Elements of monoidal topology⋆
Lecture 2: properties of the category $(\mathbb{T}, V)$-**Cat**

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Abstract

This lecture proves that the category $(\mathbb{T}, V)$-**Cat** is a topological construct (providing the explicit form of the initial structures), describes its induced preorder, and shows, how to obtain functors $(\mathbb{T}_1, V)$-**Cat** $\to$ $(\mathbb{T}_2, V)$-**Cat** as well as $(\mathbb{T}, V_1)$-**Cat** $\to$ $(\mathbb{T}, V_2)$-**Cat** for different monads and different quantales, respectively.

1. $(\mathbb{T}, V)$-Cat is a topological construct

1.1. Eilenberg-Moore algebras versus $(\mathbb{T}, V)$-categories

Definition 1.

(1) A lax extension $\hat{T}$ to V-**Rel** of a functor T on **Set** is *flat* provided that $\hat{T}1_X = T1_X$ for every set X.
(2) A lax extension $\hat{\mathbb{T}}$ to V-**Rel** of a monad $\mathbb{T} = (T, m, e)$ on **Set** is *flat* provided that the lax extension $\hat{T}$ of T is flat. ▪

Lemma 2. *Every lax extension $\hat{T}$ satisfies the following:*

$$\hat{T}(s \cdot f) = \hat{T}s \cdot Tf \quad \text{and} \quad \hat{T}(f^\circ \cdot r) = (Tf)^\circ \cdot \hat{T}r \tag{1.1}$$

for every map $X \xrightarrow{f} Y$ and every V-relations $Y \overset{s}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Z$, $Z \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$.

Proof. First, observe that $\hat{T}s \cdot Tf \leqslant \hat{T}s \cdot \hat{T}f \leqslant \hat{T}(s \cdot f) \overset{1_{TX} \leqslant (Tf)^\circ \cdot Tf}{\leqslant} \hat{T}(s \cdot f) \cdot (Tf)^\circ \cdot Tf \leqslant \hat{T}(s \cdot f) \cdot \hat{T}(f^\circ) \cdot Tf \leqslant \hat{T}(s \cdot f \cdot f^\circ) \cdot Tf \overset{f \cdot f^\circ \leqslant 1_Y}{\leqslant} \hat{T}s \cdot Tf$, which implies $\hat{T}(s \cdot f) = \hat{T}s \cdot Tf$.

Second, observe that $Tf \cdot \hat{T}(f^\circ \cdot r) \leqslant \hat{T}f \cdot \hat{T}(f^\circ \cdot r) \leqslant \hat{T}(f \cdot f^\circ \cdot r) \overset{f \cdot f^\circ \leqslant 1_Y}{\leqslant} \hat{T}r$ implies $\hat{T}(f^\circ \cdot r) \overset{1_{TX} \leqslant (Tf)^\circ \cdot Tf}{\leqslant} (Tf)^\circ \cdot Tf \cdot \hat{T}(f^\circ \cdot r) \leqslant (Tf)^\circ \cdot \hat{T}r$, i.e., $\hat{T}(f^\circ \cdot r) \leqslant (Tf)^\circ \cdot \hat{T}r$. Moreover, $(Tf)^\circ \cdot \hat{T}r \leqslant \hat{T}(f^\circ) \cdot \hat{T}r \leqslant \hat{T}(f^\circ \cdot r)$, i.e., $(Tf)^\circ \cdot \hat{T}r \leqslant \hat{T}(f^\circ \cdot r)$. Altogether, one obtains that $\hat{T}(f^\circ \cdot r) = (Tf)^\circ \cdot \hat{T}r$. □

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Theorem 3. *A lax extension $\hat{T}$ is flat iff the following diagrams commute:*

$$
\begin{array}{ccc}
\mathbf{Set} & \xrightarrow{T} & \mathbf{Set} \\
{\scriptstyle (-)_\circ}\downarrow & & \downarrow{\scriptstyle (-)_\circ} \\
V\text{-}\mathbf{Rel} & \xrightarrow[\hat{T}]{} & V\text{-}\mathbf{Rel}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbf{Set}^{op} & \xrightarrow{T^{op}} & \mathbf{Set}^{op} \\
{\scriptstyle (-)^\circ}\downarrow & & \downarrow{\scriptstyle (-)^\circ} \\
V\text{-}\mathbf{Rel} & \xrightarrow[\hat{T}]{} & V\text{-}\mathbf{Rel}
\end{array}
$$

(namely, $\hat{T}f = Tf$ and $\hat{T}(f^\circ) = (Tf)^\circ$ for every map $X \xrightarrow{f} Y$).

