**Theorem 18:** There are functors

$$1\text{-Mod}^{s.f.} \underset{\mathcal{M}}{\overset{M^{sp}}{\rightleftarrows}} \mathcal{C}C\omega R$$

s.th. $\mathcal{M} \circ M^{sp} \cong 1$.

Idea: Appl. structures are not quite monoids: not right-associative, no right units. We can remedy this when they exhibit a 1-model structure $M$, and turn them into monoids $M(M)$ which are almost ccwr's. Fixing the "almost" yields a ccwr $M(M)^{sp} =: M^{sp}(M)$.

A (s.f.) 1-model gives rise to a monoid in the following way. Let

$$M(M) := \{ c \in C \mid \sigma(x_c) = c \} = \mathrm{im}(\sigma)$$

$\uparrow$ $\sigma$ is idempotent.

For $c, d \in C$, let $c \circ d := \sigma(\underbrace{x_c \circ x_d}_{\in \mathrm{Rep}(C_i) \text{ by comb. completeness}})$,

$$= [\![ \lambda x. \mathrm{App}(v, \mathrm{App}(u, x)) ]\!] \rho[c/v][d/u] ,$$

and let $1_{M(M)} := \sigma(1_C) = [\![ \lambda x. x ]\!] = [\![ I ]\!] =: \mathbb{I}$.

**Exercise 19:** $(M(M), \circ, \mathbb{I})$ is a monoid for any 1-model $M$.

Proof: For $c \in M(M)$, we have

* $\mathbb{I} \circ c = \sigma(x_{\mathbb{I}} \circ x_c) = \sigma(1_C \circ x_c) = \sigma(x_c) = c$,
* $c \circ \mathbb{I} = \sigma(x_c \circ x_{\mathbb{I}}) = \sigma(x_c \circ 1_C) = \sigma(x_c) = c$.

For $c, d, e \in M(M)$, we have

* $(c \circ d) \circ e = \sigma(\chi_c \circ \chi_d) \circ e = \sigma(\chi_{\sigma(\chi_c \circ \chi_d)} \circ \chi_e)$
$= \sigma((\chi_c \circ \chi_d) \circ \chi_e) = \sigma(\chi_c \circ (\chi_d \circ \chi_e))$
$= \sigma(\chi_c \circ \chi_{\sigma(\chi_d \circ \chi_e)}) = \sigma(\chi_c \circ \chi_{d \circ e}) = c \circ (d \circ e).$

$\square$

**Observation 20:** $M(M)$ considered as a category with one object is almost a CCwR: It has the structure of a semi-cartesian closed category (that is, of a "cartesian closed monoid", see Koymans).

There is a universal way to turn semi-categorical structure to strictly categorical structure: the idempotent completion functor (the "Karoubi envelope")

$$\begin{tikzcd} \mathrm{Cat}_{\mathrm{semi}} \arrow[r, "i"'] \arrow[r, phantom, shift left=2, "\top"] & \mathrm{Cat} \arrow[l, bend right=50, "sp"'] \end{tikzcd}$$

See Hoofman - "A note on semi-adjunctions", Koymans - "Models of the $\lambda$-calculus" for more details.

**Construction 21:** For a category $\mathbb{C}$, let $\mathbb{C}^{sp}$ be the category given as follows.

* $\mathrm{Ob}(\mathbb{C}^{sp}) = \{ f \in \mathbb{C}_1 \mid f \circ f = f \}$,
* $\mathbb{C}^{sp}(f, g) = \{ h \in \mathbb{C}(\mathrm{cod}\, f, \mathrm{dom}\, g) \mid g \circ h \circ f = h \}$, $\quad \overset{f}{\overset{\circlearrowleft}{c}} \xrightarrow{h} , \overset{g}{\overset{\circlearrowleft}{b}} = c \xrightarrow{h} b$
* $\mathbb{C}^{sp}(f, f) \ni 1_f := f$,
* $h_1 \circ_{\mathbb{C}^{sp}} h_2 := h_1 \circ_{\mathbb{C}} h_2$.

