§3 Categorical semantics of the untyped λ-calculus.

Definition 1: A λ-theory is a tuple $(\mathbb{C}, T)$ for $\mathbb{C}$ a set and $T \subseteq \Phi_\lambda^{\mathbb{C}}$ a $\lambda^{\mathbb{C}}$-theory (in part. $\lambda\beta \subseteq T$). A preinterpretation $f : (\mathbb{C}_1, T_1) \to (\mathbb{C}_2, T_2)$ between λ-theories is a function $f : \mathbb{C}_1 \to \mathbb{C}_2$.

A preinterpretation $f : (\mathbb{C}_1, T_1) \to (\mathbb{C}_2, T_2)$ recursively induces functions

$$\Lambda(f) : \Lambda^{\mathbb{C}_1} \to \Lambda^{\mathbb{C}_2},$$

$$\Phi(f) : \Phi_\lambda^{\mathbb{C}_1} \to \Phi_\lambda^{\mathbb{C}_2}$$

s.th.

1. $\Lambda(f)(x) = x$ f.e. $x \in \mathrm{Var}$
2. $\Lambda(f)(c) = f(c)$ f.e. $c \in \mathbb{C}_1$
3. $\Lambda(f)(\mathrm{App}(s,t)) = \mathrm{App}(\Lambda(f)(s), \Lambda(f)(t))$ for $s,t \in \Lambda^{\mathbb{C}_1}$
4. $\Lambda(f)(\lambda x.t) = \lambda x. \Lambda(f)(t)$ for $t \in \Lambda^{\mathbb{C}_1}$,

and

$$* \quad \Phi(f)(s :\equiv t) = \Lambda(f)(s) :\equiv \Lambda(f)(t) \quad \text{for } s,t \in \Lambda^{\mathbb{C}_1}.$$

Definition 2: An interpretation $f : (\mathbb{C}_1, T_1) \to (\mathbb{C}_2, T_2)$ between λ-theories is a preinterpretation f s.th. $\forall \varphi \in T_1 : T_2 \vdash \Phi(f)(\varphi)$.

Write $\mathrm{Int}((\mathbb{C}_1, T_1), (\mathbb{C}_2, T_2))$ for the set of interpretations between $(\mathbb{C}_1, T_1)$ and $(\mathbb{C}_2, T_2)$.

Say that two interpretations $f, g : (\mathbb{C}_1, T_1) \to (\mathbb{C}_2, T_2)$ are equivalent up to intended identity of denotation if

$\forall t \in \Lambda^{\mathbb{C}_1} : T_2 \vdash \Lambda(f)(t) :\equiv \Lambda(g)(t)$. We then write $f \sim g$.

Remark 3: It is straight-forward to show that

* $\sim$ is an equivalence relation on each $\mathrm{Int}((\mathbb{C}_1, T_1), (\mathbb{C}_2, T_2))$
* whenever $f : (\mathbb{C}_1, T_1) \to (\mathbb{C}_2, T_2)$ is an interpretation, then $\forall \varphi \in \Phi_1^{\mathbb{C}_1} : (T_1 \vdash \varphi \Rightarrow T_2 \vdash \Phi(f)(\varphi))$

Definition 4: The category $\mathcal{I}$-Thy has as objects $\mathcal{I}$-theories, and

$\mathrm{Hom}_{\mathcal{I}\text{-Thy}}((\mathbb{C}_1, T_1), (\mathbb{C}_2, T_2)) := \mathrm{Int}((\mathbb{C}_1, T_1), (\mathbb{C}_2, T_2))/\sim$

with the obvious identities & compositions.

Exercise 5: Verify that $\mathcal{I}$-Thy indeed forms a category.

Definition 6: Given $\mathcal{I}^{\mathbb{C}_i}$-models $(C_i, \circ, [\![\cdot]\!]_i)$, a homomorphism of $\mathcal{I}$-models $(f, \alpha) : (C_1, \circ, [\![\cdot]\!]_1) \to (C_2, \circ, [\![\cdot]\!]_2)$ is a function $f : \mathbb{C}_1 \to \mathbb{C}_2$ together with a map $\alpha : C_1 \to C_2$ s.th.

$\forall \rho \in \mathrm{Val}(C_1), t \in \Lambda^{\mathbb{C}_1} : \quad \alpha([\![t]\!]_{1,\rho}) = [\![\Lambda(f)(t)]\!]_{2, \alpha \circ \rho}$.

