonal of p_κ as in Definition 21, and such that r is stably weakly left orthogonal to p_κ. That means, r is stably orthogonal to all functions with κ-small fibers! Thus, whenever $\kappa > 0$, r is weakly left orthogonal to all monos, and hence epic. As r factors the diagonal of p_κ (which is monic), it is monic itself. $\Rightarrow$ r is an isomorphism.

$$\Rightarrow \mathrm{Id} \cong \Delta_{p_\kappa}$$

So p_κ has exactly one Id-structure up to isomorphism, given by the pair

$(r = \mathrm{Id}_{\mathrm{Set}^{(k)}_\bullet}, \ \mathrm{Id} = \Delta p_\kappa)$. In particular, f.a. $X \in \mathrm{Set}^{(\kappa)}$, $x, y \in X$ we get

$$\mathrm{Id}(X, x, y) = \begin{cases} *, & \text{if } x = y \\ \emptyset, & \text{else} \end{cases}.$$

Thus, the one and only Id-type structure associated to p_κ for $\kappa > 0$ is extensional (and so the associated type theory satisfies Equality-Reflection)!

- Every sequence of cardinals $\mu_i < \kappa$, $i < \omega$, yields a sequence of internal universes in p_κ via the canonical inclusions

$$\begin{tikzcd} \mathrm{Set}_\bullet^{(\mu_i)} \arrow[r, hook] \arrow[d, "p_{\mu_i}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Set}_\bullet^{(\kappa)} \arrow[d, "p_\kappa"] \\ \mathrm{Set}^{(\mu_i)} \arrow[r, hook, "\ulcorner \mu_i \urcorner"'] & \mathrm{Set}^{(\kappa)} \end{tikzcd}$$

The internal universes $\ulcorner \mu_i \urcorner$ are closed under all type formers iff each μ_i is a strongly inaccessible cardinal.

Thus, every strictly increasing sequence $(\mu_i \mid i < \omega)$ of strongly inaccessible cardinals yields a contextual category $\mathrm{Set}_{p_{\mu_\omega}}$ equipped with all type formers we have considered. It satisfies Equality-Reflection, and hence Function Extensionality as well.

Example 26: (Logical structure in the category of groupoids)

Let Gpd be the category of groupoids. We want to interpret type families as indexed groupoids over another groupoid. We'd like to interpret proofs of identity between two terms of the same type as isomorphisms between objects of the same groupoid. In this sense, we enhance the Set-model by freely adding a family of proof-sets over each set.

(This is one of the reasons we consider groupoids rather than categories, as general arrows in a category cannot be inverted, while proofs of identity in the syntax do.)

The Grothendieck-Construction yields a canonical choice of (large) universe: The (Grothendieck/iso-) fibration $p : \mathrm{Gpd}_\bullet \twoheadrightarrow \mathrm{Gpd}$ induces an equivalence

$$\textstyle\int : \mathrm{Fun}(X, \mathrm{Gpd}) \overset{\simeq}{\rightleftarrows} \mathrm{Fib}(X) : \ulcorner \ldots \urcorner$$

The 2-categorical aspect here is not important (for the universe structure per se)

for all $X \in \mathrm{Gpd}$ of accordingly defined 2-categories.

$$\begin{tikzcd} \textstyle\int F \arrow[r, dashed] \arrow[d, dotted, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Gpd}^{\simeq}_{\bullet} \arrow[d, two heads, "p"] \\ X \arrow[r, "F = \ulcorner \int F \urcorner"'] & \mathrm{Gpd}^{\simeq} . \end{tikzcd}$$

← Exists by abstract non-sense, but computes to be the groupoid of small "pointed" groupoids: Objects are (X, x) s.th. $x \in \mathrm{Ob}\, X$, $X \in \mathrm{Gpd}$. Arrows are (F, φ) s.th. $F : X \xrightarrow{\simeq} Y$, $\varphi : F(x) \xrightarrow{\simeq} y$.

We can again stratify p along cardinal boundaries to obtain universes in Gpd. Thus, for a cardinal κ, let $\mathrm{Gpd}_\kappa \subseteq \mathrm{Gpd}$ be the full subcategory of

κ-small groupoids, and let

$$\begin{tikzcd} \mathrm{Gpd}^{(\kappa)}_{\bullet} \arrow[r] \arrow[d, "p_\kappa"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Gpd}^{\simeq}_{\bullet} \arrow[d, "p"] \\ \mathrm{Gpd}^{(\kappa)} \arrow[r, hook] & \mathrm{Gpd}^{\simeq} . \end{tikzcd}$$

The interpretation of type families as indexed groupoids intrinsically causes the dependent projections in the associated contextual category Gpd_{p_κ} to be given by the fibrations between (κ-small) groupoids.

