Recall the λ-theories $\lambda\beta$, $\lambda\beta\eta$. Example for core distinction in Def. 6.:

$$s = \lambda x. App(x, y) \rightsquigarrow s(x/y) \overset{?}{\neq} \lambda x. App(x,y)[x/y] = \lambda x. \underset{\text{closed}}{App(x,x)}$$

$$\text{``}\cdot\text{''}\quad \lambda z. App(z, x) \rightsquigarrow \text{not closed.}$$

Definition 13: In the following, the meta-variables s, t, u, v range over λ-terms, x ranges over Var, and Γ in a finite set of λ-formulas.

The calculus $\mathcal{S}_1^C$ of structural rules is given by the following "rules of inference":

1. (Monotonicity) $\dfrac{}{\Gamma, t :\equiv s \vdash t :\equiv s}$ 2. (Weakening) $\dfrac{\Gamma \vdash t :\equiv s}{\Gamma, u :\equiv v \vdash t :\equiv s}$

3. (App-Congruence 1) $\dfrac{\Gamma \vdash s :\equiv t}{\Gamma \vdash App(u, s) :\equiv App(u, t)}$ 4. (App-Cong. 2) $\dfrac{\Gamma \vdash s :\equiv t}{\Gamma \vdash App(s, u) :\equiv App(t, u)}$

5. (λ-Congruence (or ξ)) $\dfrac{\Gamma \vdash s :\equiv t}{\Gamma \vdash \lambda x. s :\equiv \lambda x. t}$ 6. (Reflexivity) $\dfrac{}{\Gamma \vdash s :\equiv s}$

7. (Symmetry) $\dfrac{\Gamma \vdash s :\equiv t}{\Gamma \vdash t :\equiv s}$ 8. (Transitivity) $\dfrac{\Gamma \vdash s :\equiv t \quad \Gamma \vdash t :\equiv u}{\Gamma \vdash s :\equiv u}$

The product of this calculus is denoted by "$\vdash$". That is, "$\vdash$" is the smallest relation on $(\Phi_1^C)^\omega$ that is closed under the rules of inference.

Notation 14: Given λ-theories $S, T \subseteq \Lambda^C$, write $T \vdash S$ iff f.a. $(s :\equiv t) \in S$ there is a finite subset $\Gamma \subseteq T$ s.th. $\Gamma \vdash s :\equiv t$.

Exercise 15: Parts 1 & 2 give a correspondence between syntactic deduction and "operational semantics". It yields a consistency proof of $\lambda\beta$ and $\lambda\beta\eta$ via reductions to "normal form"), see Church-Rosser-Theorems [HS86, 1.32, 7.16.2]. Here, a λ-theory T is consistent if $\exists s,t$ λ-terms: $T \nvdash s :\equiv t$.

Note: Normal forms do not always exist. E.g. $App(\lambda x. App(x,x), \lambda x. App(x,x)) \in \Lambda^{\mathbb{C}}$ is $\leq_\beta$-minimal but not in normal form: It contains β-reducable occurrences. In fact, β-reduction can make λ-terms longer and more complicated rather than shorter and simpler - consider e.g.

$App(\lambda x. App(App(x,x),y), \lambda x. App(App(x,x),y)) \in \Lambda^{\mathbb{C}}$.

Part 3 defines the formal pair $[t_1,t_2] \in \Lambda^{\mathbb{C}}$ of λ-terms $t_1,t_2 \in \Lambda^{\mathbb{C}}$, and projections $pr_i \in \Lambda^{\mathbb{C}}$, $i=1,2$, s.th.

$$\forall t_1,t_2 \in \Lambda^{\mathbb{C}}: \quad \lambda\beta \vdash App(pr_i,[t_1,t_2]) :\equiv t_i$$

Exercise 16: For $s \in \Lambda^{\mathbb{C}}$, $X = \{x_1,\dots,x_n\} \subset Var$, $\langle t_1,\dots,t_n\rangle \subseteq \Lambda^{\mathbb{C}}$, define the simultaneous substitution $t[{}^{t_i}/_{x_i}]_{i\leq n}$ of $(t_i \mid i\leq n)$ in s for $(x_i \mid i\leq n)$ by a recursion analogous to Definition I.6.

