In an applicative structure $(C,\cdot)$, every element $c\in C$ gives rise to a function

$$\chi_c : C \to C$$
$$d \mapsto c\cdot d$$

Definition 9: A map $f: C\to C$ is representable if $f=\chi_c$ for some $c\in C$.
We denote the set of representable functions on C by $\mathrm{Rep}(C,\cdot)\subseteq \mathrm{Fun}(C,C)$.
Moreover, two elements $c,d\in C$ are extensionally equivalent $(c\sim d)$ if

$\forall e\in C: \quad c\cdot e = d\cdot e.$

The applicative structure $(C,\cdot)$ is called extensional if the quotient projection $\pi: C\to C/\sim$ is a bijection.
By construction, we have

$$\begin{tikzcd} C \arrow[r, "\chi"] \arrow[d, "\pi"'] & \mathrm{Rep}(C,\cdot) \subseteq \mathrm{Fun}(C,C). \\ C/\sim \arrow[ur, "\cong"'] & \end{tikzcd}$$

Proposition 10: Let $M=(C,\cdot,[\![\cdot]\!])$ be a λ-model. Then

$$M\models \lambda\beta\eta \quad \text{iff} \quad (C,\cdot) \text{ is extensional.}$$

{ independent of $[\![\cdot]\!]$!

Proof: We already know that $M\models\lambda\beta$ from Corollary 8. We are thus to show that

$$\left(\forall t\in\Lambda^c, x\in \mathrm{Var}\setminus FV(t): M\models \lambda x.\, App(t,x) :\equiv t\right) \text{ iff } (C,\cdot) \text{ is extensional.}$$

"$\Leftarrow$" Let $\rho \in Val(C)$, $c \in C$, $t \in \Lambda_1^C$, $x \in Var \setminus FV(t)$. Then

$$[\![\lambda x.\, App(t,x)]\!]_\rho \cdot c = [\![App(t,x)]\!]_{\rho[c/x]} \quad \text{by Def. 1.4}$$

$$= [\![t]\!]_{\rho[c/x]} \cdot [\![x]\!]_{\rho[c/x]} \quad \text{by Def. 1.3}$$

$$= [\![t]\!]_\rho \cdot c \quad , \text{ because } x \notin FV(t).$$

Thus, $[\![\lambda x. App(t,x)]\!]_\rho \sim [\![t]\!]_\rho$ f.a. $\rho \in Val(C)$.

$\Rightarrow$ $[\![\lambda x. App(t,x)]\!]_\rho = [\![t]\!]_\rho$ f.a. $\rho \in Val(C)$ by assumption, which was to show.

"$\Rightarrow$" Let $c, d \in C$ s.th. $c \sim d$. Let $\rho: Var \to C$ with $\rho(x) = c$, $\rho(y) = d$.

Let $z \in Var \setminus \{x, y\}$. Then f.a. $e \in C$:

$$[\![App(x,z)]\!]_{\rho[e/z]} = [\![x]\!]_{\rho[e/z]} \cdot [\![z]\!]_{\rho[e/z]}$$

$$= c \cdot e$$

$$= d \cdot e = \ldots = [\![App(y,z)]\!]_{\rho[e/z]}.$$

By weak extensionality of M (Def. 1.5), we see that

$$[\![\lambda z. App(x,z)]\!]_\rho = [\![\lambda z. App(y,z)]\!]_\rho \quad \text{in } C.$$

But, by assumption, $LHS = [\![x]\!]_\rho = c$ & $RHS = [\![y]\!]_\rho = d$. So $c = d$.

$\square$

Thus, the property of $(C, \cdot, [\![\cdot]\!])$ to model $\lambda\beta\eta$ is a property of the appl. structure $(C, \cdot)$!

Question: Can we find a characterization of λ-models which is entirely syntax-free? Yes!

1st step: Given an app. structure $(C, \cdot)$ together with a function $f: \mathbb{C} \to C$, and a valuation $\rho: \mathrm{Var} \to C$, to define an interpretation $[\![\cdot]\!]: \Lambda^{\mathbb{C}} \to C$ as in Def. 1 we only need to manage λ-abstractions:

1. $[\![x]\!]_\rho := \rho(x)$ for $x \in \mathrm{Var}$, $[\![c]\!]_\rho = f(c)$ for $c \in \mathbb{C}$,
2. $[\![\mathrm{App}(s,t)]\!]_\rho := [\![s]\!]_\rho \cdot [\![t]\!]_\rho$
3. $[\![\lambda x.t]\!]_\rho := ?$

Want $a \in C$ s.th. $\forall c \in C$: $[\![t]\!]_{\rho[c/x]} = a \cdot c$

where each $[\![t]\!]_{\rho[c/x]}$ has been defined in a prior recursive step.