- In a nutshell, p_κ always has 0-structure, and 1-structure whenever $\kappa > 0$. It again has Π- and Σ-structure iff κ is strongly inaccessible. Here, the additional cardinal arithmetics is fairly straight-forward, but crucially one uses here that fibrations between groupoids are stable under composition (for Σ-structure) and under dependent products as well. Here, note that general functors between groupoids are not even exponentiable (i.e. pullback along them has no right-adjoint).
  - ↳ Fibrations between groupoids are exponentiable by Conduché's characterization of exponentiable functors between categories. Furthermore, for every (κ-small) fibration $q : X \twoheadrightarrow Y$, there is a lift of the form

$$
\begin{tikzcd}
\mathrm{Fib}(Y) \arrow[r, dotted, "q^*"] \arrow[d, hook] & \mathrm{Fib}(X) \arrow[d, hook] \\
\mathrm{Gpd}/Y \arrow[r, "q^*"'] & \mathrm{Gpd}/X
\end{tikzcd}
\qquad (*)
$$

by virtue of some model categorical properties of Gpd equipped with its canonical Quillen model structure: The acyclic cofibrations are pullback-stable along fibrations, and so the lift $(*)$ exists by a simple duality argument.

- Id-structure on p_κ: In the model categorical lingo employed above, Id-structure on p_κ exists whenever the induced path-object factorizations for fibrations between κ-small groupoids are representable in $\mathrm{Gpd}^{(\kappa)}$ (they are automatically "stable" again by pullback stability of the class of acyclic cofibrations in Gpd).

  As any diagonal is monic, and hence κ-small (whenever $\kappa > 0$), the Id-structure is given by

The (strict) iso-comma object in Gpd is again κ-small whenever $\kappa \geq \aleph_0$, or $\kappa \in \{0,1\}$.

$$
\begin{tikzcd}
\mathrm{Gpd}^{(k)}_\bullet \arrow[r, hook, "r", "\sim"'] \arrow[dr, hook, "\iota"'] & P(p_\kappa) = \Delta_!\left(\mathrm{Gpd}^{(k)}_\bullet \times_{\mathrm{Gpd}^{(k)}} \mathrm{Gpd}^{(k)}_\bullet\right) \arrow[r] \arrow[d, "\mathrm{Id}^*\mathrm{Gpd}^{(k)}_\bullet"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Gpd}^{(k)}_\bullet \arrow[d, "p_\kappa"] \\
 & \mathrm{Gpd}^{(k)}_\bullet \times_{\mathrm{Gpd}^{(k)}} \mathrm{Gpd}^{(k)}_\bullet \arrow[r, dashed, "\mathrm{Id}"'] & \mathrm{Gpd}^{(k)}
\end{tikzcd}
.
$$

Exercise 28: For $X \in \mathrm{Gpd}^{(k)}$, $x, y \in X$, we obtain an equivalence

$$
\mathrm{Id}((x,x),(x,y)) \simeq X(x,y) \in \mathrm{Gpd}^{(k)}.
$$

Corollary 29: The type theory associated to Gpd_{p_κ} violates the UIP, as $X(x,y) \not\simeq *$ for general groupoids in Gpd_κ unless $\kappa \leq 1$. $\square$

Yet, the groupoid $\mathrm{Id}(X,x,y) \simeq X(x,y)$ is essentially discrete itself. One in fact choose a path-object $P(p_\kappa)$ s.th. $\mathrm{Id}(X,x,y) \cong X(x,y)$ (see Hofmann, Streicher), so the groupoids $\mathrm{Id}(X,x,y)$ are discrete. That implies an Equality-Reflection rule for Identity-types of Identity-types.

$$\frac{x,y:A,\ p,q:x=_A y \vdash \alpha : p =_{x=_A y} q}{x,y:A,\ p,q:x=_A y \vdash p \equiv q : x=_A y} \qquad \frac{x,y:A,\ p:x=_A y \vdash \alpha : p =_{x=_A y} p}{x,y:A,\ p:x=_A y \vdash \alpha \equiv \mathrm{refl}_p : x=_A y}$$

(2-Equality-Reflection)

This suggests a pattern: n-groupoids yield a MLTT$^{\cdots}$ which negates the UIP on all iterated Id-types up to level n, but satisfies "$(n+1)$-Equality Reflection".

↳ Idea: Pushing $n \to \infty$ yields a type theory of ∞-groupoids with $\neg$UIP in all dimensions.

- Lastly, a sequence of internal universes which is closed under all type formers is again given by any sequence $(\mu_i \mid i \leq \omega)$ of strongly inaccessible cardinals, s.th.

$$\begin{tikzcd}
\mathrm{Gpd}_\bullet^{(\mu_i)} \arrow[r, hook] \arrow[d, "p_{\mu_i}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Gpd}_\bullet^{(\mu_\omega)} \arrow[d, "p_{\mu_\omega}"] \\
\mathrm{Gpd}_{\mu_i} \arrow[r, hook, "\ulcorner u_i \urcorner"'] & \mathrm{Gpd}^{(\mu_\omega)}
\end{tikzcd}$$

.