Show that $\forall s_i, t_i^j \in \Lambda^{\mathbb{C}}$ for $i\in\{1,2\}$, $j\leq n$, and f.a. $\{x_1,\dots,x_n\} \subseteq Var$, $\lambda\beta \vdash s_1 :\equiv s_2$, $\lambda\beta \vdash t_1^j :\equiv t_2^j$ implies $\lambda\beta \vdash s_1[{}^{t_1^i}/_{x_i}]_{i\leq n} :\equiv s_2[{}^{t_2^i}/_{x_i}]_{i\leq n}$

(This is the "Substitution Lemma")

Example 17: 1. $\lambda\beta \vdash App(I,t) :\equiv t$,

2. $\lambda\beta \vdash App(App(K,t),u)) :\equiv t$,

3. $\lambda\beta \vdash App(App(App(S,s),t),u) :\equiv App(App(s,u), App(t,u))$.

Exercise 18: Recall the standard combinators I, K, S from Example 3.

1. $\lambda\beta \vdash I :\equiv App(S, App(K,K))$
2. $\lambda\beta \vdash \lambda x.t :\equiv App(K,t)$ for $t \in \Lambda^{\mathbb{C}}$, $x \in Var \setminus FV(t)$.
3. $\lambda\beta \vdash \lambda x. App(s,t) :\equiv App(App(S, \lambda x.s), \lambda x.t)$ f.a. $s,t \in \Lambda^{\mathbb{C}}$, $x \in Var$.

Proposition 19: $\lambda\beta \cup \{K :\equiv S\}$ is inconsistent.

Proof: In context of the theory $T := \lambda\beta \cup \{K :\equiv S\}$, for all λ-terms s,t,u we have

$$App(s,u) :\equiv App(App(App(K,s),t),u) :\equiv App(App(App(S,s),t),u)$$
$$:\equiv App(App(s,u), App(t,u)) \quad \text{via Exple 17.}$$

For $s = u = I$, get $T \vdash I :\equiv App(t,I)$ f.a. $t \in \Lambda^{\mathbb{C}}$. In part., for any $v \in \Lambda^{\mathbb{C}}$, $x \notin FV(v)$, get $T \vdash I :\equiv App(\lambda x.v, I) :\equiv v$, and so $T \vdash v_1 :\equiv I :\equiv v_2$ f.a. $v_1, v_2 \in \Lambda^{\mathbb{C}}$. That means T is inconsistent. $\square$

Remark 20: Via Exercise 18, one can show that every pure closed λ-term can be expressed entirely in terms of K, S and $App(-,-)$ under $\lambda\beta$.
Together with Prop. 18 this means that the pair $\{K,S\}$ is a "basis" for the λ-calculus. See [Bar84, Chapter 8.1]. He shows that there even is a basis of

of cardinality 1.

Theorem 21 (Fixed Point Theorem I): For all $f \in \Lambda^{\mathbb{C}}$ there is $t \in \Lambda^{\mathbb{C}}$ s.th.

$$\lambda\beta \vdash App(f,t) :\equiv t .$$

Classic diagonal argument.

Proof: Given f, define $s := \lambda x. App(f, App(x,x))$ and $t := App(s,s)$. Then

$$\lambda\beta \vdash t :\equiv App(s,s) :\equiv App(\lambda x. App(f, App(x,x)), s) :\equiv App(f, App(s,s)) :\equiv App(f,t).$$

□

Corollary 22: The λ-calculus allows for the definition of a fixed point combinator: There is $F \in \Lambda^{\mathbb{C}}$ s.th. f.a. $f \in \Lambda^{\mathbb{C}}$

$$\lambda\beta \vdash App(f, App(F,f)) :\equiv App(F,f).$$

Proof: Define $F := \lambda f. App(\lambda x. App(f, App(x,x)), \lambda x. App(f, App(x,x)))$.

□

Closing argument for the untyped λ-calculus:

Con:
* Very alien behaviour no Model theory?
* Rather poorly behaved operational semantics: Normal forms don't generally exist; reductions sometimes don't really reduce at all.

Both are rectified by typing!

Pros: * Isolated environment to study the effects of computability.