Basically a continuity condition: To define $[\![t]\!]_\rho$ for each $\rho \in \mathrm{Val}(C)$ individually is not enough. We need to represent the function

$$[\![t]\!]_{\rho[-/x]}: C \to C.$$

In fact, more generally, if $FV(t) = \{x_1, \dots, x_n\}$ and $[\![t]\!]_\rho \in C$ is defined f.a. $\rho \in \mathrm{Val}(C)$, we need $a \in C$ s.th.

$$\forall c_1, \dots, c_n \in C: \quad [\![t]\!]_{\rho[c_1/x_1]\dots[c_n/x_n]} = (\dots((a \cdot c_1) \cdot c_2) \cdot \dots) \cdot c_n$$

to model n-fold λ-abstraction.

E.g. $\quad [\![t]\!]_{\rho[c-/x_1][c-/x_2]} : C\times C \to C$

$\forall c\in C\ \exists a_c\in C: \ [\![t]\!]_{\rho[c/x_1][c-/x_2]} = x\, a_c \overset{?}{=} x_{x(a)c}$

In fact, it suffices to assume the following.

Definition 11: Given a set $S\subseteq \Lambda^{\mathbb{C}}$ of λ-terms, the combinatory closure $\bar{S}$ of S is the smallest set T s.th.

1. $S\subseteq T$
2. $s,t\in T \Rightarrow App(s,t)\in T$.

Let $(C,\cdot)$ be an applicative structure. For $\rho\in Val(C)$ and some $f:\mathbb{C}\to C$, let $[\![\cdot]\!]_\rho : \overline{Var\cup\mathbb{C}} \to C$ be given by

1. $[\![x]\!]_\rho := \rho(x)$ f.a. $x\in Var$, $[\![c]\!]_\rho = f(c)$ f.a. $c\in\mathbb{C}$.
2. $[\![App(s,t)]\!]_\rho = [\![s]\!]_\rho \cdot [\![t]\!]_\rho$ f.a. $s,t\in\overline{Var\cup\mathbb{C}}$.

Say $(C,\cdot)$ is combinatorially complete wrt. $f:\mathbb{C}\to C$ if f.a. finite sets $X=\{x_1,\ldots,x_n\}$ of variables, and all $t\in\overline{X\cup\mathbb{C}}$,

$$\exists a_t\in C\ \forall\rho\in Val(C): \ [\![t]\!]_\rho = \underbrace{a_t\cdot\rho(x_1)\cdot\ldots\cdot\rho(x_n)}_{\substack{\text{left associative notation,}\\ (\ldots((a_t\cdot\rho(x_1))\cdot\rho(x_2))\cdot\ldots)\cdot\rho(x_n)}}$$

Definition 12: A *syntax-free λ-model* is a tuple $(C, \cdot, \sigma, f)$ where $(C, \cdot)$ is a combinatorially complete applicative structure w.r.t. $f : \mathbb{C} \to C$, and $\sigma : C/\!\sim \to C$ is a section of the projection $\pi : C/\!\sim \twoheadrightarrow \underset{\mathrm{Rep}(C,C)}{\overset{\cong}{C}}$ such that $\sigma \circ \pi : C \to C$ is representable.

Theorem 13: I. Every $\lambda^{\mathbb{C}}$-model $(C, \cdot, [\![\cdot]\!])$ gives rise to a syntax-free λ-model $(C, \cdot, \sigma, f)$ with $\sigma : C/\!\sim \to C$, $[c] \mapsto \underbrace{[\![\lambda y. \mathrm{App}(x, y)]\!]_{\rho[c/x]}}_{[\![\mathbb{1}]\!] \cdot c}$ for any $\rho \in \mathrm{Val}(C)$, and $f = [\![\cdot]\!]|_{\mathbb{C}}$.

II. Every syntax-free λ-model $(C, \cdot, \sigma, f)$ gives rise to a $\lambda^{\mathbb{C}}$-model $(C, \cdot, [\![\cdot]\!])$ s.th. for $\rho \in \mathrm{Val}(C)$,

a) $[\![x]\!]_\rho = \rho(x)$ f.a. $x \in \mathrm{Var}$, $[\![c]\!]_\rho = f(c)$ f.a. $c \in \mathbb{C}$

b) $[\![\mathrm{App}(s, t)]\!]_\rho = [\![s]\!]_\rho \cdot [\![t]\!]_\rho$

c) $[\![\lambda x. t]\!]_\rho = \sigma(\underbrace{[\![t]\!]_{\rho[-/x]}}_{\in \mathrm{Rep}(C,C)})$

