Continuation of the proof of Theorem 17: We were left to show the

Claim: $\varepsilon : C(T(\mathcal{C})) \to \mathcal{C}$ is an isomorphism in Cart.

$\ulcorner$ The function $\varepsilon_0 : Ob(C(T(\mathcal{C}))) = T_Y^{Ob(\mathcal{C})}/\mathcal{T}_{\mathcal{C}_0} \to Ob(\mathcal{C})$ is a bijection as $\mathcal{T}_{\mathcal{C}_0} \vdash A :\equiv_{T_Y} B$ iff $\varepsilon_0 A = \varepsilon_0 B$. For $A, B \in T_Y^{Ob(\mathcal{C})}/\mathcal{T}_{\mathcal{C}_0}$, we have

$$\mathrm{Hom}(A,B) = \Lambda^{Ob(\mathcal{C}), \mathcal{C}}_{A \to B} \xrightarrow{\varepsilon(A,B)} \mathcal{C}(A,B)$$

$$f \mapsto \ulcorner \varepsilon_1(f) \urcorner^{-1},$$

since $f : A \to B :\equiv \underset{\sim}{B^A}$ ($\in Ob(\mathcal{C})$), so $f \in \ulcorner (B^A)$

with

- $\varepsilon(A,A)(1_A) = \varepsilon(A,A)(\lambda x.x) = \ulcorner \ulcorner id_A \urcorner \urcorner^{-1} = id_A$,
- $\varepsilon(A,C)(g \circ f) = \varepsilon(A,C)(\lambda x.\, App(g, App(f,x))) = \dots = \varepsilon(B,C)(g) \circ \varepsilon(A,B)(f)$.

Furthermore, given $f \in \mathcal{C}(A,B)$, we have $\ulcorner f \urcorner \in \Lambda^{\circ}_{B^A}$ in $T(\mathcal{C})$, and $(B^A :\equiv_{T_S} (A \to B))$ is in $\mathcal{T}_{\mathcal{C}_0}$, so $\ulcorner f \urcorner \in \Lambda^{\circ}_{A \to B}$. We get $\varepsilon(A,B)(\ulcorner f \urcorner) = \ulcorner \varepsilon_1(\ulcorner f \urcorner) \urcorner^{-1} = \ulcorner \ulcorner f \urcorner \urcorner^{-1} = f$.

Given $f, g \in \Lambda^{\circ}_{A \to B}$ s.th. $\varepsilon(A,B)(f) = \varepsilon(A,B)(g)$, we get $\varepsilon_1(f) = \varepsilon_1(g)$ and hence $\mathcal{T}_{\mathcal{C}_1} \vdash f :\equiv_{T_m} g$. Thus, $f = g$ in $\mathrm{Hom}(A,B)$.

Thus, we have shown that $\varepsilon : C(T(\mathcal{C})) \to \mathcal{C}$ is an isomorphism of categories. It preserves the cartesian structure on the nose by construction.

$\lrcorner$

Remark 18: 1. To define ε_1 in the proof of Theorem 17, we have to take care of the fact that $FV(s)$ for terms s is an unordered set, while products in $\mathcal{C}$ are only unordered up to isomorphism. We thus have to choose an order of components $\varepsilon\circ A_1\times\dots\times\varepsilon\circ A_n$ for $FV(s)=\{x_1,\dots,x_n\}$, $x_i\in Var_{A_i}$. We can do so by enumerating the set $\coprod_{A:T_\gamma^{\mathcal{L}}} Var_A$ globally, and implicitly ordering each $FV(s)$ accordingly.

2. Lambek's over-determined formulas of the form $(X, s:=t)$ with $X=\{x_1,\dots,x_k\}\supseteq FV(s)\cup FV(t)=\{x_{i_1},\dots,x_{i_\ell}\}$, $x_i\in Var_{D_i}$, are interpreted by commutativity of the precomposition

$$\begin{tikzcd} & & A_1\times\dots\times A_n \arrow[dr, "s"] & \\ D_1\times\dots\times D_k \arrow[r, "\pi"] & D_{i_1}\times\dots\times D_{i_\ell} \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, phantom, "\circlearrowleft"] & & C \\ & & B_1\times\dots\times B_m \arrow[ur, "t"'] & \end{tikzcd}$$

Corollary 19: Under the equivalence of Theorem 17, the <u>pure</u> simply typed λ-theory $(\emptyset,\emptyset,\emptyset,\emptyset)\in S.T.T.^{\to,\times,1}$ corresponds to the initial cartesian closed category $F(\emptyset)$, freely generated over the empty set.

