Definition 10: The calculus of structural and logical rules over $(\Phi_{\mathbb{C},\Pi})^\infty$ is given as follows. The meta-variables A, B range over Ty^Π.

1. $$\frac{}{a :\equiv a}\,, \qquad \frac{\Gamma \vdash a :\equiv b}{\Gamma \vdash b :\equiv a}\,, \qquad \frac{\Gamma \vdash a :\equiv b \quad \Gamma \vdash b :\equiv c}{\Gamma \vdash a :\equiv c} \qquad \text{f.a. } a, b, c \in \Lambda_A^{\mathbb{C}}$$

(Refl, Symmetry, Transitivity)

2. $$\frac{}{\Gamma, a :\equiv b \vdash a :\equiv b} \qquad \text{f.a. } a, b \in \Lambda_A^{\mathbb{C}}$$ (Monotonicity)

3. $$\frac{\Gamma \vdash a :\equiv b}{\Gamma, c :\equiv d \vdash a :\equiv b} \qquad \text{f.a. } a, b, c, d \in \Lambda_A^{\mathbb{C}}.$$ (Weakening)

4. $$\frac{\Gamma \vdash a :\equiv b}{\Gamma \vdash App(f, a) :\equiv App(f, b)} \qquad \text{f.a. } a, b \in \Lambda_A^{\mathbb{C}},\ f \in \Lambda_{A \to B}$$

5. $$\frac{\Gamma \vdash a :\equiv b}{\Gamma \vdash \lambda x.a :\equiv \lambda x.b} \qquad \text{f.a. } a, b : \Lambda_A^{\mathbb{C}},\ x \in Var_B$$ (ξ)

6. $$\frac{\Gamma \vdash a_1 :\equiv b_1 \quad \Gamma \vdash a_2 :\equiv b_2}{\Gamma \vdash [a_1, a_2] :\equiv [b_1, b_2]} \qquad \text{f.a. } a_i, b_i \in \Lambda_{A_i}^{\mathbb{C}}.$$

7. $$\frac{\Gamma \vdash c_1 :\equiv c_2}{\Gamma \vdash pr_i(c_1) :\equiv pr_i(c_2)} \qquad \text{f.a. } c_i \in A_1 \times A_2,\ i \in \{0, 1\}.$$

Term- (Congruence rules)

8. $$\frac{}{\Gamma \vdash a :\equiv *} \qquad \text{f.a. } a \in \Lambda_1$$ (1-uniqueness)

9. $$\frac{}{\Gamma \vdash pr_i([a_1, a_2]) :\equiv a_i}$$ f.a. $a_i \in \Lambda^{\mathcal{C}}_{A_i}$ ($\times$-computation)

10. $$\frac{}{\Gamma \vdash [pr_1(c), pr_2(c)] :\equiv c}$$ f.a. $c \in \Lambda_{A \times B}$ ($\times$-uniqueness)

11. $$\frac{}{\Gamma \vdash \lambda x. App(f, x) :\equiv f}$$ f.a. $f \in \Lambda_{A \to B}$, $x \in Var_A \setminus FV(f)$

(η-congruence, or $\to$-uniqueness)

12. $$\frac{}{\Gamma \vdash App(\lambda x. b, a) :\equiv b[a/x]}$$ f.a. $b : B$, $a : A$, $x \in Var\,A$

(β-congruence, or $\to$-computation)

13. $$\frac{}{\Gamma \vdash \lambda x. a :\equiv \lambda y. a[y/x]}$$ f.a. $a \in \Lambda A$, $x, y \in Var\,A$ (α-congruence)

Definition 11: A *simply typed λ-theory* (with products and a terminal type) is a tuple $(\Pi, \mathcal{C}, \mathcal{T}_{Ty}, \mathcal{T}_{Tm})$ for arbitrary disjoint sets $\Pi, \mathcal{C}$, and subsets $\mathcal{T}_{Ty} \subseteq \Phi^{\Pi}_{Ty}$, $\mathcal{T}_{Tm} \subseteq \Phi^{\Pi, \mathcal{C}}_{Tm}$.

Remark 12: 1. There is a notion of $\beta\eta$-reduction for simply typed λ-theories. Here, every term has a normal form, and $\beta\eta$-reduction always terminates.

