Elements of monoidal topology*
Lecture 1: $(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors

Sergejs Solovjovs

Department of Mathematics, Faculty of Engineering, Czech University of Life Sciences Prague (CZU)
Kamýcká 129, 16500 Prague - Suchdol, Czech Republic

Abstract

This lecture introduces monoidal topology in the form of $(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors for a monad $\mathbb{T}$ on **Set** and a unital quantale V, and shows that the examples of these structures include preordered sets, as well as quasi-pseudo-metric, topological, approach, and closure spaces, together with their respective maps.

1. Categorical preliminaries

1.1. Monads and their algebras

Definition 1. A *category* $\mathbf{X}$ is a sextuple $(\mathcal{O}, \mathcal{M}, dom, cod, 1, \cdot)$, where $\mathcal{O}$ is a class of *objects* (denoted X, Y, Z, *etc.*), $\mathcal{M}$ is a class of *morphisms* (denoted f, g, h, *etc.*), $\mathcal{M} \overset{dom}{\underset{cod}{\rightrightarrows}} \mathcal{O}$ (*domain* and *codomain*) and $\mathcal{O} \xrightarrow{1} \mathcal{M}$ (*identity morphisms* denoted $1_X, 1_Y, 1_Z$, *etc.*) are maps, and $\cdot$ is a partial binary operation on $\mathcal{M}$ (*composition*) such that $g \cdot f$ is defined iff $cod\, f = dom\, g$. Given $X, Y \in \mathcal{O}$ and $f \in \mathcal{M}$, one uses the notation $X \xrightarrow{f} Y$ as a shorthand for "$dom\, f = X$ and $cod\, f = Y$". Additionally, one assumes the following axioms:

(1) for every object X, $X \xrightarrow{1_X} X$;

(2) for every objects X and Y, the family $\mathbf{X}(X, Y) := \{f \in \mathcal{M} \mid X \xrightarrow{f} Y\}$ is a set (*hom-set*);

(3) $h \cdot (g \cdot f) = (h \cdot g) \cdot f$ for every morphisms $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$ and $Z \xrightarrow{h} W$;

(4) $1_Y \cdot f = f = f \cdot 1_X$ for every morphism $X \xrightarrow{f} Y$. ∎

Example 2. There exist the categories **Set** of sets and maps, **Top** of topological spaces and continuous maps, **Met** of metric spaces and non-expansive maps, **Pos** of partially ordered sets and monotone maps. ∎

Definition 3. For every category $\mathbf{X} = (\mathcal{O}, \mathcal{M}, dom, cod, 1, \cdot)$, there exists the *opposite* (or *dual*) *category* of $\mathbf{X}$, namely, the category $\mathbf{X}^{op} = (\mathcal{O}, \mathcal{M}, cod, dom, 1, *)$, in which $*$ is defined by $f * g = g \cdot f$. In other words, $\mathbf{X}$ and $\mathbf{X}^{op}$ have the same objects and morphisms, whereas the domain and the codomain maps are switched, and the composition laws are the "opposites" of each other. ∎

*This lecture course was supported by the ESF Project No. CZ.1.07/2.3.00/20.0051 "Algebraic methods in Quantum Logic" of the Masaryk University in Brno, Czech Republic.

european social fund in the czech republic — EUROPEAN UNION — MINISTRY OF EDUCATION, YOUTH AND SPORTS — OP Education for Competitiveness
INVESTMENTS IN EDUCATION DEVELOPMENT

Email address: `solovjovs@tf.czu.cz` (Sergejs Solovjovs)
URL: `http://home.czu.cz/solovjovs` (Sergejs Solovjovs)

Proposition 4. *For every category* $\mathbf{X}$*,* $(\mathbf{X}^{op})^{op} = \mathbf{X}$*.*

Definition 5. A *functor* F from a category $\mathbf{X}$ to a category $\mathbf{Y}$ is a pair of maps $\mathcal{O}_{\mathbf{X}} \xrightarrow{F_{\mathcal{O}}} \mathcal{O}_{\mathbf{Y}}$ and $\mathcal{M}_{\mathbf{X}} \xrightarrow{F_{\mathcal{M}}} \mathcal{M}_{\mathbf{Y}}$ (both denoted F), which satisfy the following axioms:

(1) $F(X \xrightarrow{f} Y) = FX \xrightarrow{Ff} FY$ for every $\mathbf{X}$-morphism $X \xrightarrow{f} Y$;

(2) $F(g \cdot f) = Fg \cdot Ff$ for every $\mathbf{X}$-morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$;

(3) $F1_X = 1_{FX}$ for every $\mathbf{X}$-object X. ∎

Example 6.

(1) Given a category $\mathbf{X}$, there exists the identity functor $1_{\mathbf{X}}$ on $\mathbf{X}$ defined by $1_{\mathbf{X}}(X \xrightarrow{f} Y) = X \xrightarrow{f} Y$ for every $\mathbf{X}$-morphism $X \xrightarrow{f} Y$.

(2) There exist the forgetful functors $\mathbf{Top} \xrightarrow{|-|} \mathbf{Set}$, $\mathbf{Met} \xrightarrow{|-|} \mathbf{Set}$, and $\mathbf{Pos} \xrightarrow{|-|} \mathbf{Set}$, as well as the powerset functor $\mathbf{Set} \xrightarrow{P} \mathbf{Set}$ defined by $P(X \xrightarrow{f} Y) = PX \xrightarrow{Pf} PY$, where $PX = \{A \mid A \subseteq X\}$ and $Pf(A) = f(A) = \{f(x) \mid x \in A\}$ for every map $X \xrightarrow{f} Y$. ∎

Remark 7. Functors can be composed componentwise as pairs of maps. The composition of two functors $\mathbf{X} \xrightarrow{F} \mathbf{Y}$ and $\mathbf{Y} \xrightarrow{G} \mathbf{Z}$ is often written GF instead of $G \cdot F$. ∎

Definition 8. A *natural transformation* α from a functor $\mathbf{X} \xrightarrow{F} \mathbf{Y}$ to a functor $\mathbf{X} \xrightarrow{G} \mathbf{Y}$ is a map $\mathcal{O}_{\mathbf{X}} \xrightarrow{\alpha} \mathcal{M}_{\mathbf{Y}}$ such that $FX \xrightarrow{\alpha_X} GX$ for every $X \in \mathcal{O}_{\mathbf{X}}$, and, additionally, the diagram