Say that $(f,\alpha) \sim (g,\beta)$ if $\alpha = \beta$.

Note that this implies $[\![f(c)]\!]_2 = \alpha([\![c]\!]_1) = \beta([\![c]\!]_1) = [\![g(c)]\!]_2$ f.a. $c \in \mathcal{C}$.

Exercise 7: 1. Every homomorphism of λ-models is a morphism of underlying applicative structures, i.e., f.a. $c,d \in C_1$ $\alpha(c \cdot d) = \alpha(c) \cdot \alpha(d)$.

2. The class of λ-models together with equiv. classes of λ-model-homomorphisms forms a category λ-Mod (with the canonical comp. + units).

Definition 8: To every $\lambda^{\mathbb{C}}$-model $M = (C, \cdot, [\![\cdot]\!])$ we may assign its internal $\lambda^{\mathbb{C}}$-theory $(\mathbb{C}, Th(M))$ for $Th(M) := \{ \varphi \in \Phi_\lambda^{\mathbb{C}} \mid M \models \varphi \}$.

Proposition 9: The internal λ-theory of a λ-model yields a functor

$$\lambda\text{-Mod} \xrightarrow{T} \lambda\text{-Thy}$$

$$\begin{array}{ccc}
M_1 = (C_1, \cdot, [\![\cdot]\!]_1) & \longmapsto & (\mathbb{C}_1, Th(M_1)) \\
(f,\alpha) \downarrow & & \downarrow T(f,\alpha) = \begin{array}{c} \mathbb{C}_1 \\ \downarrow f \\ \mathbb{C}_2 \end{array} \\
M_2 = (C_2, \cdot, [\![\cdot]\!]_2) & \longmapsto & (\mathbb{C}_2, Th(M_2))
\end{array}$$

Proof: Let (f,α) be a λ-model homomorphism. If $\varphi = (s := t)$ is in $Th(M_1)$, then $[\![s]\!]_\rho = [\![t]\!]_\rho \in C_1$ f.a. $\rho \in Val(C_1)$. Hence

$\forall \varphi \in Val(C_1)$: $[\![\Lambda(f)(s)]\!]_{2,\alpha\circ\varphi} = \alpha([\![s]\!]_{1,\varphi}) = \alpha([\![t]\!]_{1,\varphi}) = [\![\Lambda(f)(t)]\!]_{2,\alpha\circ\varphi}$.

? $\Rightarrow$ $[\![\Lambda(f)(s)]\!]_{2,\sigma} = [\![\Lambda(f)(t)]\!]_{2,\sigma}$ f.a. $\sigma \in Val(C_2)$.

Solution: Let $FV(s) \cup FV(t) \subseteq \{x_1, \dots, x_n\}$. If $Th(M_1) \vdash s :\equiv t$, then

$Th(M_1) \vdash \lambda x_1. \dots . \lambda x_n . s :\equiv \lambda x_1. \dots . \lambda x_n . t$ by λ-congruence.

$$\Rightarrow \forall \varphi \in Val(C_1): \; [\![\lambda x_1. \dots . \lambda x_n . \Lambda(f)(s)]\!]_{2, \alpha \circ \varphi} = [\![\Lambda(f)(\lambda x_1. \dots . \lambda x_n . s)]\!]_{2, \alpha\circ\varphi}$$
$$= \alpha([\![\lambda x_1. \dots . \lambda x_n . s]\!]_{1, \varphi})$$
$$= \alpha([\![\lambda x_1. \dots . \lambda x_n . t]\!]_{1, \varphi})$$
$$= [\![\lambda x_1. \dots . \lambda x_n . \Lambda(f)(t)]\!]_{2, \alpha\circ\varphi}$$

$$\Rightarrow \forall \sigma \in Val(C_1): \; [\![\lambda x_1. \dots . \lambda x_n . \Lambda(f)(s)]\!]_{\sigma} = [\![\lambda x_1. \dots . \lambda x_n . \Lambda(f)(s)]\!]_{2, \alpha\circ\varphi}$$
$$= [\![\lambda x_1. \dots . \lambda x_n . \Lambda(f)(t)]\!]_{2, \alpha\circ\varphi}$$
$$= [\![\lambda x_1. \dots . \lambda x_n . \Lambda(f)(t)]\!]_{2, \sigma}$$

for any $\varphi \in Val(C_1)$. $\Rightarrow$ $Th(M_2) \vdash \lambda x_1. \dots . \lambda x_n . \Lambda(f)(s) \equiv \lambda x_1. \dots . \lambda x_n . \Lambda(f)(t)$.