2. One can define the typed Church-numerals and study definable arithmetic functions in a typed context by adding a constant type 0 with no term-rules and letting $n :\equiv \lambda x. \lambda y. x^{(n)}(y) : (0 \to 0) \to (0 \to 0)$, i.e. $\mathbb{1} :\equiv \lambda x. \lambda y. App(x, y)$ etc. See Bar84, Appendix 1 for more on both.

Definition 13: Given two simply typed λ-theories $\mathcal{L}_i = (\Pi_i, \mathcal{C}_i, \mathcal{T}^i_{Ty}, \mathcal{T}^i_{Tm})$, a preinterpretation $I : \mathcal{L}_1 \to \mathcal{L}_2$ is a pair of functions

$$\Big(I_{Ty} : \Pi_1 \to Ty^{\Pi_2},\ I_{Tm} : \coprod_{A : Ty^{\Pi_1}} \mathcal{C}_1^A \to \coprod_{A : Ty^{\Pi_2}/\mathcal{T}_2} \Lambda^{\mathcal{C}_2}_A \Big).$$

Every preinterpretation induces a pair of maps

$$\Big(I_{Ty} : Ty^{\Pi_1} \to Ty^{\Pi_2},\ I_{Tm} : \coprod_{A : Ty^{\Pi_1}/\mathcal{T}^1_{Ty}} \Lambda^{\mathcal{C}_1}_A \to \coprod_{A : Ty^{\Pi_2}/\mathcal{T}^2_{Ty}} \Lambda^{\mathcal{C}_2}_A \Big)$$

by recursion in the obvious way: E.g. $I_{Ty}(A \to B) = I_{Ty}(A) \to I_{Ty}(B)$,

$I_{Ty}(1) = 1$, ..., $I_{Tm}(\overset{\mathrm{Var}\,A}{x_u}) = \overset{\mathrm{Var}\,I_{Ty}(A)}{x_u}$ for all $x \in \mathrm{Var}\,A$,

$I_{Tm}(\lambda x.b) = \lambda x. I_{Tm}(b)$ etc.

A preinterpretation $I : \mathcal{L}_1 \to \mathcal{L}_2$ is an interpretation if

* $\forall (A :\equiv_{Ty} B) \in \mathcal{T}^1_{Ty} : \ \mathcal{T}^2_{Ty} \vdash_{Ty} I_{Ty}(A) :\equiv_{Ty} I_{Ty}(B)$
* $\forall A \in Ty^{\Pi_1}, a \in \Lambda^{\mathcal{C}_1}_A : \quad I_{Tm}(a) \in \Lambda^{\mathcal{C}_2}_{I_{Ty}(A)}$
* $\forall A \in Ty^{\Pi_1}, a, b \in \Lambda^{\mathcal{C}_1}_A : \ (a :\equiv_{Tm} b) \in \mathcal{T}^1_{Tm} \Rightarrow \mathcal{T}^2_{Tm} \vdash I_{Tm}(a) :\equiv I_{Tm}(b)$
* $FV(I_{Tm}(a)) = \emptyset$ whenever $FV(a) = \emptyset$.

Two interpretations $I_i = (I^i_{Ty}, I^i_{Tm}) : \mathcal{L}_1 \to \mathcal{L}_2$ are equivalent (denoted by $I_1 \sim I_2$) if

* $\forall A \in Ty^{\Pi_1} : \quad \mathcal{T}^2_{Ty} \vdash_{Ty} I^1_{Ty}(A) :\equiv_{Ty} I^2_{Ty}(B)$, and
* $\forall a \in \Lambda^{\mathcal{C}_1}_A : \quad \mathcal{T}^2_{Tm} \vdash I^1_{Tm}(a) :\equiv I^2_{Tm}(a)$.

Definition 14: The *category $S.T.T^{\to,\times,1}$ of simply typed λ-calculi (with products and a terminal type)* consists of simply typed λ-theories $\mathcal{L}$ and equivalence classes of interpretations

$$\mathrm{Hom}(\mathcal{L}_1,\mathcal{L}_2) := \underline{\mathrm{Int}}(\mathcal{L}_1,\mathcal{L}_2)/\sim$$

together with the obvious compositions and identities.