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow[\alpha_Y]{} & GY \end{array}$$

commutes for every $\mathbf{X}$-morphism $X \xrightarrow{f} Y$. ∎

Example 9. There exists a natural transformation $1_{\mathbf{Set}} \xrightarrow{e} P$, with $X \xrightarrow{e_X} PX$ given by $e_X(x) = \{x\}$. ∎

Remark 10. Given functors and a natural transformation as in the diagram $\mathbf{W} \xrightarrow{K} \mathbf{X} \underset{G}{\overset{F}{\rightrightarrows}}{\Downarrow\alpha}\ \mathbf{Y} \xrightarrow{H} \mathbf{Z}$, there exist the following natural transformations (*whiskering by a functor* from the left or right):

(1) $HF \xrightarrow{H\alpha} HG$ given by $HFX \xrightarrow{(H\alpha)_X} HGX := HFX \xrightarrow{H(\alpha_X)} HGX$;

(2) $FK \xrightarrow{\alpha K} GK$ given by $FKW \xrightarrow{(\alpha K)_W} GKW := FKW \xrightarrow{\alpha_{KW}} GKW$. ∎

Definition 11. A *monad* $\mathbb{T}$ on a category $\mathbf{X}$ is a triple (T, m, e), where $\mathbf{X} \xrightarrow{T} \mathbf{X}$ is a functor, and $TT \xrightarrow{m} T$, $1_{\mathbf{X}} \xrightarrow{e} T$ are natural transformations, which make the diagrams

$$\begin{array}{ccc} TTT & \xrightarrow{Tm} & TT \\ {\scriptstyle mT}\downarrow & & \downarrow{\scriptstyle m} \\ TT & \xrightarrow[m]{} & T \end{array} \qquad \begin{array}{ccccc} T & \xrightarrow{eT} & TT & \xleftarrow{Te} & T \\ & {\scriptstyle 1_T}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle 1_T} & \\ & & T & & \end{array}$$

commute. ∎

Example 12. There exists the powerset monad $\mathbb{P} = (P, m, e)$ on **Set**, with $X \xrightarrow{e_X} PX$ given by $e_X(x) = \{x\}$ and $PPX \xrightarrow{m_X} PX$ given by $m_X(\mathcal{A}) = \bigcup \mathcal{A}$. ∎

Remark 13. Monads on a category $\mathbf{X}$ are precisely the monoids in the strict monoidal category $\mathbf{X}^{\mathbf{X}}$ (see Lecture 4 for more detail on monoidal categories). ∎

Definition 14. Given a monad $\mathbb{T}$ on a category $\mathbf{X}$, a $\mathbb{T}$-*algebra* (or *Eilenberg-Moore algebra*) is a pair (X, a), where X is an $\mathbf{X}$-object, and $TX \xrightarrow{a} X$ is an $\mathbf{X}$-morphism, which makes the diagrams

$$\begin{array}{ccc} TTX & \xrightarrow{m_X} & TX \\ {\scriptstyle Ta}\downarrow & & \downarrow{\scriptstyle a} \\ TX & \xrightarrow[a]{} & X \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ & {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle a} \\ & & X \end{array}$$

commute. A $\mathbb{T}$-*homomorphism* $(X, a) \xrightarrow{f} (Y, b)$ is an $\mathbf{X}$-morphism $X \xrightarrow{f} Y$, which makes the diagram

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commute. $\mathbf{X}^{\mathbb{T}}$ is the category of $\mathbb{T}$-algebras and $\mathbb{T}$-homomorphisms (the *Eilenberg-Moore category* of $\mathbb{T}$). ∎

Example 15. $\mathbf{Set}^{\mathbb{P}}$ is isomorphic to the category **Sup** of $\bigvee$-semilattices and $\bigvee$-preserving maps. ∎

1.2. Quantale-valued relations

Definition 16. A $\bigvee$-*semilattice* is a partially ordered set $(V, \leqslant)$, which has arbitrary joins (denoted $\bigvee$). ∎

Remark 17. Every $\bigvee$-semilattice $(V, \leqslant)$ is a complete lattice, in which $\bot_V := \bigvee \varnothing$ (the smallest element) and $\top_V := \bigwedge \varnothing$ (the largest element). ∎

Definition 18. A *quantale* V is a $\bigvee$-semilattice, which is equipped with an associative binary operation $V \times V \xrightarrow{\otimes} V$ (*multiplication*) such that $a \otimes (\bigvee B) = \bigvee_{b \in B} a \otimes b$ and $(\bigvee B) \otimes a = \bigvee_{b \in B} (b \otimes a)$ for every $a \in V$ and every $B \subseteq V$. A quantale V is said to be

(1) *unital* provided that its multiplication has a unit k;
(2) *commutative* provided that $a \otimes b = b \otimes a$ for every $a, b \in V$. ∎

Example 19. There exists the two-element unital quantale $2 = (\{\bot, \top\}, \wedge, \top)$. The extended real half-line $[0, \infty]$ gives a unital quantale $\mathsf{P}_+ = ([0, \infty]^{op}, +, 0)$. ∎

Remark 20. Every unital quantale is a strict monoidal closed category (see Lecture 4 for more detail on monoidal categories). ∎

Remark 21. Given a set X, there is a one-to-one correspondence between subsets of X and maps $X \to 2$. For a subset $S \subseteq X$, one defines $X \xrightarrow{\chi_S} 2$ (*characteristic map* of S) by $\chi_S(x) = \top$ iff $x \in S$, and vice versa. ∎

Definition 22. A *relation* r *from* a set X *to* a set Y is a map $X \times Y \xrightarrow{r} 2$ (denoted $X \xrightarrow{r} \mkern-14mu\mapsto Y$). Given $x \in X$ and $y \in Y$, one uses $x\, r\, y$ as a shorthand for "$r(x, y) = \top$". The *opposite* (or *dual*) of a relation $X \overset{r}{\longrightarrow\mkern-20mu+\mkern6mu} Y$ is the relation $Y \overset{r^\circ}{\longrightarrow\mkern-20mu+\mkern6mu} X$ defined by $y\, r^\circ\, x$ iff $x\, r\, y$. ∎

Definition 23. **Rel** is the category, whose objects are sets, and whose morphisms are relations $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} Y$. Composition of relations $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} Y$ and $Y \overset{s}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} Z$ is defined by $x\,(s \cdot r)\,z$ iff there exists $y \in Y$ such that $x\,r\,y$ and $y\,s\,z$. Given a set X, the identity 1_X is the diagonal $\{(x, x) \mid x \in X\}$. ∎