* Consistent & surprisingly expressive: One can encode pairs (Ex. 5), Boolean truth values and their connectives (BAR 6.2), and all natural numbers including their basic arithmetic operations into the λ-calculus. One can even encode n-dimensional function $\mathbb{N}^n \to \mathbb{N}$ as n-long sequences of nested functions (via "Currying"):

Can define $\ulcorner\,\urcorner : \mathbb{N} \to \{\text{closed } \lambda\text{-terms}\}$ in various ways, e.g.

$n \mapsto \lambda f.\lambda x.\,App(f^n, x)$ "Church numerals"

A (partial) function $f : \mathbb{N}^n \to \mathbb{N}$ is λ-definable if

$\exists t \in \Lambda^c : \forall n_1, \dots, n_n \in \mathbb{N}^n : \lambda\beta \vdash \ulcorner f(n_1, \dots, n_n) \urcorner = App(\dots(App(App(t, \ulcorner n_1 \urcorner), \ulcorner n_2 \urcorner), \dots), \ulcorner n_n \urcorner)$

Theorem (Kleene): $f : \mathbb{N}^n \to \mathbb{N}$ is (partial) recursive iff it is λ-definable.

See [Bar84, 6.3]. Thus, the untyped λ-Calculus is "Turing complete".

This theorem fails in the typed setting. $\rightsquigarrow$ The untyped λ-calculus is in this respect more expressive.

§2 Models of the untyped λ-calculus

Goal: Interpret λ-terms as actual functions.

$$\rightsquigarrow \quad \Lambda^{\mathbb{C}} \overset{?}{\cong} \mathrm{Fun}(\Lambda^{\mathbb{C}}, \Lambda^{\mathbb{C}})$$

cannot exist by Cantor's Theorem.

Idea: $\Lambda^{\mathbb{C}} \xhookrightarrow{i} \mathrm{Fun}(\Lambda^{\mathbb{C}}, \Lambda^{\mathbb{C}})$

$\supseteq$ {"Representable" endomorphisms}, a bit like monoids: $M \to \mathrm{Fun}(M, M)$, $m \mapsto m \cdot \_$

& Axiomatize this abstractly.

Definition 1: An *applicative structure* is a tuple $(\mathcal{C}, \cdot)$ where $\mathcal{C}$ is a set which contains at least two elements, and $\cdot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a binary operation.

A *$\mathcal{C}$-valuation* is a function $\rho : \mathrm{Var} \to \mathcal{C}$; the set of all $\mathcal{C}$-valuations is denoted by $\mathrm{Val}(\mathcal{C})$.

Given a $\mathcal{C}$-valuation ρ, a variable x and an element $c \in \mathcal{C}$, define the semantical substitution of c for x in ρ by

$$\rho[c/x](y) := \begin{cases} c & \text{if } y = x, \\ \rho(y) & \text{if } y \neq x. \end{cases}$$

A $\lambda^{\mathbb{C}}$-model is a triple $M=(C, \bullet, [\![\cdot]\!])$ where $(C, \bullet)$ is an applicative structure,

$$[\![\cdot]\!] : Val(C) \to (\Lambda^{\mathbb{C}} \to C)$$
$$\rho \mapsto (t \mapsto [\![t]\!]_\rho)$$

is a function s.th. f.a. $x, y \in Var$, f.a. λ-terms s, t, and all $\rho, \sigma \in Val(C)$, the following equations hold.

1. $[\![x]\!]_\rho = \rho(x)$
2. $[\![t]\!]_\rho = [\![t]\!]_\sigma$ whenever $\rho|_{FV(t)} = \sigma|_{FV(t)}$
3. $[\![App(s,t)]\!]_\rho = [\![s]\!]_\rho \bullet [\![t]\!]_\rho$
4. $[\![\lambda x.t]\!]_\rho \bullet c = [\![t]\!]_{\rho[c/x]}$ f.a. $c \in C$ ← β-congruence
5. $[\![\lambda x.t]\!]_\rho = [\![\lambda y.t[y/x]]\!]$ whenever $y \notin FV(t)$. ← α-congruence
6. $[\![\lambda x.s]\!]_\rho = [\![\lambda x.t]\!]_\rho$ if $[\![s]\!]_{\rho[c/x]} = [\![t]\!]_{\rho[c/x]}$ f.a. $c \in C$. ← "weak extensionality"

Due to 2., if $t \in \Lambda_0^{\mathbb{C}}$, write $[\![t]\!]$ for $[\![t]\!]_\rho$ for $\rho \in Val(C)$.