$\Rightarrow$ $\underbrace{\Lambda(f)(s) :\equiv \Lambda(f)(t)}_{\Phi(f)(s :\equiv t)} \in Th(M_2)$ by β-congruence (apply both sides to $x_1, \dots, x_n$).

Thus, $T(f, \alpha)$ is an interpretation.

If $(f, \alpha) \sim (g, \beta)$, then we want $T(f) \sim T(g)$: For $c \in C_1$, have

$[\![f(c)]\!]_2 = \alpha([\![c]\!]_1) = \beta([\![c]\!]_1) = [\![g(c)]\!]_2 \in C_2$. In other words,

$Th(M_2) \vdash f(c) :\equiv g(c)$. Inductively, we get

$\forall t \in \Lambda^{C_1}, \varphi \in Val(C_2)$: $[\![\Lambda(f)(t)]\!]_{2,\varphi} = [\![\Lambda(g)(t)]\!]_{2,\varphi} \in C_2$.

(The λ-abstr.-step again follows from Berry's Ext. Prop.).

$\Rightarrow$ $\forall t \in \Lambda^{C_1}$: $\quad Th(M_2) \vdash \Lambda(f)(t) = \Lambda(g)(t)$.

Thus, T is well-defined on hom-sets. Functoriality again follows from a straight-forward computation. $\square$

Proposition 10: The term model construction from Example II.3 yields a functor

$$\begin{array}{ccccl} C: & \lambda\text{-Thy} & \longrightarrow & \lambda\text{-Mod} & \\ & (\mathbb{C}_1,T_1) & \longmapsto & C(\mathbb{C}_1,T_1) & \Lambda^{\mathbb{C}_1}/\sim_{T_1} \\ & f\downarrow & & \downarrow C(f)=(f,\alpha_f) \text{ for } \alpha_f = & \downarrow \Lambda(f) \\ & (\mathbb{C}_2,T_2) & \longmapsto & C(\mathbb{C}_2,T_2) & \Lambda^{\mathbb{C}_2}/\sim_{T_2} \end{array}$$

Proof: The fact that $\Lambda(f): \Lambda^{\mathbb{C}_1}/\sim_{T_1} \to \Lambda^{\mathbb{C}_2}/\sim_{T_2}$ is well-defined follows from Remark 3. We have to show that

i) $f \sim g \Rightarrow C(f) = C(g)$

ii) $\forall f \in \mathrm{Int}((\mathbb{C}_1,T_1),(\mathbb{C}_2,T_2)),\ t \in \Lambda^{\mathbb{C}_1},\ \varphi \in \mathrm{Val}(C(\mathbb{C}_1,T_1))$:

$$\alpha_f(\llbracket t \rrbracket_{1,\varphi}) = \llbracket \Lambda(f)(t) \rrbracket_{2,\alpha_f\circ\varphi}$$

iii) functoriality.

To i) By definition, $f \sim g$ states $\forall t \in \Lambda^{\mathbb{C}_1}: T_2 \vdash \Lambda(f)(t) :\equiv \Lambda(f)(g)$, i.e.

$\forall t \in \Lambda^{\mathbb{C}_1}: \Lambda(f)(t) = \Lambda(f)(g) \in \Lambda^{\mathbb{C}_2}/\sim_{T_1}$.

$\Rightarrow \alpha_f = \alpha_g \quad \Rightarrow (f,\alpha_f) \sim (g,\alpha_g)$ ✓

To ii) $\alpha_f(\llbracket t \rrbracket_{1,\varphi}) := \Lambda(f)(t[\varphi(\vec{x})/\vec{x}])$ for $\langle \vec{x} \rangle := FV(t)$

$= \Lambda(f)(t)[\Lambda(f)(\varphi(\vec{x}))/\vec{x}]$ by another induction

$= \llbracket \Lambda(f)(t) \rrbracket_{2,\alpha_f\circ\varphi}$ since $FV(\Lambda(f)(t)) = FV(t)$. ✓

iii) is once again left to the reader. $\square$

Corollary 11: The functors $\lambda\text{-Thy} \underset{C}{\overset{T}{\rightleftarrows}} \lambda\text{-Mod}$ exhibit $\lambda\text{-Thy}$ as a retract of $\lambda\text{-Mod}$.