Remark 19: - Combining the results of Chapter II and Theorem 17, we see that the category λ-Calc of untyped λ-calculi is a subcategory of $S.T.T.^{\to,\times,1}$. In fact, given an untyped λ-theory $\mathcal{L}=(C,T)$, we obtain $\mathcal{L}^+:=T(\mu^{op}(C(\mathcal{L},T)))\in S.T.T.^{\to,\times,1}$ as conservative extension:

$T\vdash s:=t$ iff $[s]=[t]\in C(\mathcal{L})$ iff $\underbrace{[\![k]\!]^\circ[s]}_{=[\lambda s.s]}=\underbrace{[\![k]\!]^\circ[t]}_{=[\lambda s.t]}$ iff

$[\![k]\!]^\circ[s]=[\![k]\!]^\circ[t]$ in $\Gamma(\mathbb{I})$ for $\mathbb{I}\in\mu^{op}(\mathcal{L})$ the ref. object iff

$\mathcal{L}^+\vdash[\![k]\!]^\circ[s]:=[\![k]\!]^\circ[t]$ in $\Lambda^\circ_{\mathbb{I}}$ (this does not automatically imply $\mathcal{L}\vdash\lambda\beta\eta$).

Note that $\mathcal{L}^+$ does have many types, but that they're all retracts of the "λ-term universe" $\mathbb{I}$. Clearly not every $\mathcal{C} \in \mathrm{Cart}$ has a reflexive object, so λ-Calc is a proper subcategory of $S.T.T.^{\to,\times,1}$.

There is a plethora of examples of cartesian closed categories, and most have additional categorical structure not captured by the fragment $S.T.T.^{\to,\times,1}$. Some such structure can be freely added to the definition of simply typed λ-theories, e.g.

- The natural numbers type:

$$\frac{}{\mathbb{N}} \quad (\mathbb{N}\text{-Formation})$$

$$\frac{}{0 : \mathbb{N}} \; , \; \frac{n : \mathbb{N}}{\mathrm{succ}(n) : \mathbb{N}} \quad (\mathbb{N}\text{-Introduction})$$

$$\frac{a : A \quad f : A \to A \quad n : \mathbb{N}}{I(a, f, n) : A} \quad (\mathbb{N}\text{-Elimination})$$

$$\frac{}{\Gamma \vdash I(a,f,0) := a} \; , \; \frac{}{\Gamma \vdash I(a,f,\mathrm{succ}(x)) := \mathrm{App}(f, I(a,f,x))} \quad \text{for } x \in \mathrm{Var}_{\mathbb{N}} \setminus (FV(a) \cup FV(f))$$
(ℕ-Computation)

+ acc. congruence rules, see (LS86, I.10.6).

- The type of internal predicates:

$$\frac{}{\Omega} \quad (\Omega\text{-Formation})$$

$$\frac{}{\bot : \Omega} \; , \; \frac{}{\top : \Omega} \; , \; \frac{}{\bot : \Omega} \; , \; \frac{}{\top : \Omega} \; , \; \frac{p : \Omega \quad q : \Omega}{p \wedge q : \Omega} \; , \; \frac{p : \Omega \quad q : \Omega}{p \vee q : \Omega} \; , \; \frac{p : \Omega \quad q : \Omega}{p \Rightarrow q : \Omega} \; ,$$

$$\frac{\varphi : \Omega}{\forall x{:}A.\varphi : \Omega} \text{ for all } x \in \mathrm{Var}_A, \quad \frac{\varphi : \Omega}{\exists x{:}A.\varphi : \Omega} \text{ for all } x \in \mathrm{Var}_A \quad (\Omega\text{-Introduction})$$

+ Elimination, Computation & Congruence rules, see (LS86, II.1).

One obtains extensions of Theorem 17 as follows.

Theorem: The equivalence from Theorem 17 lifts to an equivalence

$$\mathrm{S.T.T}^{\to, \times, 1, \mathbb{N}} \underset{C}{\overset{T}{\rightleftarrows}} \mathrm{Cart}_{\mathbb{N}},$$

where $\mathrm{Cart}_{\mathbb{N}}$ denotes the category of cartesian categories with weak natural numbers object. $\square$

Issue: $\mathbb{N}$ has no evident uniqueness rule; $\dfrac{}{(\Gamma \vdash u :\equiv 0 \text{ or } \Gamma \vdash u :\equiv \mathrm{succ}(m)) \text{ for some } m}$? f.a. $u \in \mathbb{N}$

In fact, $\mathbb{N}$-Uniqueness, or "canonicity", can be very hard to prove (if at all) depending on the given type theory.