Rather than listing examples and properties of s.t. λ-theories, we proceed to show the following theorem, which reduces the understanding of $S.T.T^{\to,\times,1}$ to a well-known category with a plethora of examples.

Construction 15: (The term model functor)

Given a s.t. λ-theory $\mathcal{L} = (\Pi, C, \mathcal{T}_{Ty}, \mathcal{T}_{Tm})$, let

$\mathrm{Ob}(C(\mathcal{L})) := Ty^{\Pi}/\mathcal{T}_{Ty}$, $\quad$ $A \sim B$ iff $\mathcal{T}_{Ty} \vdash A :\equiv_{Ty} B$ $\quad$ We'll drop "C" if it's clear from context.

$\mathrm{Hom}(A,B) := \Lambda^{C,0}_{A\to B}/\mathcal{T}_{Tm}$, where $\Lambda^{C,0}_{A}$ denotes the set of closed terms of type A.

Let $1_A := \lambda x.x \in \mathrm{Hom}(A,A)$, and,

given $f : A \to B$, $g : B \to C$, let $g \circ f := \lambda x.\,\mathrm{App}(g, \mathrm{App}(f,x)) : A \to C$.

Then $C(\mathcal{L}) = (\mathrm{Ob}(C(\mathcal{L})), \mathrm{Hom}, 1, \circ)$ is a category. Furthermore, $C(\mathcal{L})$ canonically comes equipped with

* a *terminal object* $1 \in C(\mathcal{L})$, as for all $A \in C(\mathcal{L})$, $\Lambda^{C,0}_{A\to 1}/\mathcal{T}_{Tm} \overset{!}{=} \{\lambda x.*\}$:

Given $f : A \to 1$, we have $\Gamma_{Th} \vdash f :\equiv \lambda x. \mathrm{App}(f, x)$ by η-congruence

In particular, $\mathrm{App}(f, x) : 1$, and hence $\Gamma_{Th} \vdash \mathrm{App}(f,x) :\equiv *$ by 1-uniqueness. Thus, $\Gamma_{Th} \vdash \lambda x. \mathrm{App}(f,x) :\equiv \lambda x. *$ by λ-congruence.

$\Rightarrow \Gamma_{Th} \vdash f :\equiv \lambda x. *$ by transitivity.

* exponentials $B^A := A \to B$ and products $A \times B$ for $A, B \in C(L)$ via

- $\pi_1 := \lambda x. \mathrm{pr}_1(x) : A \times B \to A$, $\pi_2 := \lambda x. \mathrm{pr}_2(x) : A \times B \to B$, s.th.

for $f : C \to A$, $g : C \to B$ we get

$$(f, g) := \lambda x. [\mathrm{App}(f,x), \mathrm{App}(g,x)] : C \to A \times B$$

s.th. $\pi_1 \circ (f, g) = f$, $\pi_2 \circ (f, g) = g$.

- $ev_{A,B} := \lambda x. \mathrm{App}(\mathrm{pr}_2(x), \mathrm{pr}_1(x)) : A \times B^A \to A$, s.th. for $f : A \times C \to B$ we get (a unique)

$$\ulcorner f \urcorner := \lambda c. \lambda a. \mathrm{App}(f, [c, a]) : C \to B^A$$

s.th. $ev_{A,B} \circ (\ulcorner f \urcorner, 1_C) = f$ in $\mathrm{Hom}(A \times C, B)$.

Exercise: Verify that the given data makes $C(L)$ a cartesian closed category.

Furthermore, given an interpretation $I : L_1 \to L_2$, we obtain a functor

$$C(I) : C(L_1) \to C(L_2)$$

via $\quad C(I)_0 : T_y^{\pi_1} / \Gamma^1_{Ty} \to T^{\pi_2} / \Gamma^2_{Ty}$

$$[A] \mapsto [I_{Ty}(A)] \quad ,$$

$$C(I)(A,B) : \Lambda^{\circ}_{A \to B} / \Uparrow^{1}_{T_m} \to \Lambda^{\circ}_{I_{T_y}(A) \to I_{T_y}(B)} / \Uparrow^{2}_{T_m}$$
$$f \mapsto [I_{T_m}(f)].$$

Indeed, $C(I)(A,A)(1_A) = I_{T_m}(\lambda x.x) = \lambda x.x$ for all suitably typed variables, similarly for compositions.