Remark 24. **Rel** is an involutive quantaloid (hom-sets are $\bigvee$-semilattices w.r.t. the inclusion order, and composition preserves $\bigvee$ in both variables; cf. Lecture 4). Additionally, **Rel** is isomorphic to the Kleisli category of the powerset monad $\mathbb{P}$ on **Set** (see Lecture 7 for more detail on the Kleisli category of a monad). ∎

Definition 25. Given a unital quantale V, a *V-relation r from* a set X *to* a set Y is a map $X \times Y \xrightarrow{r} V$ (denoted $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} Y$). The *opposite* (or *dual*) of a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} Y$ is the V-relation $Y \overset{r^{\circ}}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} X$ defined by $r^{\circ}(y, x) = r(x, y)$. ∎

Definition 26. Given a unital quantale V, V-**Rel** is the category, whose objects are sets, and whose morphisms are V-relations $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} Y$. Composition of V-relations $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} Y$ and $Y \overset{s}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} Z$ is defined by $(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$. Given a set X, the identity 1_X is defined by

$$1_X(x, y) = \begin{cases} k, & x = y \\ \bot_V, & \text{otherwise.} \end{cases}$$

∎

Remark 27. 2-**Rel** is isomorphic to **Rel**. V-**Rel** is a quantaloid (with hom-set $\bigvee$ given by pointwise evaluation), which is involutive (w.r.t. the dual relation operation $(-)^{\circ}$) iff V is commutative (one should observe that given V-relations $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} Y$ and $Y \overset{s}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} Z$, it follows that $(s \cdot r)^{\circ} = r^{\circ} \cdot s^{\circ}$ iff $\bigvee_{y \in Y} r(x, y) \otimes s(y, z) = (s \cdot r)(x, z) = (s \cdot r)^{\circ}(z, x) = (r^{\circ} \cdot s^{\circ})(z, x) = \bigvee_{y \in Y} s^{\circ}(z, y) \otimes r^{\circ}(y, x) = \bigvee_{y \in Y} s(y, z) \otimes r(x, y)$ for every pair $(z, x) \in Z \times X$; cf. Lecture 4). Moreover, considering V as a quantaloid with one object 1 (also thought of as a singleton set $1 = \{*\}$), we get a full quantaloid embedding $V \overset{E}{\hookrightarrow} V$-**Rel**, which is given by $E(1 \xrightarrow{a} 1) = 1 \overset{a}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} 1$, where $1 \times 1 \xrightarrow{a} V$ is the map with value a. Additionally, V-**Rel** is isomorphic to the Kleisli category w.r.t. the V-powerset monad $\mathbb{P}_V$ on **Set** (an extension of the powerset monad $\mathbb{P}$), whose Eilenberg-Moore category is the category V-**Mod** of left unital V-modules. ∎

Remark 28. To avoid trivial cases, suppose that V has at least two elements ($k \neq \bot_V$). Then there exists a non-full embedding **Set** $\overset{(-)_{\circ}}{\hookrightarrow} V$-**Rel**, which takes a map $X \xrightarrow{f} Y$ to a relation $X \overset{f_{\circ}}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} Y$ given by

$$f_{\circ}(x, y) = \begin{cases} k, & f(x) = y \\ \bot_V, & \text{otherwise.} \end{cases}$$

For the sake of simplicity, one identifies a map $X \xrightarrow{f} Y$ and its respective relation $X \overset{f_{\circ}}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} Y$, employing the notation f for both. It is easy to see that $1_X \leqslant f^{\circ} \cdot f$ and $f \cdot f^{\circ} \leqslant 1_Y$. ∎

1.3. Lax extension of monads

Definition 29. Given a unital quantale V and a functor **Set** $\xrightarrow{T}$ **Set**, a *lax extension* V-**Rel** $\xrightarrow{\hat{T}}$ V-**Rel** of T to V-**Rel** is a pair of maps $\mathcal{O}_{V\text{-}\mathbf{Rel}} \xrightarrow{\hat{T}_{\mathcal{O}}} \mathcal{O}_{V\text{-}\mathbf{Rel}}$, $\mathcal{M}_{V\text{-}\mathbf{Rel}} \xrightarrow{\hat{T}_{\mathcal{M}}} \mathcal{M}_{V\text{-}\mathbf{Rel}}$ (both denoted $\hat{T}$), which satisfy the following axioms:

(1) $\hat{T}(X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} Y) = TX \overset{\hat{T}r}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} TY$ for every V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,\,} Y$;

(2) $\hat{T}r \leqslant \hat{T}s$ for every V-relations $X \overset{r}{\underset{s}{\rightrightarrows}} Y$ such that $r \leqslant s$;

(3) $\hat{T}s \cdot \hat{T}r \leqslant \hat{T}(s \cdot r)$ for every V-relations $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} Y$ and $Y \overset{s}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} Z$;

(4) $Tf \leqslant \hat{T}f$ and $(Tf)^\circ \leqslant \hat{T}(f^\circ)$ for every map $X \xrightarrow{f} Y$. ▪

Example 30. The identity functor on V-**Rel** is a lax extension of the identity functor on **Set**. The powerset functor **Set** $\xrightarrow{P}$ **Set** has lax extensions **Rel** $\overset{\check{P}}{\underset{\hat{P}}{\rightrightarrows}}$ **Rel**, where, given a relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} Y$,

(1) $A\,\check{P}r\,B$ iff for every $x \in A$ there exists $y \in B$ such that $x\,r\,y$;
(2) $A\,\hat{P}r\,B$ iff for every $y \in B$ there exists $x \in A$ such that $x\,r\,y$.