III. The two assignments are mutually inverse.

Proof:

Part I: If $(C, \cdot)$ supports a $\lambda^{\mathbb{C}}$-model $((C, \cdot), [\![\cdot]\!])$, then $(C, \cdot)$ is combinatorially complete w.r.t. $[\![\cdot]\!]|_{\mathbb{C}}$, as for $X = \{x_1, \dots, x_n\} \in \mathrm{Var}$, $t \in \overline{X \cup \mathbb{C}}$, we can set $a_t := [\![\lambda x_1 \dots \lambda x_n. t]\!]_\rho$ for any $\rho \in \mathrm{Val}(C)$.

The function $\sigma : C/\sim \to C$ is well-defined, since $c \sim c'$ implies f.a. $d \in C$,

$$[\![\lambda y.\, App(x,y)]\!]_{\rho[c/x]} \cdot d = [\![App(x,y)]\!]_{\rho[c/x][d/y]}$$
$$= c \cdot d$$
$$= c' \cdot d = [\![\lambda y.\, App(x,y)]\!]_{\rho[c'/x]} \cdot d,$$

i.e. $\sigma(c) = [\![\lambda y.\, App(x,y)]\!]_{\rho[c/x]} \sim [\![\lambda y.\, App(x,y)]\!]_{\rho[c'/x]} = \sigma(c')$.

$\Rightarrow \sigma(c) = \sigma(c')$ by Ex. II.3 (Berry's Ext. Prop.).

To show that $\sigma : C/\sim \to C$ is a section of the projection π, we are to show that

$$\forall c \in C \qquad c \sim [\![\lambda y.\, App(x,y)]\!]_{\rho[c/x]}, \qquad (*)$$

But in the proof of Proposition 10 we computed that

$$[\![\lambda y.\, App(x,y)]\!]_{\rho[c/x]} \cdot d = c \cdot d \quad \text{f.a. } c,d \in C,$$

which proves $(*)$.

Lastly, for $c \in C$, we have

$$\sigma \circ \pi(c) = [\![\lambda y.\, App(x,y)]\!]_{\rho[c/x]} = \underbrace{[\![\lambda x.\, \lambda y.\, App(x,y)]\!]_\rho}_{[\![\mathbb{1}]\!]} \cdot c,$$

i.e. $\sigma \circ \pi : C \to C$ is represented by $[\![\mathbb{1}]\!]$.

<u>Part II</u>: First, we need to show that $[\![\cdot]\!]_\rho$ is defined on all λ-terms.

We construct $[\![\cdot]\!]_\rho$ recursively and show that f.a. $t \in \Lambda^C$,

i) $[\![t]\!]_{\rho[c/x]} = [\![t]\!]_{\rho[c'/x]}$ f.a. $\rho \in \mathrm{Val}(C)$, $c, c' \in C$ whenever $x \notin FV(t)$,

ii) $\forall \{x_1, \dots, x_n\} \supseteq FV(t)\ \exists a_t \in C : \forall c_1, \dots, c_n \in C : a_t \cdot c_1 \cdots c_n = [\![t]\!]_{\rho[c_1/x_1]\dots[c_n/x_n]}$.

Case 0: $c \in \mathbb{C}$, then $[\![c]\!]_\rho = f(c)$. i) is clear, ii) holds by comp. compl'ness wrt. f.

Case 1: $t \in \overline{\mathrm{Var}}$, then $[\![t]\!]_\rho \in C$ is defined as in Definition II.9 s.th. i) holds.

Condition ii) holds again by combinatorial completeness.

Case 2: Let $s, t \in \Lambda^{\mathbb{C}}$ s.th. $[\![s]\!]_\rho, [\![t]\!]_\rho$ are defined f.a. $\rho \in \mathrm{Val}(C)$. Then

$[\![\mathrm{App}(s,t)]\!]_\rho = [\![s]\!]_\rho \cdot [\![t]\!]_\rho \in C$ again satisfies i). For ii), let $FV(s) \cup FV(t) \subseteq \{x_i\}_{i \leq n}$,

$a_s, a_t \in C$ s.th. $\chi_{a_s} = [\![s]\!]_{\rho[-/\vec{x}]}$, $\chi_{a_t} = [\![t]\!]_{\rho[-/\vec{x}]} : C^n \to C$,

which exist by the ind. hyp. Then for $u, v \neq x_i$ for $i \leq n$, let $a_{\mathrm{App}} \in C$ s.th.