Let **Cart** be the category of cartesian closed categories and cartesian functors (Definition II.17).

Proposition 16: $C(I) : C(\mathcal{L}_1) \to C(\mathcal{L}_2)$ is a cartesian closed functor. We obtain a fully fully faithful term-model functor $C : S.T.T.^{\to,\times,1} \longrightarrow \text{Cart}$.

Proof: We are left to show fully faithfulness, thus let $\mathcal{L}_1, \mathcal{L}_2 \in S.T.T.^{\to,\times,1}$, $I_1, I_2 : \mathcal{L}_1 \to \mathcal{L}_2$ be two interpretations such that $C(I_1) = C(I_2)$.

By construction it follows that

$$\forall A \in T_y^{\mathcal{L}_1} : \; [I_{1,T_y}(A)] = [I_{2,T_y}(A)] \text{ in } T_y^{\mathcal{L}_2} / \Uparrow^{2}_{T_y}, \text{ i.e.}$$
$$\forall A \in T_y^{\mathcal{L}_1}, \; \Uparrow^{2}_{T_y} \vdash I_{1,T_y}(A) :\equiv I_{2,T_y}(A).$$

Second, for $a \in \Lambda^{C_2}_{A}$, $\{x_1, \ldots, x_n\} = FV(a)$, $x_i \in Var_{A_i}$, we get $\lambda x_1 \ldots \lambda x_n . a \in C(\mathcal{L}_1)(A_1, A^{A_n^{\cdot^{\cdot^{A_2}}}})$ and hence

$$[I_{1,T_m}(\lambda x_1 \ldots \lambda x_n . a)] = [I_{2,T_m}(\lambda x_1 \ldots \lambda x_n . a)] \text{ in}$$
$$C(\mathcal{L}_2)(I_{1,T_y}A_1, I_{1,T_y}A^{\cdot^{\cdot^{I_{1,T_y}A_2}}}) = \Lambda^{C_2,0}_{I_{1,T_m}A_2 \to I_{1,T_m}A^{I_{1,T_m}A_n^{\cdot^{\cdot^{I_{1,T_m}A_2}}}}} / \Uparrow^{2}_{T_m}.$$

$$\Rightarrow \quad \mathcal{T}^2_{T_m} \vdash \lambda x_1.\ldots\lambda x_n.\, I_{1,T_m}\, a :\equiv_{T_m} \lambda x_1.\ldots\lambda x_n.\, I_{2,T_m}\, a$$

acc'ly typed

$$\Rightarrow \quad \mathcal{T}^2_{T_m} \vdash I_{1,T_m}(a) :\equiv I_{2,T_m}(a) \quad \text{by } \beta\text{-congruence.}$$

Hence, we get that $I_1 \sim I_2$, and so C is faithful.

Towards fullness, let $f: C(\mathcal{L}_1) \to C(\mathcal{L}_2)$ be a cartesian functor, and define $I^f: \mathcal{L}_1 \to \mathcal{L}_2$ as follows:

$$I^f_{Ty}: \mathbb{T}_1 \longrightarrow Ty^{\pi_2}$$
$$T \longmapsto f_0(T)$$

Again, variables accordingly typed.

$$I^f_{Tm}: \coprod_{A : Ty^{\pi_1}} \mathcal{C}_1^A \longrightarrow \coprod_{A : Ty^{\pi_2} / \mathcal{T}^2_{Tm}} \Lambda_A^{\mathcal{C}_2}$$

$$(A, a) \longmapsto \underbrace{\left[App(\ldots App(f_1(\lambda x_1.\ldots\lambda x_n.a), x_1), \ldots), x_n) \right]}$$

for $a \in \Lambda_A^{\mathcal{C}_1}$, $FV(a) = \{x_1, \ldots, x_n\}$, $x_i \in Var_{A_i}$.