Every functor T on **Set** has the largest lax extension $\hat{T}^\top$ to V-**Rel**, where, given a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} Y$, $\hat{T}^\top r(\mathfrak{x}, \mathfrak{y}) = \top_V$ for every $\mathfrak{x} \in TX$ and every $\mathfrak{y} \in TY$. ▪

Definition 31. Given a unital quantale V and a monad $\mathbb{T}$ on **Set**, a *lax extension* $\hat{\mathbb{T}}$ of $\mathbb{T}$ to V-**Rel** is a triple $(\hat{T}, m, e)$, where $\hat{T}$ is a lax extension of T to V-**Rel**, and $\hat{T}\hat{T} \xrightarrow{m} \hat{T}$, $1_{V\text{-}\mathbf{Rel}} \xrightarrow{e} \hat{T}$ are *oplax natural transformations*, which means

$$\begin{array}{ccc} TTX & \xrightarrow{m_X} & TX \\ {\scriptstyle \hat{T}\hat{T}r}\downarrow & \leqslant & \downarrow{\scriptstyle \hat{T}r} \\ TTY & \xrightarrow[m_Y]{} & TY \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ {\scriptstyle r}\downarrow & \leqslant & \downarrow{\scriptstyle \hat{T}r} \\ Y & \xrightarrow[e_Y]{} & TY \end{array}$$

for every V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} Y$. ▪

Example 32. The identity monad $\mathbb{I}$ on V-**Rel** is a lax extension of the identity monad $\mathbb{I}$ on **Set**. The lax extensions $\check{P}$ and $\hat{P}$ of the powerset functor P provide lax extensions of the powerset monad $\mathbb{P}$ on **Set** to **Rel**. Every monad $\mathbb{T}$ on **Set** has the largest lax extension $\mathbb{T}^\top$ to V-**Rel**, which is given by $\hat{T}^\top$. ▪

2. $(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors, and their examples

2.1. $(\mathbb{T}, V)$*-categories and* $(\mathbb{T}, V)$*-functors*

Definition 33. Suppose V is a unital quantale, and $\hat{\mathbb{T}}$ is a lax extension of a monad $\mathbb{T}$ on **Set** to V-**Rel**. A $(\mathbb{T}, V)$-*category* (or $(\mathbb{T}, V)$-*algebra*, or $(\mathbb{T}, V)$-*space*, or *lax algebra*) is a pair (X, a), which comprises a set X and a V-relation $TX \overset{a}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} X$ such that

$$\begin{array}{ccc} TTX & \xrightarrow{m_X} & TX \\ {\scriptstyle \hat{T}a}\downarrow & \leqslant & \downarrow{\scriptstyle a} \\ TX & \xrightarrow[a]{} & X \end{array} \quad (\textit{transitivity}) \qquad \text{and} \qquad \begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ & {\scriptstyle 1_X}\searrow \;\leqslant & \downarrow{\scriptstyle a} \\ & & X \end{array} \quad (\textit{reflexivity}).$$

A $(\mathbb{T}, V)$-*functor* (or *lax homomorphism*) $(X, a) \xrightarrow{f} (Y, b)$ is a map $X \xrightarrow{f} Y$ such that

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\downarrow & \leqslant & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

$(\mathbb{T}, V)$-**Cat** is the category of $(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors. ▪

Definition 34. The category $(\mathbb{1}, V)$-**Cat** is denoted V-**Cat**, whose objects (resp. morphisms) are called *V-categories* (resp. *V-functors*). ▪

Remark 35. There is an analogy between $\mathbb{T}$-algebras and $\mathbb{T}$-homomorphisms w.r.t. a monad $\mathbb{T}$ on **Set**, on one hand, and $(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors w.r.t. its lax extension $\hat{\mathbb{T}}$ to V-**Rel**, on the other hand. ▪

Definition 36. *Monoidal topology* is a branch of categorical topology, which studies the properties of the categories of the form $(\mathbb{T}, V)$-**Cat**. ▪

2.2. Examples of the categories $(\mathbb{T}, V)$-**Cat**

2.2.1. Preordered sets and quasi-pseudo-metric spaces as V-categories

Remark 37. A V-category (X, a) consists of a set X and a V-relation $X \overset{a}{\longrightarrow\!\!\!\!\!\!\!+\;\;} X$ such that

(1)

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ & \searrow^{1_X} \;\leqslant & \big\downarrow a \\ & & X, \end{array}$$

which is equivalent to $1_X \leqslant a$, which is equivalent to $k \leqslant a(x, x)$ for every $x \in X$;

(2)

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ a \big\downarrow & \leqslant & \big\downarrow a \\ X & \xrightarrow[a]{} & X, \end{array}$$

which is equivalent to $a \cdot a \leqslant a$, which is equivalent to $\bigvee_{y \in X} a(x, y) \otimes a(y, z) \leqslant a(x, z)$ for every $x, z \in X$, which is equivalent to $a(x, y) \otimes a(y, z) \leqslant a(x, z)$ for every $x, y, z \in X$.

A V-functor $(X, a) \xrightarrow{f} (Y, b)$ has the property that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ a \big\downarrow & \leqslant & \big\downarrow b \\ X & \xrightarrow[f]{} & Y, \end{array}$$

which is equivalent to $f \cdot a \leqslant b \cdot f$, which is equivalent to $\bigvee_{f(z)=y} a(x, z) \leqslant b(f(x), y)$ for every $x \in X$ and every $y \in Y$, which is equivalent to $a(x, z) \leqslant b(f(x), f(z))$ for every $x, z \in X$. ▪

Example 38. A 2-category is a pair $(X, \leqslant)$ such that $x \leqslant x$ for every $x \in X$; and $x \leqslant y$, $y \leqslant z$ imply $x \leqslant z$ for every $x, y, z \in X$. A 2-functor $(X, \leqslant) \xrightarrow{f} (Y, \leqslant)$ is a map $X \xrightarrow{f} Y$ such that $x, z \in X$ and $x \leqslant z$ imply $f(x) \leqslant f(z)$. As a result, 2-**Cat** is the category **Prost** of preordered sets and monotone maps. ▪

Example 39. A P_+-category is a pair (X, ρ) such that $\rho(x, x) = 0$ for every $x \in X$; and $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$ for every $x, y, z \in X$. A P_+-functor $(X, \rho) \xrightarrow{f} (Y, \varrho)$ is a map $X \xrightarrow{f} Y$ such that $\varrho(f(x), f(z)) \leqslant \rho(x, z)$ for every $x, z \in X$. As a result, P_+-**Cat** is the category **QPMet** of quasi-pseudo-metric spaces (generalized metric spaces in the sense of F. W. Lawvere) and non-expansive maps. ▪

2.2.2. Topological spaces as $(\mathbb{T}, V)$-categories

Remark 40. It is well-known that the Eilenberg-Moore category of the ultrafilter monad on **Set** is the category of compact Hausdorff topological spaces. One cannot extend this results to the whole category **Top**, since the latter category is not of algebraic nature (e.g., bijective morphisms in **Top** are not necessarily homeomorphisms). To get the whole category **Top**, one employs a lax extension of the ultrafilter monad. ▪

Definition 41. Given a set X, a *filter on* X is a family $\mathfrak{x}$ of subsets of X such that

(1) $X \in \mathfrak{x}$;
(2) $A \in \mathfrak{x}$ and $A \subseteq B$ imply $B \in \mathfrak{x}$;
(3) $A, B \in \mathfrak{x}$ implies $A \bigcap B \in \mathfrak{x}$.