**Proposition 22:** Let $M=(C,\cdot,\sigma,f)$ be a (s.f.) $\lambda$-model. Then

a. $\mathcal{M}(M)^{sp}$ is a cart. closed category.

b. $\mathbb{I} = 1_{\mathcal{M}(M)} \in \mathcal{M}(M)^{sp}$ is a reflexive object via $\mathbb{I}^{\mathbb{I}} \underset{\mathbb{1}}{\overset{\mathbb{1}}{\leftrightarrows}} \mathbb{I}$.

Proof: Note that for $c \in C$, if $c = c\circ c = \sigma(\chi_c \circ \chi_c)$, then $c \in \mathcal{M}(M)$.

First, let's show the object $t := [\![\lambda x.\lambda y.y]\!] \in \mathcal{M}(M)^{sp}$ is terminal. For any $c \in \mathcal{M}(M)$, we have

$$t \circ c = \sigma(\chi_{[\![\lambda x.\lambda y.y]\!]} \circ \chi_c) = \sigma([\![\lambda y.y]\!]_{\varphi[c/x]}) = [\![\lambda x.\lambda y.y]\!] = t.$$

In particular, $t\circ t = t$, so $t \in \mathcal{M}(M)^{sp}$, and f.a. $c \in \mathcal{M}(M)^{sp}$,

$$\mathcal{M}(M)^{sp}(c,t) = \{d \in \mathcal{M}(M) \mid t\circ d \circ c = d\} = \{d \in \mathcal{M}(M) \mid t = d\} = \{*\}.$$

Second, to define products, recall the pair-terms $[t_1,t_2] \in \Lambda^{\mathbb{C}}$ and the projections $pr_i \in \Lambda^{\mathbb{C}}$ from Exercise I.15.3.

For $c_1, c_2 \in \mathcal{M}(M)$, let

$$c_1 \times c_2 := [\![\lambda z.[App(x_1, App(pr_1,z)), App(x_2, App(pr_2,z))]]\!]_{\varphi[c_i/x_i]_{i=1,2}}$$

Then $c_1 \times c_2 \xrightarrow{\pi_i} c_i$ is given by $\pi_i := c_i \circ [\![pr_i]\!]$.

For $d \xrightarrow{f_i} c_i$, we set $(f_1,f_2): d \to c_1 \times c_2$ as

$[\![\lambda z.[App(x,z), App(y,z)]]\!]_{\varphi[f_1/x][f_2/y]}$. The according universal property of the product is verified by a straight-forward computation (Exercise).

Third, to define exponentials for $c, d \in \mathcal{M}(M)^{sp}$, let

* $d^c := \sigma(e \mapsto d \circ e \circ c)$

$$= [\![ \lambda z.\lambda x.\, App(v, App(z, App(u, x))) ]\!]_{\varphi[d/v][c/u]}$$

* $ev_{c,d} : d^c \times c \to d$ by $ev_{c,d} := \sigma(e \mapsto d^\circ((\pi_1^\circ e)^\circ(c^\circ(\pi_2^\circ e))))$

$$= [\![ \lambda z. \dots ]\!]_{\varphi[-]}$$

* For $f : a \times c \to d$, we get $\ulcorner f \urcorner : a \to d^c$ via

$$\ulcorner f \urcorner := [\![ \lambda z.\lambda x.\lambda y.\, App(z, [x,y]) ]\!]_{\varphi[f/z]}$$

represents $C^2 \to C$

$(e_1, e_2) \mapsto f^\circ [\![ [x,y] ]\!]_{\varphi[e_1/x, e_2/y]}$

and verify the due identities (see Exercise sheet 2).

Lastly, the unit $\mathbb{I} = 1_{\mathcal{M}(M)}$ is a reflexive object in $\mathcal{M}(M)^{sp}$ via

$$\mathbb{I}^{\mathbb{I}} = \sigma(e \mapsto \underbrace{\mathbb{I} \cdot e \circ \mathbb{I}}_{=\sigma(\chi_e)}) = \sigma(\sigma \circ \pi) = \mathbb{1},$$

$$\mathbb{1} \underset{F = \mathbb{1}}{\overset{G = \mathbb{1}}{\rightleftarrows}} \mathbb{I} \quad \text{with}$$