It is straight-forward to verify that $I^f: \mathcal{L}_1 \to \mathcal{L}_2$ is an interpretation s.th.

$$f = C(I^f): C(\mathcal{L}_1) \to C(\mathcal{L}_2)$$

$$\begin{array}{ccc} A & \mapsto & f_0(A) \\ g \downarrow & & \downarrow f_1(g) \\ B & \mapsto & f_0(B) \end{array}$$

$\square$

Theorem 17 (Lambek): The functor $C: S.T.T.^{\to,\times,1} \to Cart$ is part of an equivalence of categories. Its inverse is the "internal language" functor.

Proof: Given $\mathcal{C} \in \mathrm{Cart}$, we define the typed λ-theory

$T(\mathcal{C}) := (\mathrm{Ob}(\mathcal{C}), \underbrace{\coprod_{C:\mathrm{Ob}(\mathcal{C})} \mathcal{C}(C)}_{=:\,\mathcal{C}(\mathcal{C})}, T_{\mathcal{C}_0}, T_{\mathcal{C}_1})$ as follows. First, let

$$T_{\mathcal{C}_0} := \{ 1 :\equiv_{Ty} 1_{\mathcal{C}} \} \cup \{ A \times B :\equiv_{Ty} \overbrace{A \times_{\mathcal{C}} B}^{\in \mathrm{Ob}(\mathcal{C})} \mid A, B \in \mathrm{Ob}(\mathcal{C}) \} \cup \{ A \to B :\equiv_{Ty} \overbrace{B^A}^{\in \mathrm{Ob}(\mathcal{C})} \mid A, B \in \mathrm{Ob}(\mathcal{C}) \}.$$

To define $T_{\mathcal{C}_1}$, let $\varepsilon_0 : Ty^{\mathrm{Ob}(\mathcal{C})} \twoheadrightarrow \mathrm{Ob}(\mathcal{C})$ be recursively defined by

$A \in \mathrm{Ob}(\mathcal{C}) \mapsto A$, $A \times B \mapsto A \times B$, $A \to B \mapsto B^A$, $1 \mapsto 1$. So,

$T_{Ty} \vdash A :\equiv_{Ty} B$ iff $\varepsilon_0(A) = \varepsilon_0(B)$ in $\mathcal{C}$ (by induction).

Again by recursion we define a map $\varepsilon_1 : \coprod_{A : Ty^{\mathrm{Ob}(\mathcal{C})}/T_{\mathcal{C}_0}} \Lambda_A^{\mathcal{C}(\mathcal{C})} \to \mathcal{C}$ s.th.

whenever $t : A$ has $FV(t) = \{x_1, \dots, x_n\}$ with $x_i \in \mathrm{Var}_{A_i}$, then

$\varepsilon_1(t) : A_1 \times \dots \times A_n \to A$ as follows.

* For $x \in \mathrm{Var}_A$, let $\varepsilon_1(x) := 1_A : A \to A$
* For $C \in \mathrm{Ob}(\mathcal{C})$, $c \in \mathcal{C}(\mathcal{C})$, let $\varepsilon_1(c) = c : 1 \to C$.
* For $b : B$ with $FV(b) = \{x_1, \dots, x_n\}$, $x_i \in \mathrm{Var}_{A_i}$, $x \in \mathrm{Var}_A$ for some $A \in Ty^{\mathrm{Ob}(\mathcal{C})}$.

$$\varepsilon_1(\lambda x . b) := \begin{cases} \varepsilon_0(A_1) \times \dots \times \varepsilon_0(A_{\hat{i}}) \times \dots \times \varepsilon_0(A_n) \xrightarrow{\ulcorner b \urcorner} \varepsilon_0(B)^{\varepsilon_0(A_i)}, & \text{if } x = x_i, \\ \varepsilon_0(A_1) \times \dots \times \varepsilon_0(A_n) \xrightarrow{\varepsilon_1(b)} \varepsilon_0(B) \xrightarrow{\ulcorner \pi_1 \urcorner} \varepsilon_0(B)^{\varepsilon_0(A)} & \text{if } x \notin FV(b). \end{cases}$$

* For $f : A \to B$ with $FV(f) = \{x_1, \dots, x_n\}$, $x_i \in \mathrm{Var}_{A_i}$, $a : A$, $FV(a) = \{y_1, \dots, y_m\}$, $y_i \in \mathrm{Var}_{C_i}$.