A filter $\mathfrak{x}$ is called *proper* provided that $\varnothing \notin \mathfrak{x}$. An *ultrafilter* $\mathfrak{x}$ on a set X is a maximal element in the set of proper filters on X, ordered by inclusion. ▪

Example 42. Given a set X, every $x \in X$ provides the *principal* ultrafilter $\dot{x} = \{A \subseteq X \mid x \in A\}$ on X. ▪

Remark 43. A proper filter $\mathfrak{x}$ on X is an ultrafilter on X iff for every $A \subseteq X$, either $A \in \mathfrak{x}$ or $X \backslash A \in \mathfrak{x}$. ▪

Definition 44. The *ultrafilter monad* $\beta\!\!\beta = (\beta, m, e)$ on **Set** is given by

(1) a functor $\mathbf{Set} \xrightarrow{\beta} \mathbf{Set}$, where $\beta X = \{\mathfrak{x} \mid \mathfrak{x} \text{ is an ultrafilter on } X\}$ for every set X, and $\beta X \xrightarrow{\beta f} \beta Y$ is defined by $\beta f(\mathfrak{x}) = \{B \subseteq Y \mid f^{-1}(B) \in \mathfrak{x}\}$ for every map $X \xrightarrow{f} Y$;
(2) a natural transformation $1_{\mathbf{Set}} \xrightarrow{e} \beta$, where $X \xrightarrow{e_X} \beta X$ is defined by $e_X(x) = \dot{x}$;
(3) a natural transformation $\beta\beta \xrightarrow{m} \beta$, where $\beta\beta X \xrightarrow{m_X} \beta X$ is defined by $m_X(\mathfrak{X}) = \Sigma\, \mathfrak{X}$ (*filtered sum* or *Kowalsky sum*), where $A \in \Sigma\, \mathfrak{X}$ iff $\{\mathfrak{x} \in \beta X \mid A \in \mathfrak{x}\} \in \mathfrak{X}$. ▪

Theorem 45. *Given a relation* $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\,} Y$, *define* $\mathfrak{x}\, \hat{\beta} r\, \mathfrak{y}$ *iff for every* $A \in \mathfrak{x}$ *and every* $B \in \mathfrak{y}$, *there exist* $x \in A$ *and* $y \in B$ *such that* $x\, r\, y$. *Then* $\hat{\beta\!\!\beta} = (\hat{\beta}, m, e)$ *is a lax extension to* **Rel** *of the ultrafilter monad* $\beta\!\!\beta$, *in which, additionally,* $\mathbf{Rel} \xrightarrow{\hat{\beta}} \mathbf{Rel}$ *is a functor, and* $\hat{\beta}\hat{\beta} \xrightarrow{m} \hat{\beta}$ *is a natural transformation.*

Remark 46. Every $(\beta\!\!\beta, 2)$-category (X, a) has the following two properties:

(1) $1_X \leqslant a \cdot e_X$, which is equivalent to $\dot{x}\, a\, x$ for every $x \in X$;
(2) $a \cdot \hat{\beta} a \leqslant a \cdot m_X$, which is equivalent to $\mathfrak{X}\, (\hat{\beta} a)\, \mathfrak{x}$ and $\mathfrak{x}\, a\, x$ imply $(\Sigma\, \mathfrak{X})\, a\, x$ for every $\mathfrak{X} \in \beta\beta X$, every $\mathfrak{x} \in \beta X$, and every $x \in X$.

Every $(\beta\!\!\beta, 2)$-functor $(X, a) \xrightarrow{f} (Y, b)$ satisfies the condition $f \cdot a \leqslant b \cdot \beta f$, which is equivalent to $\mathfrak{x}\, a\, x$ implies $\beta f(\mathfrak{x})\, b\, f(x)$ for every $\mathfrak{x} \in \beta X$ and every $x \in X$. ▪

Definition 47. Given a set X, a *closure operation* on X is a monotone map $PX \xrightarrow{c} PX$ (w.r.t. the inclusion order) such that $1_{PX} \leqslant c$ and $c \cdot c \leqslant c$ (pointwise evaluation as maps). A closure operation c on X is *finitely additive* provided that $c(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} c(A_i)$ for every finite family $\{A_i \mid i \in I\} \subseteq PX$ (equivalently, $c(\varnothing) = \varnothing$ and $c(A \bigcup B) = c(A) \bigcup c(B)$ for every $A, B \in PX$). ▪

Proposition 48. *A closure operation* c *on a set* X *is finitely additive iff the family* $\tau = \{X \backslash A \mid c(A) = A\}$ *is a topology on* X, *i.e., there exists a one-to-one correspondence between finitely additive closure operations on* X *and topologies on* X.

Proof. As an illustration, one could verify that τ is closed under finite intersections provided that c is finitely additive. Given a finite family $\{X \backslash A_i \mid i \in I\} \subseteq \tau$, it follows that $\bigcap_{i \in I}(X \backslash A_i) = X \backslash (\bigcup_{i \in I} A_i) = X \backslash (\bigcup_{i \in I} c(A_i)) \overset{(\dagger)}{=} X \backslash c(\bigcup_{i \in I} A_i) \overset{(\dagger\dagger)}{\in} \tau$, where $(\dagger)$ relies on finite additivity of c, and $(\dagger\dagger)$ follows from the property $c \cdot c = c$ of every closure operation c on X (observe that $1_{PX} \leqslant c$ implies $c \leqslant c \cdot c$).