Proof: Straight-forward. $\square$

But, in fact, we can do better than that, see Exercise 16.

Definition 12: A *homomorphism of s.f. λ-models* $(C_1, \cdot, \sigma_1, f_1) \to (C_2, \cdot, \sigma_2, f_2)$ is a tuple of functions $(\mathbb{A}_1 \xrightarrow{g} \mathbb{A}_2, C_1 \xrightarrow{\alpha} C_2)$ s.th.

1.
$$\begin{tikzcd} \mathbb{A}_1 \arrow[r, "f_1"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & C_1 \arrow[d, "\alpha"] \\ \mathbb{A}_2 \arrow[r, "f_2"'] & C_2 \end{tikzcd}$$

2.
$$\begin{tikzcd} C_1 \arrow[r, two heads, "\pi_1"'] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Rep}(C_1, \cdot) \arrow[l, bend right, "\sigma_1"'] \arrow[d, dashed, "\exists \bar{\alpha}"] & \ni \sigma_1 \circ \pi_1 \arrow[d, dashed, mapsto] \\ C_2 \arrow[r, two heads, "\pi_2"] & \mathrm{Rep}(C_2, \cdot) \arrow[l, bend left, "\sigma_2"] & \ni \sigma_2 \circ \pi_2 \end{tikzcd}$$

3. $C_1 \xrightarrow{\alpha} C_2$ is a morphism of comb. complete appl. structures, i.e.

$\forall X = \{x_1, \dots, x_n\} \subset \mathrm{Var}, \forall t \in \overline{X \cup \mathbb{A}_1}, \varphi \in \mathrm{Val}(C_2), a_t \in C_1$ as in Def. II.11:

$[\![\Lambda(g)(t)]\!]_\varphi = \alpha(a_t) \cdot \varphi(x_1) \cdot \ldots \cdot \varphi(x_n)$, i.e. $\forall \vec{c} \in C_2^n : \alpha(a_t) \cdot \vec{c} = \alpha_{\Lambda(g)(t)} \cdot \vec{c}$.

Say that $(g, \alpha) \sim (h, \beta)$ if $\alpha = \beta$.

The category $\lambda\text{-Mod}^{sf}$ consists of s.f. λ-models & homomorphisms.

Remark 13: Whenever $(f,\alpha):(C_1,\cdot,\sigma_1,f_1)\to(C_2,\cdot,\sigma_2,f_2)$ is a hom. of s.f. $\mathcal{L}$-models, we have

* $\alpha:(C_1,\cdot)\to(C_2,\cdot)$ preserves "$\_\cdot\_$".
* $\alpha(\mathbb{1}_1)=\mathbb{1}_2$ for $\mathbb{1}_i:=\sigma_i(\sigma_i\circ\pi_i)$ by condition 2 (In fact, $\alpha(\mathbb{1}_1)=\mathbb{1}_2$ iff $\sigma_2\circ\bar{\alpha}=\alpha\circ\sigma_1$.)
* $\bar{\alpha}(1_{C_1})=1_{C_2}$ via Def. 12.1.b applied to $[\![x]\!]\in C[-/x]\in Rep(C_i,\cdot)$: Have $a_x\in C_i$ s.th. $a_x\cdot c=c$ f.a. $c\in C_i$, and
$$\bar{\alpha}(1_{C_1})=\bar{\alpha}(\pi_1(a_x))=\pi_2(\alpha(a_x))=\pi_2(a_x)=1_{C_2}.$$

Theorem 14: The 1-1 correspondence from Theorem II.11 extends to an isomorphism
$$\mathcal{L}\text{-Mod}\xrightarrow{\cong}\mathcal{L}\text{-Mod}^{sf}.$$

Proof: Let $(g,\alpha):M_1\to M_2$ be a morphism in $\mathcal{L}$-Mod, $M_i=(C_i,\cdot,[\![\cdot]\!]_i)$. We get
$$(g,\alpha):(C_1,\cdot,\sigma_1,[\![\cdot]\!]|_{\mathbb{C}_1})\longrightarrow(C_2,\cdot,\sigma_2,[\![\cdot]\!]_2|_{\mathbb{C}_2}),\text{ as}$$

1.
$$\begin{array}{ccc} \mathbb{C}_1 & \xrightarrow{[\![\cdot]\!]|_{\mathbb{C}_1}} & C_1 \\ g\downarrow & \circlearrowleft & \downarrow\alpha \\ \mathbb{C}_2 & \xrightarrow[{[\![\cdot]\!]|_{\mathbb{C}_2}}]{} & C_2 \end{array}$$
by def. of (g,α).