f.a. $\rho \in \mathrm{Val}(C)$,

$$a_{\mathrm{App}} \cdot \rho(u) \cdot \rho(v) \cdot \rho(x_1) \cdots \rho(x_n) = [\![\underbrace{\mathrm{App}((\dots(\mathrm{App}(u,x_1),x_2),\dots),x_n),(\dots(\mathrm{App}(v,x_1),x_2)\dots x_n))}_{\in \overline{\mathrm{Var}}}]\!]_\rho .$$

Let $a_{\mathrm{App}(s,t)} = a_{\mathrm{App}} \cdot a_s \cdot a_t$. Then for $c_1, \dots, c_n \in C$, and any $\rho \in \mathrm{Val}(C)$, we get

$$\begin{aligned}
a_{\mathrm{App}(s,t)} \cdot c_1 \cdots c_n &= a_{\mathrm{App}} \cdot a_s \cdot a_t \cdot c_1 \cdots c_n \\
&= [\![\mathrm{App}(\mathrm{App}(u,\vec{x}), \mathrm{App}(v,\vec{x}))]\!]_{\rho[a_s/u][a_t/v][\vec{c}/\vec{x}]} \\
&= [\![\mathrm{App}(u,\vec{x})]\!]_{\rho[\dots]} \cdot [\![\mathrm{App}(v,\vec{x})]\!]_{\rho[\dots]} \\
&= (a_s \cdot c_1 \cdots c_n) \cdot (a_t \cdot c_1 \cdots c_n) \\
&= [\![s]\!]_{\rho[\vec{c}/\vec{x}]} \cdot [\![t]\!]_{\rho[\vec{c}/\vec{x}]} = [\![\mathrm{App}(s,t)]\!]_{\rho[\vec{c}/\vec{x}]}.
\end{aligned}$$

Case 3: Suppose $[\![t]\!]_\rho$ has been defined f.a. $\rho \in \mathrm{Val}(C)$.

Let $FV(t) \subseteq \{x_1, \dots, x_n, x\}$. By the ind. hyp. we have $a_t \in C$ s.th.

f.a. $\rho \in \mathrm{Val}(C)$, $a_t \cdot \rho(x_1) \cdot \dots \cdot \rho(x_n) \cdot \rho(x) = [\![t]\!]_\rho$.

For $\rho \in \mathrm{Val}(C)$, let

$$[\![\lambda x.t]\!]_\rho := \sigma(\chi_{\overbrace{a_t \cdot \rho(x_1) \cdot \dots \cdot \rho(x_n)}^{= [\![t]\!]_{\rho[-/x]}}})$$

To verify i), for any $\rho \in \mathrm{Val}(C)$, $z \notin FV(t)$, we have

$$\begin{aligned}[\![\lambda x.t]\!]_{\rho[d/z]} &= \sigma(\chi_{a_t \cdot \rho[d/z](x_1) \cdot \dots \cdot \rho[d/z](x_n)}) \\ &= \sigma(\underbrace{[\![t]\!]_{\rho[d/z][-/x]}}_{= [\![t]\!]_\rho \text{ whenever } z \notin FV(t) \text{ by the ind. hyp.}}) \\ &= \sigma([\![t]\!]_{\rho[-/x]}) = [\![\lambda x.t]\!]_\rho .\end{aligned}$$

For ii), let $*$ $\mathbb{1} \in C$ be s.th. $\chi_{\mathbb{1}} = \sigma \circ \pi : C \to C$

$*$ $a_1 = [\![\underbrace{\mathrm{App}(u, ((\dots(\mathrm{App}(\mathrm{App}(\sigma, x_1), x_2), \dots), x_n))}_{\in \overline{\mathrm{Var}}}]\!]$ via combinatory compl.

$*$ $a_{\lambda x.t} := a_1 \cdot \mathbb{1} \cdot a_t$

Then f.a. $c_1, \dots, c_n \in C$, $\rho \in \mathrm{Val}(C)$,

$$\begin{aligned}a_{\lambda x.t} \cdot c_1 \cdot \dots \cdot c_n &= a_1 \cdot \mathbb{1} \cdot a_t \cdot c_1 \cdot \dots \cdot c_n \\ &= \mathbb{1} \cdot (a_t \cdot c_1 \cdot c_2 \cdot \dots \cdot c_n) \\ &= \sigma(\chi_{a_t \cdot c_1 \cdot \dots \cdot c_n}) = [\![\lambda x.t]\!]_{\rho[\vec{c}/\vec{x}]} .\end{aligned}$$

By recursion, for any $\rho \in Val(C)$ we obtain a map $[\![\cdot]\!]_\rho : \Lambda^C \to C$ s.th.