Adding rules freestyle, such as

$$\frac{\Gamma \vdash \mathrm{App}(g, c) :\equiv a, \quad \Gamma \vdash \mathrm{App}(g, \mathrm{succ}(x)) :\equiv \mathrm{App}(f, I(a, fx))}{\Gamma \vdash g :\equiv \lambda x. I(a, fx)} \quad \text{for all } a : A,\ f : A \to A,\ g : \mathbb{N} \to A,\ x \in \mathrm{Var}_{\mathbb{N}} \setminus FV(\dots)$$

is generally avoided, because it can lead to unintended consequences in the (operational) semantics (viz. univalence).

Theorem: There is a 2-categorical adjunction

$$\mathrm{S.T.T.}^{\to,\times,1,\mathbb{N},\Omega} \underset{C}{\overset{T}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} \mathrm{Top},$$

where Top is the 2-category of elementary toposes (i.e. cartesian closed categories with subobject-classifier and natural numbers object) and logical functors. The counit $\varepsilon: C \circ T \to 1$ is a pointwise equivalence of categories, and the unit $\eta: 1 \to T \circ C$ is a pointwise "conservative" extension. See (LS86, II.13 + 14).

$\square$

Warning: The functor $C: \mathrm{S.T.T.}^{\dots} \to \mathrm{Top}$ is not a straight-forward lift of C in case of Cart and CartN. Instead, it is in spirit very much related to the interpretation of 1st order intuitionistic theories via the internal logic provided by Ω.

Proposition: Every cartesian closed category $\mathcal{C}$ can be fully faithfully embedded into an el. topos via the Yoneda embedding $\mathcal{C} \xrightarrow{\mathcal{Y}} \hat{\mathcal{C}}$. It follows that every $\mathcal{L} \in \mathrm{S.T.T.}^{\to,\times,1}$ can be conservatively extended to some $\mathcal{L}^+ \in \mathrm{S.T.T.}^{\to,\times,1,\mathbb{N},\Omega}$.

$\square$

Even in the case of $S.T.T.^{\times, \to, 1}$, there are at least two ways to think of predicates in a λ-theory:

1. Types are sorts, predicates are λ-terms (of type Ω if present)

   We obtain a higher order theory, as we can quantify over families of predicates arbitrarily often. For instance, recall that

   - $(\mathbb{N}, +, \cdot, 0, 1)$ is the 1st order theory of arithmetics, and as such the study of naturals, primes, etc.
   - $(\mathbb{N}, \mathbb{N} \to \mathbb{N}, +, \cdot, 0, 1)$ is the 2nd order " " "

     the study of sets of naturals, in particular the rationals. Can study functions of rationals, continuous functions of reals, open sets of reals, ...
   - $(\mathbb{N}, \mathbb{N} \to \mathbb{N}, \mathbb{N} \to (\mathbb{N} \to \mathbb{N}), +, \cdot, 0, 1)$ can express statements about sets of reals, and hence address e.g. CH etc.

   This POV is adapted in the last theorem.

2. The Curry-Howard-Lambek Correspondence:

   Types are predicates, λ-terms are their proofs!

   We thus obtain a proof-relevant propositional theory.

1st order prop'l calc.	S.T.T
$\varphi \wedge \psi$	$\varphi \times \psi$
$\varphi \Rightarrow \psi$	$\varphi \to \psi$
$\top$	1

Gentzen's rules of natural deduction furthermore translate exactly to the logical rules of the simply typed λ-calculus with products and a terminal type, e.g.

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \quad \rightsquigarrow \quad \frac{a_1 : A_1 \quad a_2 : A_2}{[a_1, a_2] : A_1 \times A_2} \; (\times\text{-Introd.})$$

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \alpha} \quad \rightsquigarrow \quad \frac{c : A_1 \times A_2}{\pi_i c : A_i} \; (\times\text{-Elim.})$$

etc. To construct a proof on the LHS is to construct a term of acc. type on the LHS. If one takes care of the due formalities with the syntactic book-keeping, then a term itself represents the proof without loss of information, because the steps of construction can be read off its length and the order of operators in it.

Dependently typed λ-theories allow for the definition of type forming operations which extend the CHL-Correspondence to quantifiers as well.

CHAPTER IV: DEPENDENT TYPE THEORIES

§1: Syntax

Idea: Introduce types dependent on term variables and according quantifiers to formalize families of types indexed by other types. We will think of types both as sorts/set-like structures/spaces and as propositions where terms represent their proofs.

Technical remarks: 1. We will introduce Martin-Löf type theory with dependent sums, dependent products, identitity types, a terminal and an initial type, and Tarski-universes. Given the context of what we've done so far and where we're headed, this choice appears the most reasonable.

2. In reference to the technical remarks in Chapter III, we will present the type theories in the Curry-de Bruijn version. That means, we work with a single set of variables and introduce well-formed types and their well-formed terms in context of local variable declarations. We thus set up the whole theory by way of a calculus for pretypes and preterms, and then introduce a calculus on top for contexts, types and terms.