and $\varepsilon_1 f : \varepsilon_0 A_1 \times \dots \times \varepsilon_0 A_n \to \varepsilon_0(A \to B) = \varepsilon_0 B^{\varepsilon_0 A}$, $\varepsilon_1 a : \varepsilon_0 C_1 \times \dots \times \varepsilon_0 C_m \to \varepsilon_0 A$ given,

let $FV(\mathrm{App}(f, a)) = \{z_1, \dots, z_k\}$, $z_i \in \mathrm{Var}_{D_i}$ for $\max(m, n) \leq k \leq m+n$. The projection

$\{x_1, \dots, x_n\} \sqcup \{y_1, \dots, y_m\} \twoheadrightarrow \{z_1, \dots, z_k\}$ induces

$\varepsilon_1(\mathrm{App}(f,a)) : \varepsilon_0 D_1 \times \_ \times \varepsilon_0 D_k \xrightarrow{\Delta} \varepsilon_0 A_1 \times \_ \times \varepsilon_0 A_n \times \varepsilon_0 C_1 \times \_ \times \varepsilon_0 C_m \xrightarrow{\varepsilon_1 f \times \varepsilon_1 a} \varepsilon_0 B^{\varepsilon_0 A} \times \varepsilon_0 A \xrightarrow{ev} \varepsilon_0 B$

* for $c : B_1 \times B_2$ with $FV(c) = \langle x_1, \_, x_n \rangle$, $x_i \in \mathrm{Var}\, A_i$, and

$\varepsilon_1(c) : \varepsilon_0 A_1 \times \_ \times \varepsilon_0 A_n \to \varepsilon_0(B_1 \times B_2) = \varepsilon_0 B_1 \times \varepsilon_0 B_2$, let

$\varepsilon_1(\mathrm{pr}_i(c)) : \varepsilon_0 A_1 \times \_ \times \varepsilon_0 A_n \xrightarrow{c} \varepsilon_0 B_1 \times \varepsilon_0 B_2 \xrightarrow{\pi_i} B_i$.

* For $* : 1$, let $\varepsilon_1(*) = id_1 : 1 \to 1$.

This finishes the recursion, as Term-Conversion is clearly satisfied: $A :\equiv B \in T_{\mathcal{E}_0}$, $a : A$ implies $\mathrm{codom}(\varepsilon_1(a)) = \varepsilon_0(A) = \varepsilon_0(B)$.

We define

$$T_{\mathcal{E}_1} := \left\{ s :\equiv_{T_n} t \;\middle|\; \begin{tikzcd} & \varepsilon_0 A_1 \times \_ \times \varepsilon_0 A_n \arrow[dr, "\varepsilon_1(s)"] & \\ \varepsilon_0 C_1 \times \_ \times \varepsilon_0 C_k \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, phantom, "\circlearrowleft"] & & \varepsilon_0 A \\ & \varepsilon_0 B_1 \times \_ \times \varepsilon_0 B_m \arrow[ur, "\varepsilon_1(t)"'] & \end{tikzcd} \right\}$$

for $s, t : A$, $FV(s) = \langle x_1, \_, x_n \rangle$, $x_i \in \mathrm{Var}\, A_i$, $FV(t) = \langle y_1, \_, y_m \rangle$, $y_i \in \mathrm{Var}\, B_i$, $FV(s) \cup FV(t) = \langle z_1, \_, z_k \rangle$.

Claim: 1. $\varepsilon_{\mathcal{C}} : C(T(\mathcal{C})) \to \mathcal{C}$ is a cartesian closed functor.

$$A \mapsto \varepsilon_0 A$$

$$f \in \Lambda^{c(\mathcal{C}),0}_{A \to B} \mapsto \ulcorner \varepsilon_1(f) \urcorner^{-1} : \varepsilon_0 A \to \varepsilon_0 B$$

2. $\varepsilon_{\mathcal{C}}$ is a isomorphism of categories.

If the claim is true, then the functor $C : S.T.T.^{\to,\times,1} \to \mathrm{Cart}$ is essentially surjective. In Proposition 16 we have seen that C is fully faithful. We are thus left to show the claim.