A triple $(C, \bullet, [\![\cdot]\!])$ is a λ-model if it is a $\lambda^{\mathbb{C}}$-model for some $\mathbb{C}$. Henceforth, the $\mathbb{C}$ will be suppressed if not explicitly relevant.

Definition 2: Given a λ-model $M = (C, \bullet, [\![\cdot]\!])$, define

1. For $s := t$ a λ-formula and $\rho \in Val(C)$,

$$(M, \rho) \models s := t \quad \text{if} \quad [\![s]\!]_\rho = [\![t]\!]_\rho .$$

2. For $s :\equiv t$ a λ-formula,

$$M \models s :\equiv t \quad \text{if} \quad [\![s]\!]_\rho = [\![t]\!]_\rho \text{ f.a. } \rho \in Val(C).$$

Given some λ-theory T, we write $M \models T$ if $M \models \varphi$ f.a. $\varphi \in T$.

3. For a set T of λ-formulas, and a λ-formula $s :\equiv t$, write

$T \models s :\equiv t$ if f.a. λ-models $M = (C, \cdot, [\![\cdot]\!])$ with $M \models T$, also $C \models s :\equiv t$.

Exercise 3:

1. Let $(C, \cdot, [\![\cdot]\!])$ be as in Definition 2 which satisfies conditions 1, 2, 4.

Show that it satisfies conditions 3+5+6 iff

(Berry's extensionality) F.a. $s, t \in \Lambda^C$, all $\sigma, \rho \in Val(C)$, and all $x, y \in Var$,

$$\forall c \in C : [\![\lambda x.s]\!]_\rho \cdot c = [\![\lambda x.t]\!]_\sigma \cdot c \implies [\![\lambda x.s]\!]_\rho = [\![\lambda x.t]\!]_\sigma.$$

is satisfied.

2. Let $(C, \cdot, [\![\cdot]\!])$ be a λ-model. Then f.a. $s, t \in \Lambda^C$, $x \in Var$, $\rho \in Val(C)$,

a) $[\![s[t/x]]\!]_\rho = [\![s]\!]_{\rho[[\![t]\!]_\rho / x]}$

b) $[\![App(\lambda x.s, t)]\!]_\rho = [\![s[t/x]]\!]_\rho$.

Theorem 4: (Soundness Theorem)

$\vdash \subseteq \models$. In other words, whenever $\Gamma \subset \Phi^C_\lambda$ is a finite set, and $\Gamma \vdash s :\equiv t$, then $\Gamma \models s :\equiv t$.

Proof: (Let $(C, [\![\cdot]\!])$ be a λ-model. First, f.a. $\rho \in \mathrm{Val}(C)$, the map $[\![\cdot]\!]_\rho : \Lambda^{\mathbb{C}} \to C$ is defined on α-quotients by Def. 1.5.)

We want to show that

$$\vdash := \mathrm{Prod}(\mathcal{S}_\lambda^{\mathbb{C}}) \overset{!}{\subseteq} \{(\Gamma, s :\equiv t) \in (\Phi_\lambda^{\mathbb{C}})^{\infty} \mid \Gamma \models s :\equiv t\}.$$

Therefore, we only need to show that the RHS is closed under the rules of Definition I.13.

1. Monotonicity: $(\Gamma, s :\equiv t) \models s :\equiv t$ is immediate.

2. Weakening: If $\Gamma \models s :\equiv t$, then $\Gamma, u :\equiv v \models s :\equiv t$ is immediate.

So are all rules except λ-congruence: $\Gamma \models s :\equiv t \overset{!}{\Rightarrow} \Gamma \models \lambda x.s :\equiv \lambda x.t$.

But this is ensured by weak extensionality (Def. 1.6)

$\square$

Example 5: (The term model)

Given a set $\mathbb{C}$ of constants, and a consistent $\lambda^{\mathbb{C}}$-theory T, we can define a λ-model $C(\mathbb{C}, T)$ as follows.