Proof. The sufficiency is clear. For the necessity, notice that given a map $X \xrightarrow{f} Y$, it follows that $\hat{T}f = \hat{T}(1_Y \cdot f) \overset{\text{Lemma 2}}{=} \hat{T}1_Y \cdot Tf = Tf$ and $\hat{T}(f^\circ) = \hat{T}(f^\circ \cdot 1_Y) \overset{\text{Lemma 2}}{=} (Tf)^\circ \cdot \hat{T}1_Y = (Tf)^\circ$. □

Example 4.

(1) The identity monad $\mathbb{I}$ on V-**Rel** is a flat lax extension of the identity monad $\mathbb{I}$ on **Set**.

(2) The lax extension $\hat{\mathbb{\beta}}$ of the ultrafilter monad $\mathbb{\beta}$ on **Set** is flat, since given a set X, $\beta X \xrightarrow{\hat{\beta}1_X} \beta X$ is defined by $\mathfrak{x}\,\hat{\beta}1_X\,\mathfrak{y}$ iff for every $A \in \mathfrak{x}$ and every $B \in \mathfrak{y}$, there exist $x \in A$ and $y \in B$ such that $x\,(1_X)_\circ\,y$ iff for every $A \in \mathfrak{x}$ and every $B \in \mathfrak{y}$, there exist $x \in A$ and $y \in B$ such that $x = y$ iff for every $A \in \mathfrak{x}$ and every $B \in \mathfrak{y}$, $A \bigcap B \neq \varnothing$ iff $\mathfrak{x} = \mathfrak{y}$ (since $\mathfrak{x}$ and $\mathfrak{y}$ ultrafilters).

(3) The lax extensions $\check{\mathbb{P}}$, $\hat{\mathbb{P}}$ of the powerset monad $\mathbb{P}$ on **Set** are non-flat, since given a set X, $PX \xrightarrow{\check{P}1_X} PX$ is defined by $A\,\check{P}1_X\,B$ iff for every $x \in A$ there exists $y \in B$ such that $x\,(1_X)_\circ\,y$ iff for every $x \in A$ there exists $y \in B$ such that $x = y$ iff $A \subseteq B$ ($PX \xrightarrow{\hat{P}1_X} PX$ is defined by $A\,\hat{P}1_X\,B$ iff for every $y \in B$ there exists $x \in A$ such that $x\,(1_X)_\circ\,y$ iff for every $y \in B$ there exists $x \in A$ such that $x = y$ iff $B \subseteq A$).

(4) The largest lax extension $\mathbb{T}^\top$ of a monad $\mathbb{T}$ on **Set** is (in general) non-flat, since given a set X, $TX \xrightarrow{\hat{T}^\top 1_X} TX$ is defined by $\hat{T}^\top 1_X(\mathfrak{x}, \mathfrak{y}) = \top_V$ for every $\mathfrak{x}, \mathfrak{y} \in TX$, which provides the identity on TX in V-**Rel** iff TX is at most a singleton and $k = \top_V$. ▪

Theorem 5. *Every flat lax extension $\hat{\mathbb{T}}$ of a monad $\mathbb{T}$ on* **Set** *has a full embedding* $\mathbf{Set}^{\mathbb{T}} \xhookrightarrow{E} (\mathbb{T}, V)$-**Cat**, *which is given by $E((X, a) \xrightarrow{f} (Y, b)) = (X, a) \xrightarrow{f} (Y, b)$.*

Proof. Given a $\mathbb{T}$-algebra (X, a), it follows that $a \cdot e_X = 1_X$ and $a \cdot Ta = a \cdot m_X$. Since $\hat{T}$ is flat, $a \cdot \hat{T}a = a \cdot m_X$, and therefore, (X, a) is a $(\mathbb{T}, V)$-category. To show that the embedding E is full, notice that given a $(\mathbb{T}, V)$-functor $E(X, a) \xrightarrow{f} E(Y, b)$, the inequality $f \cdot a \leqslant b \cdot Tf$ yields that the graph of the first map is contained in the graph of the second map. Sharing the same domain, the maps thus must coincide. □

Remark 6. By Theorem 5, a flat lax extension $\hat{\mathbb{T}}$ of a monad $\mathbb{T}$ gives the category $(\mathbb{T}, V)$-**Cat**, which (in general) is larger than its respective Eilenberg-Moore category $\mathbf{Set}^{\mathbb{T}}$. Since the latter category is monadic (or, more generally, essentially algebraic), one could ask about the nature of the former category. ▪