- $\mathbb{I} \circ \mathbb{1} \circ \mathbb{1} = \mathbb{1} \circ \mathbb{1} = \mathbb{1}$, so $\mathbb{1} \in \mathcal{M}(M)^{sp}(\mathbb{1}, \mathbb{I})$,
- $\mathbb{1} \circ \mathbb{1} \circ \mathbb{I} = \mathbb{1} \circ \mathbb{1} = \mathbb{1}$, so $\mathbb{1} \in \mathcal{M}(M)^{sp}(\mathbb{I}, \mathbb{1})$,
- $F \circ G = \mathbb{1} \circ \mathbb{1} = \mathbb{1} = id_{\mathbb{1}}$. $\square$

Remark 23: In fact, every object $c \in \mathcal{M}(M)^{sp}$ is a retract of $\mathbb{I}$ via

$$c \underset{c}{\overset{c}{\rightleftarrows}} \mathbb{I}.$$

**Corollary 24:** The assignment from Proposition 22 assembles to a functor

$$\mathcal{M}^{SP} : \lambda\text{-Mod} \longrightarrow \mathrm{CCwR}.$$

Proof:

$$\begin{tikzcd}
M_1 = (C_1, \cdot, \sigma_1, f_1) \arrow[r, mapsto] \arrow[d, "\alpha"] & \mathcal{M}(M_1)^{SP} \arrow[d, "\mathcal{M}(\alpha)^{SP}"] & \ni & c \arrow[r, "f"] \arrow[d, mapsto] & d \\
M_2 = (C_2, \cdot, \sigma_2, f_2) \arrow[r, mapsto] & \mathcal{M}(M_2)^{SP} & & \alpha(c) \arrow[r, "\alpha(f)"'] & \alpha(d)
\end{tikzcd}$$

First, each $\mathcal{M}(\alpha)^{SP}$ is a functor:

$$\alpha(c \circ d) = \alpha([\![\lambda x. App(v, App(u, x))]\!] \rho[c/v][d/u])$$

$$= [\![ \_ ]\!] \rho[\alpha c/v][\alpha d/u] = \alpha(c) \circ \alpha(d),$$

$$\alpha(id_c) = \alpha(c) = id_{\alpha(c)} \in \mathcal{M}(M_2)^{SP}(c, c).$$

Second, $\mathcal{M}(\alpha)^{SP}$ is cartesian:

- $\alpha(c_1 \times c_2) = \alpha([\![\lambda z. \text{----} ]\!]\rho[c_i/x_i]) = [\![\lambda z. \_ ]\!]\rho[\alpha c_i/x_i] = \alpha c_1 \times \alpha c_2.$
- Same for exponentials and evaluations.

That $\mathcal{M}(\alpha)^{SP}$ furthermore preserves the reflexive object $(I, 1, 1)$ is immediate. The fact that $\mathcal{M}^{SP}$ takes compositions of $\lambda$-mod. homomorphisms to compositions of cartesian functors is immediate, as is unitality. $\square$

Vice versa, let's try to assign a (s.f.) $\lambda$-model to a ccwr $(\mathcal{C}, U, F, G)$. Therefore, consider

* $\mathcal{P}(U) := \mathcal{C}(1, U) \in \mathbf{Set}$,
* $(-)^\cdot(-) : \mathcal{P}(U) \times \mathcal{P}(U) \longrightarrow \mathcal{P}(U)$ be given by the composition

$$\mathcal{P}(U) \times \mathcal{P}(U) \cong \mathcal{P}(U \times U) \xrightarrow{(F\times 1)_*} \mathcal{P}(U^U \times U) \xrightarrow{ev_*} \mathcal{P}(U).$$

Then $(\mathcal{P}(U), \cdot)$ is an appl. structure, with

$$\begin{tikzcd}
\mathcal{P}(U) \arrow[r, "F_*"] \arrow[rr, bend left, "\chi"] \arrow[dr, two heads, bend right] & \mathcal{P}(U^U) \arrow[r, "\ulcorner ev_* \urcorner"] \arrow[d, dotted, two heads] & \mathrm{Fun}(\mathcal{P}(U), \mathcal{P}(U)), \\
& \mathrm{Rep}(\mathcal{P}(U), \cdot) \arrow[ur, hook] & \\
& = \mathrm{im}\, \chi &
\end{tikzcd}$$

The dotted surjection is given via the retract $\mathcal{P}(U) \underset{G_*}{\overset{F_*}{\rightleftarrows}} \mathcal{P}(U^U)$.