1. $[\![x]\!]_\rho = \rho(x)$ f.a. $x \in Var$ by construction.

2. $[\![t]\!]_\rho = [\![t]\!]_\sigma$ f.a. $\sigma \in Val(C)$ if $\rho|_{FV(t)} = \sigma|_{FV(t)}$ via i).

4. $[\![\lambda x.t]\!]_\rho \cdot c = [\![t]\!]_{\rho[c/x]}$ f.a. $t \in \Lambda^C$, $c \in C$ by case 3.

(Berry's Ext. Prop.) If $[\![\lambda x.s]\!]_\rho \sim [\![\lambda x.t]\!]_\sigma$, then

$$\sigma(a_s \cdot \rho(x_1) \cdot \ldots \cdot \rho(x_n)) \sim \sigma(a_t \cdot \sigma(y_1) \cdot \ldots \cdot \sigma(y_m))$$

$$\Rightarrow \quad '' \quad = \quad '' \qquad \text{because } \sigma \circ \overline{\sigma} \circ \sigma = \sigma$$

$$\Rightarrow [\![\lambda x.s]\!]_\rho = [\![\lambda x.t]\!]_\sigma.$$

Thus, by Ex. II. 3, we deduce that $(C, \cdot, [\![\cdot]\!])$ is a λ-model. $=$

Part III: Given a syntax-free λ-model $(C, \cdot, \sigma, f)$, let $[\![\cdot]\!]_\rho$ be as in Part II, and $\overline{\sigma} : C/\sim \to C$ as in Part I. Then f.a. $c \in C$

$$\begin{aligned}\sigma'(x_c) = [\![\lambda y. App(x,y)]\!]_{\rho[c/x]} &= \sigma([\![App(x,y)]\!]_{\rho[c/x][-/y]}) \\ &= \sigma([\![x]\!]_{\rho[c/x][-/y]} \cdot [\![y]\!]_{\rho[c/x][-/y]}) \\ &= \sigma(x_c).\end{aligned}$$

Vice versa, if $(C, \cdot, [\![\cdot]\!])$ is a λ-model, we obtain a λ-model $(C, \cdot, \overline{[\![\cdot]\!]})$ via Parts I & II.

Then $[\![\cdot]\!] = \overline{[\![\cdot]\!]}$ is shown by another induction along the complexity of λ-terms.

The variable & App-cases are immediate. For $t \in \Lambda^C$ s.th. $[\![t]\!]_\rho = \overline{[\![t]\!]}_\rho$

f.e. $\rho \in \mathrm{Val}(\mathbb{C})$ and $t \in \mathrm{Var}$, we get

$$\overline{\llbracket \lambda z.t \rrbracket_\rho} = \sigma(\overline{\llbracket t \rrbracket}_{\rho[-/z]})$$
$$= \sigma(\llbracket t \rrbracket_{\rho[-/z]})$$
$$= \sigma(\chi_{\llbracket \lambda z.t \rrbracket_\rho})$$
$$= \llbracket \lambda y. App(x,y) \rrbracket_{\rho[\llbracket \lambda x.t \rrbracket_\rho / x]}$$
$$= \llbracket \lambda y. App(\lambda z.t, y) \rrbracket_\rho \quad \text{by Berry Ext.}$$
$$= \llbracket \lambda y. t[s/z] \rrbracket_\rho \quad \text{by } \beta\text{-congruence (remember } C \models \lambda\beta)$$
$$= \llbracket \lambda z.t \rrbracket_\rho \quad \text{by } \alpha\text{-congruence.} \qquad \square$$

Corollary 14: If $(C, \cdot)$ is combinatorially complete wrt. some $f: C \to C$ and extensional, then there is exactly one way to extend $(C, \cdot)$ to a $\lambda^{\mathbb{C}}$-model.

Proof: If $(C, \cdot)$ is comb. complete wrt. $f: C \to C$ and extensional, then $(C, \cdot, \pi, f)$ is the unique syntax-free $\lambda^{\mathbb{C}}$-model on $(C, \cdot, f)$.
(Here, note that $\mathrm{Id}: C \to C$ is representable via $\llbracket x \rrbracket_{\rho[-/x]}$ & comb. completeness.)

$\square$

Remark 15: Comb. complete applicative structures wrt. $\mathbb{C} = \emptyset$ are exactly combinatory algebras in the sense of Curry. That is, appl. structures $(C, \cdot)$ together with distinguished elements $k, s \in C$ s.th.

* $\forall c, d \in C: \quad k \cdot c \cdot d = c$
* $\forall c, d, e \in C: \quad s \cdot c \cdot d \cdot e = (c \cdot e)(d \cdot e)$.