Observe that τ is closed under arbitrary unions for every closure operation c on X, since given a family $\{X\backslash A_i \,|\, i \in I\} \subseteq \tau$, it follows that $\bigcup_{i\in I}(X\backslash A_i) = X\backslash(\bigcap_{i\in I} A_i) = X\backslash(\bigcap_{i\in I} c(A_i)) \overset{(\dagger)}{=} X\backslash c(\bigcap_{i\in I} A_i) \in \tau$, where $(\dagger)$ relies on the fact that $\bigcap_{i\in I} c(A_i) = c(\bigcap_{i\in I} A_i)$, since, on the one hand, $\bigcap_{i\in I} A_i \subseteq A_j$ for every $j \in I$ implies $c(\bigcap_{i\in I} A_i) \subseteq c(A_j)$ for every $j \in I$ implies $c(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} c(A_i)$; and, on the other hand, $\bigcap_{i\in I} c(A_i) \overset{(\dagger\dagger)}{=} \bigcap_{i\in I} A_i \subseteq c(\bigcap_{i\in I} A_i)$, where $(\dagger\dagger)$ relies on the fact that $c(A_i) = A_i$ for every $i \in I$. □

Definition 49. Given a topological space (X, τ), a filter $\mathfrak{x}$ on X *converges* to an element $x \in X$ provided that $U \in \mathfrak{x}$ for every $U \in \tau$ such that $x \in U$. If $\mathfrak{x}$ converges to x, then x is called a *limit* of $\mathfrak{x}$. The set of limits of a filter $\mathfrak{x}$ is denoted $\lim \mathfrak{x}$. ▪

Proposition 50. *Given a finitely additive closure operation c on a set X, the following hold:*
(1) *for every $A \subseteq X$ and every $x \in X$, $x \in c(A)$ iff there exists $\mathfrak{x} \in \beta X$ such that $A \in \mathfrak{x}$ and $x \in \lim \mathfrak{x}$;*
(2) *for every $\mathfrak{x} \in \beta X$ and every $x \in X$, $x \in \lim \mathfrak{x}$ iff $x \in c(A)$ for every $A \in \mathfrak{x}$.*

Proof. As an illustration, one can show "⇒" of (2). Given $A \in \mathfrak{x}$, for every $U \in \tau$ (cf. Proposition 48) such that $x \in U$, it follows that $U \in \mathfrak{x}$ (since $x \in \lim \mathfrak{x}$) and, therefore, $U \bigcap A \in \mathfrak{x}$ (since $\mathfrak{x}$ is a filter), which implies $U \bigcap A \neq \varnothing$ (since $\mathfrak{x}$ is an ultrafilter). Thus, $x \in c(A)$. In a similar way, one can show "⇐" of (1). □

Theorem 51. *The category* $(\mathbb{\beta}, 2)$-**Cat** *is isomorphic to the category* **Top**.

Proof. The isomorphism between $(\mathbb{\beta}, 2)$-categories and topological spaces is based in the idea that given a set X, a $(\mathbb{\beta}, 2)$-category structure $\beta X \overset{a}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} X$ on X represents a convergence relation between ultrafilters on X and elements of X (i.e., a specifies which ultrafilter converges to which element). One then associates with a a finitely additive closure operation c on X, and also shows that every finitely additive closure operation c on X determines a convergence relation $\beta X \overset{a}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} X$, which is a $(\mathbb{\beta}, 2)$-category structure on X.

Given a $(\mathbb{\beta}, 2)$-category (X, a), one defines a closure operation $PX \xrightarrow{\texttt{clos}(a)} PX$ on X by $(\texttt{clos}(a))(A) = \{x \in X \mid \text{there exists } \mathfrak{x} \in \beta X \text{ such that } A \in \mathfrak{x} \text{ and } \mathfrak{x}\, a\, x\}$ (cf. Proposition 50 (1)). Given a finitely additive closure operation c on X, one defines a $(\mathbb{\beta}, 2)$-category structure $\beta X \overset{\texttt{conv}(c)}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} X$ on X by $\mathfrak{x}\, \texttt{conv}(c)\, x$ iff $x \in c(A)$ for every $A \in \mathfrak{x}$ (cf. Proposition 50 (2)).

To show that, e.g., $1_{PX} \leqslant \texttt{clos}(a)$, notice that given $A \subseteq X$, for every $x \in A$, it follows that $A \in \dot{x}$ and $\dot{x}\, a\, x$, i.e., $x \in (\texttt{clos}(a))(A)$, which implies $A \subseteq (\texttt{clos}(a))(A)$. To show that, e.g., $1_X \leqslant \texttt{conv}(c) \cdot e_X$, notice that given $x \in X$, it follows that $\dot{x}\, \texttt{conv}(c)\, x$, since given $A \subseteq X$, $A \in \dot{x}$ implies $x \in A \subseteq c(A)$, i.e., $x \in c(A)$. □

2.2.3. Approach spaces as $(\mathbb{T}, V)$-categories

Definition 52. An *approach space* is a pair (X, δ), where X is a set, and $X \times PX \xrightarrow{\delta} [0, \infty]$ is a map (*approach distance*) such that
(1) $\delta(x, \{x\}) = 0$ for every $x \in X$;
(2) $\delta(x, \varnothing) = \infty$ for every $x \in X$;
(3) $\delta(x, A \bigcup B) = \min\{\delta(x, A), \delta(x, B)\}$ for every $x \in X$ and every $A, B \subseteq X$;
(4) $\delta(x, A) \leqslant \delta(x, A^{(u)}) + u$, where $A^{(u)} = \{y \in X \mid \delta(y, A) \leqslant u\}$ for every $x \in X$, $A \subseteq X$, $u \in [0, \infty]$.

A morphism $(X, \delta) \xrightarrow{f} (Y, \sigma)$ of approach spaces is a *non-expansive map* $X \xrightarrow{f} Y$, i.e., $\sigma(f(x), f(A)) \leqslant \delta(x, A)$ for every $x \in X$ and every $A \subseteq X$. **App** is the category of approach spaces and non-expansive maps. ▪

Remark 53. Approach spaces provide a unifying framework for topological, metric, and uniform spaces. ▪

Remark 54. Every topological space (X, τ) gives an approach space (X, δ), in which

$$\delta(x, A) = \begin{cases} 0, & x \in cl(A) \text{ (the closure of the set } A \text{ w.r.t. } \tau) \\ \infty, & \text{otherwise} \end{cases}$$

for every $x \in X$ and every $A \in PX$. One gets thus a full embedding **Top** ↪ **App**. ▪

Remark 55. Every quasi-pseudo-metric space (X,ρ) gives an approach space (X,δ), in which $\delta(x,A) = \inf\{\rho(y,x)\,|\,y \in A\}$ for every $x \in X$ and every $A \in PX$. One gets thus a full embedding $\mathbf{QPMet} \hookrightarrow \mathbf{App}$. ■