Thus, every strictly increasing sequence $(\mu_i \mid i \leq \omega)$ of strongly inaccessible cardinals yields a contextual category $\mathrm{Gpd}_{\mu_\omega}$ equipped with all type formers we have considered. It satisfies "2-Equality-Reflection". Function Extensionality can be shown to hold independently (in fact, it holds in every "type theoretic model category with cofibrant-fibrant objects in the sense of Shulman).

Remark 31: The universes $\mathcal{P}_K$ in Gpd are not univalent:

For any $X, Y \in \mathrm{Gpd}^{(K)}$, we have

$$\mathrm{Id}(\mathrm{Gpd}^{(K)}, X, Y) \simeq \mathrm{Gpd}^{(K)}(X, Y) = \mathrm{Iso}(X, Y) \in \mathrm{Set},$$

and

$$(X \simeq Y) = \begin{cases} \mathrm{Obj} = \{\text{Equivalences } X \xrightarrow{\simeq} Y\} \\ \mathrm{Mor} = \{\text{Natural isomorphisms}\} \end{cases}.$$

The map

$$\begin{aligned} \mathrm{isotoequiv}(X,Y) : \mathrm{Iso}(X,Y) &\longrightarrow (X \simeq Y) \\ e &\longmapsto e \end{aligned}$$ is generally not an

equivalence of groupoids!

Issue: In finite dimensions n, there is an intrinsic difference between identities (i.e. isomorphisms between n-groupoids) and equivalences between such. We have seen this now explicitly for 0-groupoids (i.e. sets) where identities ($\in \{0,1\}$) are not the same as equivalences (i.e. bijections), and for 1-groupoids.

Example 30: (Logical structure in the category of ∞-groupoids)

The approach here is essentially the same as in Example 27, only that we have to make sense of the 'category of ∞-groupoids' in a way which allows us to extract a universe closed under all type formers as before.

We already have made use of the canonical model structure on Gpd to manage the type-formers for the p_k's. The fact that every object in Gpd is fibrant wasn't strictly necessary for the arguments to works, as p_k filters out only those objects which are fibrant in the first place.

Fact: ∞-groupoids are the fibrant objects in Quillen's model structure "Kan" on the category sSet of simplicial sets.

- The model category (sSet, Kan) is an abstract combinatorial model for the homotopy theory of topological spaces, i.e. there is a Quillen-equivalence $(sSet, Kan) \xrightarrow{\sim} Top$

equipped with an accordingly canonical model-structure (on CW-complexes to be precise).

- Unlike Top, the model category (sSet, Kan) has a plethora of fabulous categorical and homotopical algebraic properties.

The following is due to Kapulkin, Lumsdaine and Voevodsky.

Idea: → $\mathrm{Sset} = \mathrm{Set}^{\Delta^{op}}$ is a presheaf category and as such allows a construction of universes $p_\kappa : \tilde{\mathcal{U}}_\kappa \to \mathcal{U}_\kappa$ which classify κ-small maps between presheaves much as the universes in Set and Gpd do (this construction is due to Hofmann & Streicher).

→ Define the sub-universe

$$\begin{tikzcd} V_\bullet^{(\kappa)} \arrow[r] \arrow[d, "\pi_\kappa"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U}_\bullet^{(\kappa)} \arrow[d, "p_\kappa"] \\ V^{(\kappa)} \arrow[r, hook] & \mathcal{U}^{(\kappa)} \end{tikzcd}$$

which contains only the κ-small ∞-groupoids.

Show that $V^{(\kappa)} \in \mathrm{Sset}$ is an ∞-groupoid itself (i.e. fibrant in (sSet, Kan)) and that π_κ is a Kan fibration (i.e. an isofibration of ∞-groupoids).

→ Show that π_κ has all the type theoretic structure (which again follows mostly from the fine model categorical properties of (sSet, Kan)).

Here, sequences of strongly inaccessible cardinals (μ_i ($i \leq \omega$)) again yield an annotated type theory $\mathrm{Sset}_{\pi_{\mu_\omega}}$ with all type-formers.

The type theory $\mathrm{Sset}_{\pi_{\mu_\omega}}$ has various interesting properties:

- It is univalent! Essentially, because the intrinsic "dimension-shift" at finite levels is obliterated at ∞. I.e.

"∞-categorical id's / iso's = ∞-cat'l equivalences".

- It suggests the concept of "higher inductive types" such as synthesized versions of spheres, homotopy-pushouts, n-truncations etc. which allow to study "synthetic algebraic topology" and more.

And this is barely just the start! There is a lot of current research being done on

- HoTT as an internal language for not only spaces, but all ∞-toposes.
- HoTT + ... as a synthetic environment for algebraic topology + ...
  (e.g. cohesive structure, geometric structure, ...)
- HoTT with directed Id-types to capture (higher) category theory more generally.
- Theorem verification of highly involved proofs in topology etc, with an outlook towards computational tools to be used in higher category theory and homotopy theory.
- the development of these fields within constructive proof-relevant mathematics
- ...

$\longrightarrow \infty$.

$\square$