2.
$$\begin{array}{ccc} C_1 & \xrightarrow{\pi_1} & Rep(C_1,\cdot) \\ \alpha\downarrow & \circlearrowleft & \downarrow\bar{\alpha} \\ C_2 & \xrightarrow{\pi_2} & Rep(C_2,\cdot) \end{array}$$
exists if $c\sim d\Rightarrow\alpha c\sim\alpha d$. Indeed,

$c\sim d\Rightarrow\mathbb{1}_1\cdot c=\mathbb{1}_1\cdot d\Rightarrow\alpha(\mathbb{1}_1\cdot c)=\alpha(\mathbb{1}_1\cdot d)$

$\Rightarrow\mathbb{1}_2\cdot\alpha(c)=\alpha\mathbb{1}_1\cdot\alpha(c)=\alpha(\mathbb{1}_1\cdot c)=\ldots=\mathbb{1}_2\cdot\alpha(d)$

$\Rightarrow\alpha(c)\sim\alpha(d)$, so $\bar{\alpha}(\chi_c):=\chi_{\alpha(c)}$ is well-def'd.

Furthermore, for $c \in C_1$,

$$\alpha(\sigma_1(x_c)) = \alpha(\llbracket \lambda y.\, App(x,y) \rrbracket_{1,\varphi[c/x]})$$
$$= \llbracket \lambda y.\, App(x,y) \rrbracket_{2,\alpha\circ\varphi[c/x]}$$
$$= \llbracket \lambda y.\, App(x,y) \rrbracket_{2,(\alpha\circ\varphi)[\alpha c/x]} = \sigma_2(x_{\alpha(c)})$$

$$\Rightarrow \quad \begin{tikzcd} \mathbb{C} \arrow[r, "\sigma_1"] \arrow[dr, "\sigma_2"'] & C_1 \arrow[d, "\alpha"] \\ & C_2 \end{tikzcd} \quad + \ \alpha(\sigma_1\circ\pi_1) = \alpha(\pi_1(\llbracket M \rrbracket_1)) = \pi_2(\llbracket M \rrbracket_2) = \sigma_2\circ\pi_2.$$

3. $\forall X = \{x_1, \ldots, x_n\} \subset Var,\ t \in \overline{X \cup \mathbb{C}_1},\ \vec{c} \in C_2^n$:

$$\alpha(a_t)\,\dot{}\,\vec{c} = \alpha(\llbracket \lambda x_1. \ldots \lambda x_n.\, t \rrbracket_1)\,\dot{}\,\vec{c} = \llbracket \lambda x_1. \ldots \lambda x_n.\, \Lambda(g)(t) \rrbracket_2\,\dot{}\,\vec{c} = \alpha\, \Lambda(g)(t)\,\dot{}\,\vec{c}$$

$$\big[\, a_t\,\dot{}\,\vec{d} = \llbracket \lambda x_1. \ldots \lambda x_n. t \rrbracket_1\,\dot{}\,\vec{d} \ \text{ f.a. } \vec{d} \in C_1^n$$

$\Rightarrow \alpha(a_t), \alpha(\llbracket \lambda \vec{x}.\, t \rrbracket_1)$ both represent the same function $C_2^n \to C_2$ by Def. 12.3.

We thus obtain a functor λ-Mod $\to$ λ-Mod$^{s.f.}$ (functoriality is straight-forward).

Vice versa, given a hom. $(C_1, \dot{}, \sigma_1, f_1) \xrightarrow{(f,\alpha)} (C_2, \dot{}, \sigma_2, f_2)$ of s.f. λ-models,

consider $(f, \alpha) : (C_1, \dot{}, \llbracket \cdot \rrbracket_1) \to (C_2, \dot{}, \llbracket \cdot \rrbracket_2)$ w/ $\llbracket \cdot \rrbracket_i$ as given in

Thm II.11.2. Let $t \in \Lambda^{\mathbb{C}_1}$, $FV(t) \subseteq \{x_1, \ldots, x_n\}$, and $a_t \in C_i$ as in ii) in the proof

of Thm II.11.2. Then, by induction, we show $\alpha(a_t)\,\dot{}\,\vec{c} = a_{\Lambda(f)(t)}\,\dot{}\,\vec{c}$ f.a. $\vec{c} \in C_2^n$.