For $u, v \in \mathcal{P}(U)$, we have $1 \xrightarrow{u} U \xrightarrow{F} U^U$, so $1 \xrightarrow[(Fu, v)]{} U^U \times U \xrightarrow{ev} U$ computes

$u \cdot v := ev(Fu, v)$.

**Lemma 25:** Let $\mathcal{C}$ be a cart. closed category.

1. For $f : C \to D$ in $\mathcal{C}$ and $c : 1 \to C$, we have $f \circ c = ev \circ (\ulcorner f \urcorner, c)$.
2. If $U \underset{F}{\overset{G}{\leftrightarrows}} U^U$ is a reflective object in $\mathcal{C}$, then every $f : U \to U$ induces an element $G \circ \ulcorner f \urcorner : 1 \to U^U \to U$ s.th.

$\forall u \in \mathcal{P}(U) : (G \circ \ulcorner f \urcorner) \cdot u = f \circ u$.

Proof: Exercise. □

Lemma 26: Whenever $U \in \mathcal{C}$ has enough points, the surjection

$$\Gamma(U^U) \longrightarrow \mathrm{Repl}(\Gamma(U), \cdot)$$

is a bijection.

Proof: Suppose $\ulcorner f \urcorner, \ulcorner g \urcorner : 1 \to U^U$ are global sections with $\ulcorner ev_* \urcorner(\ulcorner f \urcorner) = \ulcorner ev_* \urcorner(\ulcorner g \urcorner)$.

Since $\forall u \in \Gamma(U)$: $\quad \ulcorner ev_* \urcorner(\ulcorner f \urcorner)(u) = ev_*(\ulcorner f \urcorner, u) = ev \circ (\ulcorner f \urcorner, u) = f \circ u$

$\qquad\qquad\qquad\qquad\quad\ \ulcorner ev_* \urcorner(\ulcorner g \urcorner)(u) = \ -\ = ev \circ (\ulcorner g \urcorner, u) = g \circ u$

it follows that $f \circ u = g \circ u$ f.a. $u \in \Gamma(U)$.

$\Rightarrow f = g$ by assumption, and so $\ulcorner f \urcorner = \ulcorner g \urcorner$. □

Hence, whenever $\mathcal{C}$ has enough points, we get a retract

$$\begin{tikzcd} \Gamma(U) \arrow[r, "\pi"] \arrow[rr, bend right, "F_*"'] & \mathrm{Repl}(\Gamma(U), \cdot) \cong & \Gamma(U^U) \arrow[ll, bend right, "G = G_*"'] \end{tikzcd}.$$

Let $CCwR^+ \subseteq CCwR$ denote the full subcategory consisting of the CCwR's whose reflexive object has enough points. Let $CCwR^+_{rel}$ be the cat. of objects $(\mathcal{C}, U, F, G, f)$ where $(\mathcal{C}, U, F, G) \in CCwR^+$, $f : \mathbb{C} \to \Gamma(U)$, and according arrows.

**Proposition 27:** The global sections functor induces a functor

$$
\begin{array}{ccc}
\Gamma : \mathcal{CCwR}^{+}_{rel} & \longrightarrow & \mathcal{A}\text{-}\mathrm{Mod}^{s.f.} \\[1em]
(\mathcal{C}_1, U_1, F_1, G_1, f_1) & \longmapsto & (\Gamma(U_1), \cdot_1, (F_1)_*, f_1) \\
\Big\downarrow (\alpha, g) & & \Big\downarrow (\Gamma(\alpha), g) \\
(\mathcal{C}_2, U_2, F_2, G_2, f_2) & \longmapsto & (\Gamma(U_2), \cdot_2, (F_2)_*, f_2)
\end{array}
$$

**Proof:** To show that $\Gamma(\mathcal{C}, U, F, G)$ is a s.f. $\mathcal{A}$-model, we are left to show that $\Gamma(U)$ is comb. complete, and that $\sigma \circ \pi : \Gamma(U) \to \Gamma(U)$ is representable. The latter holds by construction, since

$\sigma \circ \pi = G_* \circ F_* = (G \circ F)_* = \ulcorner ev_* \urcorner (\underbrace{G \circ F}_{\in \Gamma(U^U)}) \in \mathrm{Rep}(\Gamma(U), \cdot)$ via Lemma 24.