The second of these calculi cannot be split as neatly into a calculus of contexts followed by a calculus of types followed by a calculus of terms, because all three notions depend on each other. They therefore will be introduced by one simultaneous recursion.

We will introduce the language implicitly along the way, rather than to dump a page of decontextualized symbols at the start.

3. One can set this up in the Church version, see e.g. Martin-Löf "An intuitionistic theory of types". This is rather inconvenient however and leads to technical difficulties which can be avoided.

For the rest of this section let Var be a countable set of variables, $\mathbb{T}$ a set of constant type symbols, $\mathbb{C}$ a set of constant term symbols s.th. $\mathbb{T} \cap \mathbb{C} = \emptyset$.

Definition 1: Let Pretype be the product of the following calculus.

$\dfrac{}{T}$ for $T \in \mathbb{T}$, $\dfrac{}{0}$, $\dfrac{}{1}$, $\dfrac{A \quad B}{\prod_{x:A} B}$ for $x \in Var$, $\dfrac{A \quad B}{\sum_{x:A} B}$ for $x \in Var$,

$\dfrac{A}{a =_A b}$ for $a, b \in$ Preterm, $\dfrac{}{\mathcal{U}_i}$ for $i \in \mathbb{N}$, $\dfrac{}{El(s)}$ for $s \in$ Preterm.

Let Preterm be the product of the following calculus.

$\dfrac{}{c}$ for $c \in \mathcal{C}$, $\dfrac{}{x}$ for $x \in Var$, $\dfrac{}{*}$, $\dfrac{s \quad t \quad u}{ind_0(s,t,u)}$, $\dfrac{s \quad t \quad u}{ind_1(s,t,u)}$,

$\dfrac{t}{\lambda(x:A).t}$ for $x \in Var$, $A \in Pretype$, $\dfrac{s \quad t}{App(s,t)}$, $\dfrac{s \quad t}{[s,t]}$, $\dfrac{s \quad t \quad u}{ind_{\Sigma_{x:A} B}(s,t,u)}$ for $A,B \in Pretype$, $x \in Var$,

$\dfrac{s}{refl_s}$, $\dfrac{s \quad t \quad u \quad v \quad w}{ind_{=A}(s,t,u,v,w)}$ for $A \in Pretype$, $\dfrac{}{0}$, $\dfrac{}{\bullet}$, $\dfrac{}{t(T)}$ for $T \in \mathcal{T}$

$\dfrac{s \quad t}{\pi_i(s,t)}$, $\dfrac{s \quad t}{\sigma_i(s,t)}$, $\dfrac{a \quad b \quad c}{id_a(b,c)}$, $\dfrac{}{u_i}$ for $0 \leq i < \infty$, $\dfrac{a}{t_i(a)}$ for $0 \leq n < \infty$, $(\ldots)$

A type assignment is of the form $a:A$ where $a \in Preterm$ and $A: Pretype$. Here, a is said to be the subject of $a:A$.

A declaration is a type assignment with a variable for subject.

A precontext is a finite sequence of declarations of pairwise distinct variables. I.e. Precontext is the product of the following calculus on the language $(Preterm \sqcup Pretype \sqcup \{, ;\})^{\infty}$:

$\dfrac{}{\emptyset}$ $\dfrac{\Gamma}{\Gamma, x:A}$ for $A \in Pretype$, $x \in Var$ which is not subject of Γ.

We will think of precontexts as consistent sequences of variable declarations.

Remark 2: λ-abstractions carry the type-assignment explicitly bc. variables have no intrinsic type-assignment. Otherwise, for instance,

$x:A, y:A \to B \vdash App(y,x):B$ would imply "polymorphism":

$x:A \vdash \lambda y. App(y,x):(A \to B) \to B$ f.a. types B.

Definition 3: The set of free variables is defined recursively for both pretypes and preterms in habitual straight-forward fashion. Here, the only variable-binding operations are λ, Σ and Π: i.e.

- $FV(x) = \{x\}$ f.a. $x \in Var$
- $FV(\top) = FV(c) = FV(0) = FV(1) = FV(*) = FV(\circ_i) = FV(\bullet_i)$
  $= FV(u_i) = FV(\iota_i) = \emptyset$
- $FV(\lambda(x:A).\, t) = FV(A) \cup FV(t) \setminus \{x\}$
- $FV(\sum_{x:A} B) = FV(\prod_{x:A} B) = FV(A) \cup FV(B) \setminus \{x\}$

- All other operations are evaluated at the obvious unions of sets of free variables (here, note that indices are formal inputs as well).