Categorical Models of Type Theory
Exercise Sheet 1

23.02.2022

Exercise I.15

1. Given a λ-term s with an occurrence of $\mathrm{App}(\lambda x.t_1, t_2)$, we obtain another term s' with an occurrence of $t_1[t_2/x]$ by replacing $\mathrm{App}(\lambda x.t_1, t_2)$ in s by $t_1[t_2/x]$. In this case we say that s β-contracts to s'. We say that s β-reduces to s' if s' can be obtained from s by finitely many β-contractions and changes of bound variables (α-congruences). We then write $s' \leq_\beta s$. A β-conversion is a finite sequence of β-reductions and reversed β-reductions. Two λ-terms s, t are β-congruent if there is a β-conversion from s to t.

   (a) Show that two λ-terms s, t are β-congruent if and only if $\lambda\beta \vdash s :\equiv t$.

   (b) Show that two λ-terms s, t are η-congruent (defined accordingly by a finite sequence of (reversed) α-congruences, β-reductions and η-reductions) if and only if$\lambda\beta\eta \vdash s :\equiv t$.

2. For a λ-theory $\lambda\beta \subseteq T \subseteq \Lambda^{\mathbb{C}}$, show that $T \vdash \lambda\beta\eta$ if and only if $T \vdash I :\equiv \mathbb{1}$, where $\mathbb{1} = \lambda x.\lambda y.\mathrm{App}(x, y)$ denotes the first Church numeral.

3. For $t_1, t_2 \in \Lambda^{\mathbb{C}}$, let $[t_1, t_2] = \lambda z.\mathrm{App}(\mathrm{App}(z, t_1), t_2)$ for a variable $z \notin \mathrm{FV}(t_1) \cup \mathrm{FV}(t_2)$. For $i = 1, 2$, let $\mathrm{pr}_i = \lambda y.\mathrm{App}(y, \lambda x_1.\lambda x_2.x_i)$. Show that

   $$\lambda\beta \vdash \mathrm{App}(\mathrm{pr}_i, [t_1, t_2]) :\equiv t_i$$

   for $i = 0, 1$ (Pair-Elimination). Vice versa, one can show that it is not true that $\lambda\beta \vdash t :\equiv [\mathrm{App}(\mathrm{pr}_1, t), \mathrm{App}(\mathrm{pr}_2, t)]$ for all $t \in \Lambda^{\mathbb{C}}$ (Pair Uniqueness). This is in Barendregt's Book, Exercise 5.4.4, and more involved. That means, not every λ-term is (provably) a pair of terms.

Exercise I.16 For $s \in \Lambda^{\mathbb{C}}$, a finite set $X = \{x_1, \ldots, x_n\}$ of variables, and a finite set $\{t_1, \ldots, t_n\}$ of λ-terms, define the *simultaneous substitution* of $(t_i | i \leq n)$ in s for $(x_i \mid i \leq n)$ recursively as follows.

1. $x_j[t_i/x_i]_{i \leq n} = t_j$ for all $j \leq n$.
2. $a[t_i/x_i]_{i \leq n} = a$ for all $a \in \mathbb{C} \cup (\mathrm{Var} \setminus X)$.

3. $(\lambda x.s)[t_i/x_i]_{i\leq n} = \lambda x_j.s[\overrightarrow{t}/\overrightarrow{x}]_{i\leq n, i\neq j}$ for $j \leq n$.

4. $(\lambda x.s)[t_i/x_i]_{i\leq n} = \lambda y.s[t_i/x_i]_{i\leq n}$ if $y \in \text{Var} \setminus (X \cup \bigcup_{i\leq n} \text{FV}(t_i))$ or $X \cap \text{FV}(s) = \emptyset$.

5. $(\lambda x.s)[t_i/x_i]_{i\leq n} = \lambda z.s[z/y][t_i/x_i]_{i\leq n}$ if $y \in \bigcup_{i\leq n} \text{FV}(t_i) \setminus X$ and $X \cap \text{FV}(s) = \emptyset$, for $z \in \text{Var} \setminus (X \cup \text{FV}(s) \cup \bigcup_{i\leq n} \text{FV}(t_i))$ minimal.

6. $\text{App}(s_1, s_2)[t_i/x_i]_{i\leq n} = \text{App}(s_1[t_i/x_i]_{i\leq n}, s_2[t_i/x_i]_{i\leq n})$.

Show the Substitution Lemma, i.e. for all λ-terms s_i, t_i^j for $i = 1, 2$, $j \leq n$, and for all pairwise distinct variables $x_1, \ldots, x_n$, show that if $\lambda\beta \vdash s_1 :\equiv s_2$ and $\lambda\beta :\equiv t_1^j :\equiv t_2^j$ for all $j \leq n$, then $\lambda\beta \vdash s_1[t_i/x_i]_{i\leq n} :\equiv s_2[t_i/x_i]_{i\leq n}$.

Exercise II.3

1. Let $(C, ^\bullet)$ be an applicative structure and $[\![\cdot]\!]\colon \text{Val}(C) \times \Lambda^{\mathbb{C}} \to C$ be a function which satisfies Conditions 1,2 and 4 from the definition of a λ-model (Definition II.1). The triple $(C, ^\bullet, [\![\cdot]\!])$ is said to satisfy *Berry's Extensionality Property* if for all $s, t \in \Lambda^{\mathbb{C}}$, all $x, y \in \text{Var}$, and all $\rho, \sigma \in \text{Val}(C)$, the following implication holds.

$$(\forall c \in C : [\![\lambda x.s]\!]_\rho \, ^\bullet c = [\![\lambda x.t]\!]_\sigma \, ^\bullet c) \Rightarrow [\![\lambda x.s]\!]_\rho = [\![\lambda x.t]\!]_\sigma$$

   Show that the conjunction of Conditions 3,5 and 6 is equivalent to Berry's Extensionality Property.

2. Let $(C, ^\bullet, [\![\cdot]\!])$ be a λ-model. Then for all $s, t \in \Lambda^{\mathbb{C}}$, $x \in \text{Var}$, $\rho \in \text{Val}(C)$, we have

   (a) $[\![s[t/x]]\!]_\rho = [\![s]\!]_{\rho[[\![t]\!]_\rho/x]}$,

   (b) $[\![\text{App}(\lambda x.s, t)]\!]_\rho = [\![s[t/x]]\!]_\rho$.