Towards the former, for a finite set $X = \{x_1, \dots, x_n\} \subset \mathrm{Var}$, $t \in \overline{X \cup \mathcal{C}}$, we want to show that

$$\exists a_t \in \Gamma(U) \; \forall \rho \in \mathrm{Val}(\Gamma(U)) : [\![ t ]\!]_\rho = a_t \,'\, \rho(x_1) \,'\, \dots \,'\, \rho(x_n) \quad (*).$$

Let

- $a_{x_i} : 1 \xrightarrow{\ulcorner \dots \ulcorner \pi_i \urcorner \dots \urcorner} (\underbrace{(U^U)'}_{n\text{-times}})^U \xrightarrow{G'^{\dots U}} (\underbrace{(U^U)'}_{(n-1)\text{-times}})^U \xrightarrow{G'^{\dots U}} \dots \xrightarrow{G^U} U^U \xrightarrow{G} U$
- $f(c) \circ !_{U^n} : U^n \to 1 \to U$ yields $1 \xrightarrow{\ulcorner f(c) \circ !_{U^n} \urcorner} (U^U)'^{U} \xrightarrow{G'^{U}} \dots \xrightarrow{G} U$
- $\Gamma(U)^n \xrightarrow{(x_{as}^{(n)}, x_{at}^{(n)})} \Gamma(U) \times \Gamma(U) \xrightarrow{\cdot} \Gamma(U)$ yields

$a_{App(s,t)} : 1 \xrightarrow{\ulcorner \dots \ulcorner x_{as}^{(n)} \cdot x_{at}^{(n)} \urcorner \dots \urcorner} \dots \xrightarrow{G'} U$ analogously.

Then $(*)$ holds by induction on $n \geq 0$ and the complexity of $t \in \Lambda^{\mathcal{C}}$. E.g.

for $n=2$, $\pi_i : \mathcal{U}\times\mathcal{U} \to \mathcal{U}$ yields $\ulcorner \pi_i \urcorner : \mathcal{U} \to \mathcal{U}^{\mathcal{U}}$ and $\ulcorner\ulcorner \pi_i \urcorner\urcorner : 1 \to (\mathcal{U}^{\mathcal{U}})^{\mathcal{U}}$.

We thus get $a_{x_i} : 1 \xrightarrow{\ulcorner\ulcorner \pi_i \urcorner\urcorner} (\mathcal{U}^{\mathcal{U}})^{\mathcal{U}} \xrightarrow{G^{\mathcal{U}}} \mathcal{U}^{\mathcal{U}} \xrightarrow{G} \mathcal{U}$ s.th.

$\forall \varphi \in \mathrm{Val}(\mathcal{P}(\mathcal{U}))$: $\quad (G \circ G^{\mathcal{U}} \circ \ulcorner\ulcorner \pi_i \urcorner\urcorner \cdot \varphi x_1) \cdot \varphi x_2 = \mathrm{ev}(G^{\mathcal{U}} \circ \ulcorner\ulcorner \pi_i \urcorner\urcorner, \varphi x_1) \cdot \varphi x_2$

$$= \mathrm{ev}(\ulcorner (G \circ \ulcorner \pi_i \urcorner) \urcorner, \varphi x_1) \cdot \varphi x_2$$

$$= (\ulcorner G \circ \pi_i \urcorner \circ \varphi x_1) \cdot \varphi x_2$$

$$= \mathrm{ev}(\ulcorner \pi_i \urcorner \circ \varphi x_1, \varphi x_2)$$

$$= \mathrm{ev}(\ulcorner \pi_i \circ (1 \times \varphi x_1) \urcorner, \varphi x_2)$$

$$= \pi_i \circ (1 \times \varphi x_1) \circ \varphi x_2$$

$$= \pi_i(\varphi x_1 \times \varphi x_2) = \varphi x_i = [\![ x_i ]\!]_\varphi$$