Theorem 56. *Given a* P_+*-relation* $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$, *define a map* $\beta X \times \beta Y \xrightarrow{\bar{\beta} r} \mathsf{P}_+$ *by*

$$\bar{\beta} r(\mathfrak{x},\mathfrak{y}) = \bigwedge_{A\in\mathfrak{x},B\in\mathfrak{y}} \bigvee_{x\in A,y\in B} r(x,y).$$

Then $\bar{\mathbb{\beta}} = (\bar{\beta}, m, e)$ *is a lax extension to* P_+-$\mathbf{Rel}$ *of the ultrafilter monad* $\mathbb{\beta}$, *in which, additionally,* V-$\mathbf{Rel} \xrightarrow{\bar{\beta}} V$-$\mathbf{Rel}$ *is a functor, and* $\bar{\beta}\bar{\beta} \xrightarrow{m} \bar{\beta}$ *is a natural transformation.*

Remark 57. Every $(\mathbb{\beta},\mathsf{P}_+)$-category (X,a) has the following two properties:

(1) $1_X \leqslant a \cdot e_X$, which is equivalent to $a(\dot{x},x) = 0$ for every $x \in X$;
(2) $a \cdot \bar{\beta} a \leqslant a \cdot m_X$, which is equivalent to $a(\Sigma\,\mathfrak{X},x) \leqslant \bar{\beta} a(\mathfrak{X},\mathfrak{x}) + a(\mathfrak{x},x)$ for every $\mathfrak{X} \in \beta\beta X$, $\mathfrak{x} \in \beta X$, $x \in X$.

Every $(\mathbb{\beta},\mathsf{P}_+)$-functor $(X,a) \xrightarrow{f} (Y,b)$ satisfies the condition $f \cdot a \leqslant b \cdot \beta f$, which is equivalent to $b(\beta f(\mathfrak{x}), f(x)) \leqslant a(\mathfrak{x},x)$ for every $\mathfrak{x} \in \beta X$ and every $x \in X$. ■

Theorem 58. *The category* $(\mathbb{\beta},\mathsf{P}_+)$-$\mathbf{Cat}$ *is isomorphic to the category* $\mathbf{App}$.

PROOF. Following the analogy of Theorem 51, given a $(\mathbb{\beta},\mathsf{P}_+)$-category (X,a), one defines an approach distance $X \times PX \xrightarrow{\mathtt{clos}(a)} [0,\infty]$ by $(\mathtt{clos}(a))(x,A) = \inf\{a(\mathfrak{x},x)\,|\,\mathfrak{x} \in \beta A\}$. Given an approach space (X,δ), one defines a $(\mathbb{\beta},\mathsf{P}_+)$-category structure $\beta X \overset{\mathtt{conv}(\delta)}{\longrightarrow\!\!\!\!\!\!\!+\;\;} X$ by $(\mathtt{conv}(\delta))(\mathfrak{x},x) = \sup\{\delta(x,A)\,|\,A \in \mathfrak{x}\}$. □

Remark 59. Theorem 58 actually says that approach spaces provide "numerified topological spaces", since a classical convergence relation is replaced with a numerified "degree of convergence". ■

2.2.4. Closure spaces as $(\mathbb{T},V)$*-categories*

Definition 60. A *closure space* is a pair (X,c), where X is a set, and $PX \xrightarrow{c} PX$ is a closure operation on X. A map $(X,c) \xrightarrow{f} (Y,d)$ between closure spaces is *continuous* provided that $f(c(A)) \subseteq d(f(A))$ for every $A \subseteq X$. $\mathbf{Cls}$ is the category of closure spaces and continuous maps. ■

Theorem 61. *The lax extension* $\hat{\mathbb{P}}$ *of the powerset monad* $\mathbb{P}$ *provides the category* $(\mathbb{P},2)$-$\mathbf{Cat}$, *which is isomorphic to the category* $\mathbf{Cls}$.

PROOF. Every $(\mathbb{P},2)$-category (X,a) has the following two properties:

(1) $1_X \leqslant a \cdot e_X$, which is equivalent to $\{x\}\,a\,x$ for every $x \in X$;
(2) $a \cdot \hat{P} a \leqslant a \cdot m_X$, which is equivalent to $\mathcal{A}\,(\hat{P}a)\,B$ (i.e., for every $y \in B$, there exists $A \in \mathcal{A}$ such that $A\,a\,y$) and $B\,a\,x$ imply $(\bigcup \mathcal{A})\,a\,x$ for every $\mathcal{A} \in PPX$, every $B \in PX$, and every $x \in X$.

Every $(\mathbb{P},2)$-functor $(X,a) \xrightarrow{f} (Y,b)$ satisfies the condition $f \cdot a \leqslant b \cdot Pf$, which is equivalent to $A\,a\,x$ implies $Pf(A)\,b\,f(x)$ (i.e., $f(A)\,b\,f(x)$) for every $A \in PX$ and every $x \in X$.

Given a set X, there exists a bijective correspondence between $(\mathbb{P},2)$-category structures $PX \overset{a}{\longrightarrow\!\!\!\!\!\!\!+\;\;} X$ and closure operations $PX \xrightarrow{c} PX$, which is given by $A\,a\,x$ iff $x \in c(A)$ for every $A \in PX$, $x \in X$. Additionally, a $(\mathbb{P},2)$-functor $(X,a) \xrightarrow{f} (Y,b)$ provides a continuous map $(X,c) \xrightarrow{f} (Y,d)$ and vice versa.

To show that, e.g., $1_{PX} \leqslant c$ (for a given $(\mathbb{P},2)$-category structure $PX \overset{a}{\longrightarrow\!\!\!\!\!\!\!+\;\;} X$ on X), observe that for every $A \in PX$, $x \in A$ implies $\{x\}\,a\,x$ (by item (1) above) implies $\{\{y\}\,|\,y \in A\}\,\hat{P}a\,\{x\}$ and $\{x\}\,a\,x$ implies $(\bigcup\{\{y\}\,|\,y \in A\})\,a\,x$ (by item (2) above) implies $A\,a\,x$ implies $x \in c(A)$, which results in $A \subseteq c(A)$.

To show that, e.g., $1_X \leqslant a \cdot e_X$ (for a given closure operation $PX \xrightarrow{c} PX$ on X), observe that for every $x \in X$, $x \in \{x\} \subseteq c\{x\}$ implies $x \in c\{x\}$ implies $\{x\}\, a\, x$, which is exactly the condition of item (1) above.