1.2. $(\mathbb{T}, V)$-**Cat** *is a topological construct*

1.2.1. Topological categories

Definition 7. A functor $\mathbf{X} \xrightarrow{F} \mathbf{Y}$ is called *faithful* provided that $Ff \neq Fg$ for every $\mathbf{X}$-morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ such that $f \neq g$. ▪

Example 8. The powerset functor P on **Set** is faithful, since given maps $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ such that $f \neq g$, there exists $x \in X$ with $f(x) \neq g(x)$ and then $Pf(\{x\}) = \{f(x)\} \neq \{g(x)\} = Pg(\{x\})$, which implies $Pf \neq Pg$. ▪

Definition 9.

(1) Given a category $\mathbf{X}$, a *concrete category* over $\mathbf{X}$ is a pair $(\mathbf{A}, U)$, where $\mathbf{A}$ is a category and $\mathbf{A} \xrightarrow{U} \mathbf{X}$ is a faithful functor. The functor U is called the *forgetful* (or *underlying*) *functor* of the concrete category, and $\mathbf{X}$ is called the *base category* for $(\mathbf{A}, U)$.

(2) A concrete category over **Set** is called a *construct*. ▪

Example 10. There exist the constructs $(\mathbf{Prost}, U)$ of preordered sets, $(\mathbf{QPMet}, U)$ of quasi-pseudo-metric spaces, $(\mathbf{Top}, U)$ of topological spaces, $(\mathbf{App}, U)$ of approach spaces and $(\mathbf{Cls}, U)$ of closure spaces. ▪

Definition 11.

(1) A *source* in a category $\mathbf{X}$ is a pair $(X, (f_i)_{i\in I})$, which contains an $\mathbf{X}$-object X, and a family $(f_i)_{i\in I}$ of $\mathbf{X}$-morphisms $X \xrightarrow{f_i} X_i$ indexed by a class I.

(2) Dual concept: a *sink* in a category $\mathbf{X}$ is a pair $((f_i)_{i\in I}, X)$, which contains an $\mathbf{X}$-object X, and a family $(f_i)_{i\in I}$ of $\mathbf{X}$-morphisms $X_i \xrightarrow{f_i} X$ indexed by a class I. ▪

Remark 12. A more concise notation $(X \xrightarrow{f_i} X_i)_I$ for sources is sometimes used, and dually, for sinks. ▪

Example 13. Products and sums of sets are examples of sources and sinks, respectively. ▪

Definition 14. Let $(\mathbf{A}, U)$ be a concrete category over $\mathbf{X}$.

(1) A source $(A \xrightarrow{f_i} A_i)_I$ in $\mathbf{A}$ is called *(U-)initial* provided that for every source $(B \xrightarrow{g_i} A_i)_I$ in $\mathbf{A}$ and every $\mathbf{X}$-morphism $UB \xrightarrow{h} UA$, which makes the diagram

$$\begin{array}{ccc} UB & & \\ {\scriptstyle h}\downarrow & \searrow^{Ug_i} & \\ UA & \xrightarrow[Uf_i]{} & UA_i \end{array}$$

commute for every $i \in I$, there exists a (necessarily unique) $\mathbf{A}$-morphism $B \xrightarrow{\bar{h}} A$ such that $U\bar{h} = h$. Dual concept: *(U-)final* sink.

(2) An $\mathbf{A}$-object A is called *indiscrete* provided that the empty source $(A, \varnothing)$ is initial. Dual concept: *discrete* object. ▪

Example 15.

(1) A source $((X, \leqslant) \xrightarrow{f_i} (X_i, \leqslant_i))_I$ in **Prost** is initial iff the preorder $\leqslant$ on X is defined by $x \leqslant y$ iff $f_i(x) \leqslant_i f_i(y)$ for every $i \in I$. A preordered set $(X, \leqslant)$ is indiscrete iff $\leqslant \, = X \times X$.