Etc. Similarly, for the App-case, $n \geq 0$, and $s, t \in \Lambda^C$ with $a_s, a_t$ given,

$$a_{\mathrm{App}(s,t)} \cdot \varphi x_1 \cdots \varphi x_n = (G \circ \ulcorner \cdot \circ (\chi^{(n)}_{a_s}, \chi^{(n)}_{a_t}) \urcorner) \cdot \varphi x_1 \cdots \varphi x_n$$

$$= (\cdot \circ \chi^{(n)}_{a_s}, \chi^{(n)}_{a_t})(\varphi x_1, \ldots, \varphi x_n)$$

$$= \chi^{(n)}_{a_s}(\varphi x_1, \ldots, \varphi x_n) \cdot \chi^{(n)}_{a_t}(\varphi x_1, \ldots, \varphi x_n)$$

$$= (a_s \cdot \varphi x_1 \cdots \varphi x_n)(a_t \cdot \varphi x_1 \cdots \varphi x_n)$$

$$= [\![ s ]\!]_\varphi \cdot [\![ t ]\!]_\varphi = [\![ \mathrm{App}(s,t) ]\!]_\varphi .$$

The remaining computations are left to the reader. $\qquad =$

Given a functor $\alpha : (\mathcal{C}_1, \mathcal{U}_1, F_1, G_1) \to (\mathcal{C}_2, \mathcal{U}_2, F_2, G_2)$ in CCwR, we get

$\Gamma(\alpha) = \alpha_1 : \Gamma(\mathcal{U}_1) \to \Gamma(\mathcal{U}_2)$ s.th.

1. $\Gamma(\alpha)$ is a homomorphism of comb. complete applicative structures by

induction, since the $\alpha_t$'s are entirely defined via operations which are preserved by $\alpha$ on the nose.

2.
$$\begin{tikzcd}
\mathcal{P}(u_1) \arrow[r, bend left, "\mathcal{P}(G_1)"] \arrow[d, "\alpha_1"'] & \mathcal{P}(u_1^{u_1}) \arrow[l, bend left, "\mathcal{P}(F_1)"] \arrow[d, "\alpha_1"] \\
\mathcal{P}(u_2) \arrow[r, bend left, "\mathcal{P}(G_2)"] & \mathcal{P}(u_2^{u_2}) \arrow[l, bend left, "\mathcal{P}(F_2)"]
\end{tikzcd}$$
commutes in both directions by functoriality of $\alpha$.

3. Commutativity for the assignments of constant symbols is immediate, functoriality as well. $\square$

**Proposition 28:** The functor $\mathcal{M}^{sp}: 1\text{-Mod} \to CCwR$ factors through $CCwR^+_{rel}$.

The pair
$$\begin{tikzcd}
1\text{-Mod} \arrow[r, bend right, "\mathcal{M}^{sp}"'] & CCwR^+_{rel} \arrow[l, bend right, "\mathcal{P}"']
\end{tikzcd}$$
exhibits 1-Mod as a retract of $CCwR^+_{rel}$ up to equivalence (i.e. $\mathcal{P}\circ\mathcal{M}^{sp} \simeq 1$).

Proof: First, let $M = (G', \mathbb{T}\cdot\mathbb{J}) \in 1\text{-Mod}$, and $f, g : \mathbb{I} \to \mathbb{I}$ in $\mathcal{M}^{sp}(M)$ s.th. $f\circ c = g\circ c \in \mathcal{P}(\mathbb{I})$ f.a. $c \in \mathcal{P}(\mathbb{I})$. Then, by Lemma 22.b, f.a. $c \in C$ $(G\circ\ulcorner f\urcorner)'c = (G\circ\ulcorner g\urcorner)'c$, i.e. $G\circ\ulcorner f\urcorner \sim G\circ\ulcorner g\urcorner$. Hence $1'(G\circ\ulcorner f\urcorner) = 1'(G\circ\ulcorner g\urcorner)$, where $x_1 = (G\circ F)*$.

$\Rightarrow (G\circ F)\circ(G\circ\ulcorner f\urcorner) = (G\circ F)\circ(G\circ\ulcorner g\urcorner)$, and as $F\circ G = 1$, we get $G\circ\ulcorner f\urcorner = G\circ\ulcorner g\urcorner$. Again applying $F$ on both sides gives

$\ulcorner f\urcorner = \ulcorner g\urcorner$. $\Rightarrow f = g$. $=$

To show that $\Gamma\circ M^{sp} \cong 1$, let $(C_1;\sigma, f) \in \mathcal{A}\text{-Mod}^{s.f.}$.