So, in particular, $\alpha(\llbracket t \rrbracket_\varphi) = \llbracket \Lambda(f)(t) \rrbracket_{\alpha\circ\varphi}$ f.a. $\varphi \in Val(C_1)$.

a. If $t \in \overline{Var \cup \mathbb{C}_1}$: $\alpha(a_t)\,\dot{}\,\vec{c} = a_t\,\dot{}\,\vec{c}$ f.a. $\vec{c} \in \vec{C}_2$ by Def. 12.3. Hence,

$\alpha(\llbracket t \rrbracket_{1,\varphi}) = \alpha(a_t\,\dot{}\,\varphi(\vec{x})) = a_{\Lambda(f)(t)}\,\dot{}\,\alpha\circ\varphi(\vec{x}) = \llbracket t \rrbracket_{2,\alpha\circ\varphi}$ f.a. $\varphi \in Val(C_1)$.

b. If $\alpha(a_{t_i})^{\cdot}\vec{c} = a_{\Lambda(\gamma)(t_i)}\vec{c}$ f.a. $\vec{c} \in C_2^n$, then f.a. $\vec{c} \in C_2^n$,

$$\alpha(a_{App(t_1,t_2)})^{\cdot}\vec{c} = \alpha(a_{App}{}^{\cdot} a_{t_1}{}^{\cdot} a_{t_2})^{\cdot}\vec{c} = \alpha(a_{App})^{\cdot}\,\alpha(a_{t_1})^{\cdot}\,\alpha(a_{t_2})^{\cdot}\vec{c}$$

$$= a_{App}{}^{\cdot}\,\alpha(a_{t_1})^{\cdot}\,\alpha(a_{t_2})^{\cdot}\vec{c} = (\alpha(a_{t_1})^{\cdot}\vec{c})(\alpha(a_{t_2})^{\cdot}\vec{c})$$

$$= (a_{\Lambda(\gamma)(t_1)}{}^{\cdot}\vec{c})(a_{\Lambda(\gamma)(t_2)}{}^{\cdot}\vec{c}) = a_{App}{}^{\cdot}\,a_{\Lambda(\gamma)(t_1)}{}^{\cdot}\,a_{\Lambda(\gamma)(t_2)}{}^{\cdot}\vec{c}$$

$$= a_{App(\Lambda(\gamma)(t_1),\Lambda(\gamma)(t_2))}{}^{\cdot}\vec{c} = a_{\Lambda(\gamma)(App(t_1,t_2))}{}^{\cdot}\vec{c}.$$

Thus, $\alpha([\![App(t_1,t_2)]\!]\varphi) = \ldots = [\![\Lambda(\gamma)(App(t_1,t_2))]\!]_{\alpha\circ\varphi}$ f.a. $\varphi \in Val(C_1)$.

c. If $\alpha(a_t)^{\cdot}\vec{c} = a_{\Lambda(\gamma)(t)}{}^{\cdot}\vec{c}$ f.a. $\vec{c} \in C_2^n$, then for any $x \in Var$, $\vec{c} \in C_2^n$,

$$\alpha(a_{\lambda x.t})^{\cdot}\vec{c} = \alpha(a_\lambda{}^{\cdot}\,\Lambda_1{}^{\cdot}\,a_t)^{\cdot}\vec{c} \qquad \text{for } \Lambda_1 = \Gamma_1(\Gamma_1 \circ \pi_1)$$

$$= \alpha(a_\lambda)^{\cdot}\,\alpha(\Lambda_1)^{\cdot}\,\alpha(a_t)^{\cdot}\vec{c}$$

$$= \Lambda_2{}^{\cdot}(\alpha(a_t)^{\cdot}\vec{c}) = \Lambda_2{}^{\cdot}(a_{\Lambda(\gamma)(t)}{}^{\cdot}\vec{c})$$

$$= a_{\lambda x.\Lambda(\gamma)(t)}{}^{\cdot}\vec{c} = a_{\Lambda(\gamma)(\lambda x.t)}{}^{\cdot}\vec{c}.$$

Again, functoriality is immediate. That both functors are mutually inverse is immediate as well. $\square$

Exercise 15: Every λ^c-model $M = (C_1{}^{\cdot}, [\![\cdot]\!])$ can be extended functorially to a λ^C-model $M^+ = (C_1{}^{\cdot}, [\![\cdot]\!]^+)$. That means, there is a functor

$$+: \lambda\text{-Mod} \longrightarrow \lambda\text{-Mod}$$

s.th.