(2) A sink $((X_i, \leqslant_i) \xrightarrow{f_i} (X, \leqslant))_I$ in **Prost** is final iff the preorder $\leqslant$ on X is generated by the union $(\bigcup_{i\in I} f_i \times f_i(\leqslant_i)) \bigcup \Delta_X$, where $\Delta_X = \{(x,x) \mid x \in X\}$. A preordered set $(X, \leqslant)$ is discrete iff $\leqslant \, = \Delta_X$. ▪

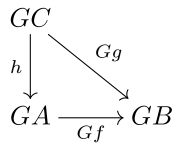
Definition 16. Given a functor $\mathbf{X} \xrightarrow{F} \mathbf{Y}$, an *(F-)structured source* is a source in $\mathbf{Y}$, which has the form $(Y \xrightarrow{f_i} FX_i)_I$. Dual concept: *(F-)costructured sink.* ▪

Definition 17. Let $(\mathbf{A}, U)$ be a concrete category over $\mathbf{X}$.

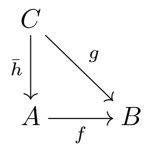
(1) An *initial lift* of a structured source $(X \xrightarrow{f_i} UA_i)_I$ is a source $(A \xrightarrow{\bar{f}_i} A_i)_I$ in $\mathbf{A}$, which is initial, and, moreover, $U(A \xrightarrow{\bar{f}_i} A_i) = X \xrightarrow{f_i} UA_i$ for every $i \in I$. Dual concept: *final lift* of a costructured sink.
(2) $(\mathbf{A}, U)$ is called *topological* provided that every structured source has a unique initial lift. ∎

Definition 18.

(1) Given a functor $\mathbf{A} \xrightarrow{G} \mathbf{X}$, an $\mathbf{A}$-morphism $A \xrightarrow{f} B$ is called *G-initial* (or *G-Cartesian*) provided that for every $\mathbf{A}$-morphism $C \xrightarrow{g} B$ and every $\mathbf{X}$-morphism $GC \xrightarrow{h} GA$, which makes the diagram



commute, there exists a unique $\mathbf{A}$-morphism $C \xrightarrow{\bar{h}} A$ such that the diagram



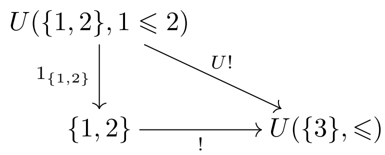
commutes and, moreover, $G\bar{h} = h$. Dual concept: *G-final* (or *G-co-Cartesian*) morphism.
(2) A functor $\mathbf{A} \xrightarrow{G} \mathbf{X}$ is called a *fibration* provided that every G-structured morphism $X \xrightarrow{f} GB$ has a G-initial lift, i.e., there exists a G-initial morphism $A \xrightarrow{\bar{f}} B$ such that $G\bar{f} = f$. Dual concept: *cofibration.* ∎

Remark 19. Forgetful functors of topological categories are particular cases of fibrations. ∎

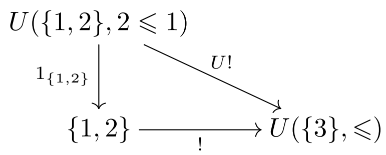
Theorem 20. *Every topological category has unique final lifts of costructured sinks.*

Example 21.

(1) The constructs **Prost**, **QPMet**, **Top**, **App**, and **Cls** are topological.
(2) The construct **Pos** of partially ordered sets is not topological. Observe that there exists no initial lift of, e.g., the unique map $\{1,2\} \xrightarrow{!} U(\{3\}, \leqslant)$, since



and



cannot be both lifted to **Pos** by the same partial order $\leqslant$ on $\{1,2\}$ ($1 \leqslant 2$ and $2 \leqslant 1$ will imply $1 = 2$). ∎

1.2.2. Initial structures in the category $(\mathbb{T}, V)$-**Cat**

Remark 22. The category $(\mathbb{T}, V)$-**Cat** is a construct, where the forgetful functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{U}$ **Set** is given by $U((X,a) \xrightarrow{f} (Y,b)) = X \xrightarrow{f} Y$. ∎

Lemma 23. *Given a V-relation $Y \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!\!\;\mapsto} Z$ and maps $X \xrightarrow{f} Y$, $W \xrightarrow{h} Z$, it follows that*

$$r \cdot f(x,z) = r(f(x), z) \quad \textit{and} \quad h^\circ \cdot r(y,w) = r(y, h(w)) \tag{1.2}$$

for every $x \in X$, $z \in Z$, $w \in W$.