Claim: $\chi_{[\![K]\!]} : (C_1^{\cdot}, \sigma, f) \longrightarrow (\Gamma(\mathbb{I}), \circ, \Gamma(\mathbb{1}), \chi_{[\![K]\!]}\circ f)$

$$c \longmapsto [\![K]\!]^{\cdot} c = [\![\lambda x.\lambda_g.x]\!]^{\cdot} c \quad \text{represents constant}$$
$$= [\![\lambda_g.x]\!]_{x[c/x]} \quad \text{endomorphism with}$$
$$= \sigma(\mathrm{cnst}(c)) \quad \text{value } c.$$

is an isomorphism in $\mathcal{A}\text{-Mod}^{s.f.}$.

Proof of the claim: First, note that

$$\Gamma(\mathbb{I}) = M^{sp}(M)(t, \mathbb{I}) = \{c \in \mathrm{im}\,\sigma \mid \underbrace{\mathbb{I}\circ c\circ t} = c\}$$
$$= \sigma(\chi_{\mathbb{I}}\circ\chi_c\circ\chi_t)$$
$$= \sigma(\mathrm{cnst}(c^{\cdot}\mathbb{I}))$$
$$= \{c \in \mathrm{im}\,\sigma \mid \chi_c \text{ is constant}\},$$

so $\chi_{[\![K]\!]}$ is well-typed. Furthermore, the square

$$\begin{tikzcd}
C \arrow[d, "\chi_{[\![K]\!]}"'] \arrow[r, shift right, "\pi"'] & \mathrm{Rep}(C_1^{\cdot}) \arrow[l, shift right, "\sigma"'] \arrow[d, dashed, "\overline{\chi_{[\![K]\!]}}"] \\
\Gamma(\mathbb{I}) \arrow[r, shift right, "\Gamma(\mathbb{1})"'] & \Gamma(\mathbb{1}) \arrow[l, shift right, "\Gamma(\mathbb{1})"']
\end{tikzcd}$$

exists since for $c \in C_1$

$$\Gamma(\mathbb{1})(\chi_{[\![K]\!]}(c)) = \mathbb{1}\circ([\![K]\!]^{\cdot}c) = \sigma(\sigma\circ\pi\circ\chi_{[\![K]\!]^{\cdot}c})$$
$$= \sigma(\mathrm{cnst}(\sigma(\chi_c)) = \sigma(\mathrm{cnst}(\mathbb{1}^{\cdot}c))) = [\![K]\!]^{\cdot}(\mathbb{1}^{\cdot}c)$$

Thus $c \sim d$ implies $\Gamma(\mathbb{1})(\chi_{[\![K]\!]}(c)) = \Gamma(\mathbb{1})(\chi_{[\![K]\!]}(d))$, and the dotted arrow exists, making the square wrt. the pair $(\pi, \Gamma(\mathbb{1}))$ commute. Additionally,

f.a. $c \in C_1$ $[\![K]\!]^{\cdot}\sigma(\chi_c) = [\![K]\!]^{\cdot}(\mathbb{1}^{\cdot}c) = \sigma(\chi_{\mathbb{1}}\circ[\![K]\!]^{\cdot}(\mathbb{1}^{\cdot}c)) = \mathbb{1}\circ[\![K]\!]^{\cdot}(\mathbb{1}^{\cdot}c)$

$$= \Gamma(\mathbb{I})(\overline{\chi_{[\![K]\!]}}(\chi_c)).$$

$=$

Furthermore, note that

Exercise: * $\forall c,d \in \mathcal{T}(\mathcal{I})$: $\quad c \,^{\odot} d := ev \circ (1 \circ c, d) \overset{!}{=} [\![S]\!]^{\cdot} c^{\cdot} d \in C$.

* $\forall c \in C, d \in \mathcal{T}(\mathcal{I})$: $([\![K]\!]^{\cdot} c) \,^{\odot} d = [\![K]\!]^{\cdot}(c^{\cdot}(d^{\cdot}\mathbb{I}))$.

Then we can show that $\chi_{[\![K]\!]}$ is a morphism of comb. complete appl. structures by induction along the complexity of terms, using the comb. closed structure of $(\mathcal{T}(\mathcal{I}), \odot)$ in the proof of Prop. 27. I.e.