To verify that a $(\mathbb{P}, 2)$-functor $(X, a) \xrightarrow{f} (Y, b)$ provides a continuous map $(X, c) \xrightarrow{f} (Y, d)$, observe that for every $A \in PX$, $x \in c(A)$ implies $A\, a\, x$ implies $f(A)\, b\, f(x)$ (since f is a $(\mathbb{P}, 2)$-functor) implies $f(x) \in d(f(A))$. As a consequence, one obtains that $f(c(A)) \subseteq d(f(A))$.

To check that a continuous map $(X, c) \xrightarrow{f} (Y, d)$ provides a $(\mathbb{P}, 2)$-functor $(X, a) \xrightarrow{f} (Y, b)$, observe that for every $A \in PX$ and every $x \in X$, $A\, a\, x$ implies $x \in c(A)$ implies $f(x) \in d(f(A))$ (since f is continuous) implies $f(A)\, b\, f(x)$, which is exactly the above-mentioned condition of $(\mathbb{P}, 2)$-functors. □

Definition 62. There exists the *finite-powerset monad* $\mathbb{P}_{\mathrm{fin}} = (P_{\mathrm{fin}}, m, e)$ on **Set**, in which the functor $\mathbf{Set} \xrightarrow{P_{\mathrm{fin}}} \mathbf{Set}$ is given on objects by $P_{\mathrm{fin}}X = \{A \subseteq X \mid A \text{ is finite}\}$. The natural transformations m and e are the restrictions of the respective natural transformations of the powerset monad $\mathbb{P}$ on **Set**. ▪

Definition 63. A closure space (X, c) is called *finitary* (or *algebraic*) provided that $c(A) = \bigcup_{B \in P_{\mathrm{fin}}A} c(B)$ for every $A \in PX$. $\mathbf{Cls}_{\mathrm{fin}}$ is the full subcategory of **Cls** of finitary closure spaces. ▪

Theorem 64. *The lax extension $\hat{\mathbb{P}}_{\mathrm{fin}}$ of the finite-powerset monad $\mathbb{P}_{\mathrm{fin}}$ provides the category $(\mathbb{P}_{\mathrm{fin}}, 2)$-**Cat**, which is isomorphic to the category $\mathbf{Cls}_{\mathrm{fin}}$.*

Proof. One uses the following modification of the proof of Theorem 61.

Given a set X, there exists a bijective correspondence between $(\mathbb{P}_{\mathrm{fin}}, 2)$-category structures $P_{\mathrm{fin}}X \overset{a}{\longrightarrow\!\!\!\!\!\!\!\!\mapsto} X$ and closure operations $P_{\mathrm{fin}}X \xrightarrow{c_{\mathrm{fin}}} P_{\mathrm{fin}}X$, which is given by $A\, a\, x$ iff $x \in c_{\mathrm{fin}}(A)$ for every $A \in P_{\mathrm{fin}}X$, $x \in X$. Moreover, a $(\mathbb{P}_{\mathrm{fin}}, 2)$-functor $(X, a) \xrightarrow{f} (Y, b)$ provides a continuous map $(X, c_{\mathrm{fin}}) \xrightarrow{f} (Y, d_{\mathrm{fin}})$ and vice versa.

Given a set X, there also exists a bijective correspondence between closure operations $P_{\mathrm{fin}}X \xrightarrow{c_{\mathrm{fin}}} P_{\mathrm{fin}}X$ and algebraic closure operations $PX \xrightarrow{c} PX$, which is given in one direction by $x \in c(A)$ iff there exists $B \in P_{\mathrm{fin}}A$ such that $x \in c_{\mathrm{fin}}(B)$ for every $A \in PX$, $x \in X$, and the opposite direction is the restriction of c to $P_{\mathrm{fin}}X$. This correspondence respects continuity of maps, e.g., given a continuous map $(X, c_{\mathrm{fin}}) \xrightarrow{f} (Y, d_{\mathrm{fin}})$, for every $x \in X$ and every $A \in PX$, $x \in c(A)$ implies the existence of $B \in P_{\mathrm{fin}}A$ such that $x \in c_{\mathrm{fin}}(B)$ implies $f(x) \in d_{\mathrm{fin}}(f(B))$ (since f is continuous) implies $f(x) \in d(f(B))$ (since $f(B) \in P_{\mathrm{fin}}(f(A))$). □

References

[1] M. Barr, *Relational algebras*, Reports of the Midwest Category Seminar, vol. IV, Lect. Notes Math. 137, 1970, pp. 39–55.

[2] M. M. Clementino and D. Hofmann, *On extensions of lax monads*, Theory Appl. Categ. **13** (2004), 41–60.

[3] M. M. Clementino, D. Hofmann, and W. Tholen, *One setting for all: Metric, topology, uniformity, approach structure*, Appl. Categ. Struct. **12** (2004), no. 2, 127–154.

[4] R. Engelking, *General Topology*, Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, 1989.

[5] H. Herrlich, *Topologie I. Topologische Räume. Unter Mitarb. von H. Bargenda*, vol. 2, Heldermann Verlag, Lemgo, 1986.

[6] D. Hofmann, G. J. Seal, and W. Tholen (eds.), *Monoidal Topology: A Categorical Approach to Order, Metric and Topology*, Cambridge University Press, 2014.

[7] D. Kruml and J. Paseka, *Algebraic and Categorical Aspects of Quantales*, Handbook of Algebra (M. Hazewinkel, ed.), vol. 5, Elsevier, 2008, pp. 323–362.

[8] H. P. Künzi, *Complete quasi-pseudo-metric spaces*, Acta Math. Hung. **59** (1992), no. 1-2, 121–146.

[9] R. Lowen, *Approach spaces - a common supercategory of TOP and MET*, Math. Nachr. **141** (1989), 183–226.

[10] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., Springer-Verlag, 1998.

[11] J. MacDonald and M. Sobral, *Aspects of Monads*, Categorical Foundations: Special Topics in Order, Topology, Algebra, and Sheaf Theory (M. C. Pedicchio and W. Tholen, eds.), Cambridge University Press, 2004, pp. 213–268.

[12] E. G. Manes, *A triple theoretic construction of compact algebras*, Semin. Triples and Categorical Homology Theory, ETH, Zurich, 1966/67, Lect. Notes Math., vol. 80, 1969, pp. 91–118.

[13] K. I. Rosenthal, *The Theory of Quantaloids*, Addison Wesley Longman, 1996.