$* \ \forall M = (C, \cdot, [\![\cdot]\!]) \in \lambda\text{-Mod}:$

$$\begin{tikzcd} \Lambda^{\mathbb{C}} \times \mathrm{Val}(C) \arrow[r, "\Lambda([\![\cdot]\!]_C) \times 1"] \arrow[dr, "{[\![\cdot]\!]}"'] & \Lambda^C \times \mathrm{Val}(C) \arrow[d, "{[\![\cdot]\!]^+}"] \arrow[dl, phantom, "G"] \\ & C \end{tikzcd} \quad (*)$$

So, in particular, we obtain an inclusion

$M \xhookrightarrow{\eta^+} M^+$ in λ-mod.

$* \ \forall (\alpha : M_1 \to M_2) \in \lambda\text{-mod}:$

$$\begin{tikzcd} M_1 \arrow[r, "\eta^+"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "G"] & M_1^+ \arrow[d, "\alpha^+"] \\ M_2 \arrow[r, "\eta^+"'] & M_2^+ \end{tikzcd}$$

Remark 16: The pair $\lambda\text{-Thy} \underset{\mathrm{T} \circ +}{\overset{C}{\rightleftarrows}} \lambda\text{-Mod}$ does not quite exhibit λ-Thy as a retract of λ-Mod, but for every $(\mathbb{C}, T) \in \lambda\text{-Thy}$, the natural interpretation $(\mathbb{C}, T) \xrightarrow{\eta} T(C(\mathbb{C}, T)^+) = (\Lambda^{\mathbb{C}}/_{\sim T}, \mathrm{Th}(\Lambda^{\mathbb{C}}/_{\sim T}^{+}))$ satisfies

$\Lambda(\eta) = \pi_T : \Lambda^{\mathbb{C}} \twoheadrightarrow \Lambda^{\mathbb{C}}/_{\sim T}$, and

$\forall (s :\equiv t) \in \Phi_\lambda^{\mathbb{C}}: \quad T \vdash s :\equiv t \iff \mathrm{Th}(\Lambda^{\mathbb{C}}/_{\sim T}^{+}) \vdash \Phi(\eta)(s :\equiv t).$

$\leadsto$ May be thought of as "weak equivalences"?

We can use the syntax-free description of λ-models to construct categorical models of the λ-calculus.

Definition 17: A category $\mathcal{C}$ is *cartesian closed* if $\mathcal{C}$ has finite products (and thus a terminal object), and $\forall C \in \mathcal{C}$, $C \times \_ : \mathcal{C} \to \mathcal{C}$ has a right-adjoint $(-)^C$. We will consider a cart. closed category $\mathcal{C}$ to come equipped with a distinguished choice of a terminal object, of a product $C \times D$ and of an exponential D^C for every pair of objects $C, D \in \mathcal{C}$.

A reflexive object in a cart. closed category $\mathcal{C}$ is a triple (U, F, G) s.th. $U \in \mathcal{C}$ and $U^U \underset{G}{\overset{F}{\leftrightarrows}} U$ is a retract in $\mathcal{C}$.

The category CCwR of cart. closed cat's w/ reflexive objects is given by according tuples $(\mathcal{C}, U, F, G)$, and morphisms $(\mathcal{C}_1, U_1, F_1, G_1) \to (\mathcal{C}_2, U_2, F_2, G_2)$ being cart. functors $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ (i.e. functors which preserve the distinguished choices of a terminal object, products and exponentials) s.th. $\Phi(U_1) = U_2$, $\Phi(F_1) = F_2$, $\Phi(G_1) = G_2$.

Theorem 18: There are functors

$$1\text{-Mod}^{s.f.} \underset{\mathfrak{M}}{\overset{\mathcal{M}^{SP}}{\rightleftarrows}} CCwR$$

s.th. $\mathfrak{M} \circ \mathcal{M}^{SP} \cong 1$.

We close the chapter on untyped λ-calculi with the construction of these two functors, and Dana Scott's construction of a non-syntactic CCwR.