Claim: $\forall X = \{x_1, \dots, x_n\} \subset Var, t \in \overline{X \cup C}$, $a_t \in C$ as in Def. 11, $\varphi \in Val(\mathcal{T}(\mathcal{I}))$:

$([\![K]\!]^{\cdot} a_t) \,^{\odot} \varphi x_1 \,^{\odot} \dots \,^{\odot} \varphi x_n = [\![t]\!]^{\odot}_{\varphi} \in \mathcal{T}(\mathcal{I})$.

$\hat{=}$ Interpretation in $(\mathcal{T}(\mathcal{I}), \odot)$!

Given $t \in \overline{X \cup C}$, $\varphi \in Val(\mathcal{T}(\mathcal{I}))$,

$([\![K]\!]^{\cdot} a_t) \,^{\odot} \varphi(\vec{x}) = \dots = [\![\lambda z. t[{}^{App(\vec{x}, \mathcal{I})}/_{\vec{x}}]]\!]_{\varphi} \overset{!}{=} [\![t]\!]^{\odot}_{\varphi}$.

The last equation can be shown for variables, constant symbols, and syntactic applications using the proof of Prop. 27. E.g. for $x \in X$,

$[\![\lambda z. x[{}^{App(\vec{x}, \mathcal{I})}/_{\vec{x}}]]\!]_{\varphi} = [\![\lambda z. App(x, \mathcal{I})]\!]_{\varphi} = [\![K]\!]^{\cdot}(\varphi(x)^{\cdot}\mathbb{I}) = \varphi(x)$.

Etc.. ┘

We are left to show that $\chi_{[\![K]\!]}$ is an isomorphism. One can show that $\mathcal{A}$-Mod is the full subcategory of a variety of algebras, thus it suffices to show that its underlying function of sets is bijective.

- Injectivity: $[\![K]\!]^{\cdot} c = [\![K]\!]^{\cdot} d \Rightarrow c = ([\![K]\!]^{\cdot} c)^{\cdot}\mathbb{I} = ([\![K]\!]^{\cdot} d)^{\cdot}\mathbb{I} = d$.

- Surjectivity: Given $c \in \Gamma(\mathbb{I})$, we have $\llbracket K D \rrbracket \cdot (c \cdot \mathbb{I}) = c$. $=$

□

Corollary 29: Let $(\mathcal{C}, \mathcal{U}, F, G, f) \in \mathcal{CC}wR^{+}_{rel}$. Then $G \circ F = 1_{\mathcal{U}}$ (and so $\mathcal{U} \cong \mathcal{U}^{\mathcal{U}}$) iff $\Gamma(\mathcal{C}, \mathcal{U}, F, G, f) \models \lambda\beta\eta$.

Proof: Recall that $\Gamma(\mathcal{C}, \mathcal{U}, F, G, f)$ is a s.f. $\lambda$-model via the retract

$$\mathrm{Rep}(\Gamma(\mathcal{U}), \cdot) = \Gamma(\mathcal{U}^{\mathcal{U}}) \underset{\sigma = G_*}{\overset{\pi = F_*}{\rightleftarrows}} \Gamma(\mathcal{U}).$$

Hence, $\Gamma(\mathcal{C}, \mathcal{U}, F, G, f) \models \lambda\beta\eta$ iff $\Gamma(\mathcal{C}, \mathcal{U}, F, G, f)$ is an extensional appl. structure (Proposition 2.10) iff $F_*$ is a bijection iff $G_* \circ F_* = 1_{\Gamma(\mathcal{U})}$ iff $G \circ F = 1_{\mathcal{U}}$ because $\mathcal{U}$ has enough points.

□

Hence, in summary, we obtain a composite retract

$$\lambda\text{-Thy} \underset{C}{\overset{T}{\rightleftarrows}} \lambda\text{-Mod} \underset{\mu^{sp}}{\overset{\Gamma}{\rightleftarrows}} \mathcal{CC}wR^{+}_{rel}.$$

Note that $\mu^{sp} \circ \Gamma \neq \mathrm{Id}$, since $\Gamma(\mathcal{C}, \mathcal{U}, F, G, f)$ only captures the global elements of $\mathcal{U}$ and $\mathcal{U}^{\mathcal{U}}$. The category $\mathcal{C}$ may have many more objects. $\rightsquigarrow$ $\mathcal{C}$ as a whole is captured by a typed $\lambda$-calculus, see next chapter.