For $s, t \in \Lambda^{\mathbb{C}}$ write $s \sim_T t$ iff $T \vdash s :\equiv t$. Then the set $\Lambda^{\mathbb{C}}/{\sim_T}$ together with the binary operation

$$\begin{aligned} \bullet : \Lambda^{\mathbb{C}}/{\sim_T} \times \Lambda^{\mathbb{C}}/{\sim_T} &\to \Lambda^{\mathbb{C}}/{\sim_T} \\ (s, t) &\mapsto s \bullet t := \mathrm{Appl}(s, t) \end{aligned}$$

is an applicative structure. Consistency of T corresponds precisely to the condition that $|C(\mathbb{C}, T)| \geq 2$.

Together with the interpretation

$$\llbracket \cdot \rrbracket : \mathrm{Val}(C(\mathbb{C},T)) \times \Lambda^{\mathbb{C}}/_{\sim_T} \to \Lambda^{\mathbb{C}}/_{\sim_T}$$

$$(\rho, t) \longmapsto \llbracket t \rrbracket_\rho := \underbrace{t[\rho(x_1)/x_1, \ldots, \rho(x_n)/x_n]}_{\text{Simultaneous substitution from Ex. I.15.3.}}$$

$$\text{for } FV(t) = \{x_1, \ldots, x_n\}$$

we obtain a λ-model.

Exercise 6: Verify conditions 1.–6. in Definition 1.
(Mostly follows from the Substitution Lemma Ex. I.16).

Theorem 7: (Completeness)

1. Given a $\lambda^{\mathbb{C}}$-theory T, and $s, t \in \Lambda^{\mathbb{C}}$, then
$T \vdash s :\equiv t$ if and only if $C(\mathbb{C}, T) \models s :\equiv t$.

2. $\models \subseteq \vdash$. In other words, whenever Γ is a (finite) sequence of λ-formulas, and s, t are λ-terms, then $\Gamma \models s :\equiv t$ implies $\Gamma \vdash s :\equiv t$.

Proof: 1. The only if direction follows from the Soundness theorem, since $C(\mathbb{C},T) \models T$.
If $C(\mathbb{C}, T) \models s :\equiv t$, then $\llbracket s \rrbracket_\rho = \llbracket t \rrbracket_\rho$ in $C(\mathbb{C}, T)$ for all
$\rho \in \mathrm{Val}(C(\mathbb{C},T))$. By definition, that means $T \vdash s[\rho(\vec{x})/\vec{x}] :\equiv t[\rho(\vec{y})/\vec{y}]$
for $FV(s) = \{x_1, \ldots, x_n\}$, $FV(t) = \{y_1, \ldots, y_m\}$ f.a. $\rho : \mathrm{Var} \to \Lambda/_{\sim_T}$.
For $\rho : \mathrm{Var} \to \Lambda/_{\sim_T}$, $x \mapsto x$, we obtain $T \vdash s :\equiv t$.

2. If $\Gamma \models s :\equiv t$, then in particular $Cl(\mathbb{C}, \Gamma) \models s :\equiv t$.

$\Rightarrow \Gamma \vdash s :\equiv t$ by Part 1.

$\square$

Corollary 8: $\forall s, t \in \Lambda^{\mathbb{C}} : \lambda\beta \vdash s :\equiv t$ iff $\emptyset \models s :\equiv t$

(for $\emptyset \subseteq \Phi_{\lambda}^{\mathbb{C}}$ the empty theory).

Proof: We observe that $\emptyset \models s :\equiv t$ f.a. $s :\equiv t \in \lambda\beta$ by Ex. 3.2.b).

If $\lambda\beta \vdash s :\equiv t$, then $\exists \Gamma \subseteq \lambda\beta$ finite : $\Gamma \vdash s :\equiv t$.

$\Rightarrow \Gamma \models s :\equiv t$ by the Soundness Theorem.

Thus, if M is a λ-model s.th. $M \models \lambda\beta$, then $M \models \Gamma$, and hence $M \models s :\equiv t$.

Hence, $\lambda\beta \vdash s :\equiv t$ implies $\emptyset \models s :\equiv t$.

Vice versa, $\emptyset \models s :\equiv t$ implies $Cl(\mathbb{C}, \lambda\beta) \models s :\equiv t$.

$\Rightarrow \lambda\beta \vdash s :\equiv t$ by Thm. 7.1.

$\square$