Proof. Observe that $r \cdot f(x,z) = \bigvee_{y\in Y} f_\circ(x,y) \otimes r(y,z) = \bigvee_{f(x)=y} r(y,z) = r(f(x),y)$ and $h^\circ \cdot r(y,w) = \bigvee_{z\in Z} r(y,z) \otimes h^\circ(z,w) = \bigvee_{z\in Z} r(y,z) \otimes h_\circ(w,z) = \bigvee_{h(w)=z} r(y,z) = r(y,h(w))$. $\square$

Lemma 24. *Given a family of V-relations $\{X \overset{r_i}{\longrightarrow\!\!\!\!\!\!\!\!\;\;|\;\;} Y \mid i \in I\}$ and a map $Z \xrightarrow{f} X$, it follows that*

$$(\bigwedge_{i\in I} r_i) \cdot f = \bigwedge_{i\in I} (r_i \cdot f), \tag{1.3}$$

i.e., V-relational composition with maps is distributive over $\bigwedge$ from the right.

Proof. For every $z \in Z$ and every $y \in Y$, it follows that $(\bigwedge_{i\in I} r_i) \cdot f(z,y) \overset{(1.2)}{=} (\bigwedge_{i\in I} r_i)(f(z),y) = \bigwedge_{i\in I} r_i(f(z),y) \overset{(1.2)}{=} \bigwedge_{i\in I} r_i \cdot f(z,y) = (\bigwedge_{i\in I} r_i \cdot f)(z,y)$. $\square$

Theorem 25. *The category $(\mathbb{T},V)$-**Cat** is a topological construct.*

Proof. Given a structured source $(X \xrightarrow{f_i} U(X_i,a_i))_I$, the required initial structure on X is given by $a = \bigwedge_{i\in I} f_i^\circ \cdot a_i \cdot Tf_i$, or, in pointwise notation,

$$a(\mathfrak{x},x) = \bigwedge_{i\in I} a_i(Tf_i(\mathfrak{x}), f_i(x)) \tag{1.4}$$

for every $\mathfrak{x} \in TX$ and every $x \in X$.

To show $1_X \leqslant a \cdot e_X$ (reflexivity), observe that $a \cdot e_X = (\bigwedge_{i\in I} f_i^\circ \cdot a_i \cdot Tf_i) \cdot e_X \overset{(1.3)}{=} \bigwedge_{i\in I} (f_i^\circ \cdot a_i \cdot Tf_i \cdot e_X) \overset{(\dagger)}{=} \bigwedge_{i\in I} (f_i^\circ \cdot a_i \cdot e_{X_i} \cdot f_i) \overset{(\dagger\dagger)}{\geqslant} \bigwedge_{i\in I} f_i^\circ \cdot 1_X \cdot f_i = \bigwedge_{i\in I} f_i^\circ \cdot f_i \geqslant \bigwedge_{i\in I} 1_X \geqslant 1_X$, where $(\dagger)$ relies on the fact that $1_{\mathbf{Set}} \xrightarrow{e} T$ is a natural transformation, and $(\dagger\dagger)$ uses the fact that (X_i,a_i) is a $(\mathbb{T},V)$-category for every $i \in I$.

To show $a \cdot \hat{T}a \leqslant a \cdot m_X$ (transitivity), observe that $a \cdot \hat{T}a = (\bigwedge_{i\in I} f_i^\circ \cdot a_i \cdot Tf_i) \cdot \hat{T}(\bigwedge_{i\in I} f_i^\circ \cdot a_i \cdot Tf_i) \leqslant (\bigwedge_{i\in I} f_i^\circ \cdot a_i \cdot Tf_i) \cdot (\bigwedge_{i\in I} \hat{T}(f_i^\circ \cdot a_i \cdot Tf_i)) \overset{(1.1)}{=} (\bigwedge_{i\in I} f_i^\circ \cdot a_i \cdot Tf_i) \cdot (\bigwedge_{i\in I} (Tf_i)^\circ \cdot \hat{T}a_i \cdot TTf_i) \leqslant \bigwedge_{i\in I} (f_i^\circ \cdot a_i \cdot Tf_i \cdot (Tf_i)^\circ \cdot \hat{T}a_i \cdot TTf_i) \leqslant \bigwedge_{i\in I} (f_i^\circ \cdot a_i \cdot 1_{TX_i} \cdot \hat{T}a_i \cdot TTf_i) = \bigwedge_{i\in I} (f_i^\circ \cdot a_i \cdot \hat{T}a_i \cdot TTf_i) \overset{(\dagger)}{\leqslant} \bigwedge_{i\in I} (f_i^\circ \cdot a_i \cdot m_{X_i} \cdot TTf_i) \overset{(\dagger\dagger)}{=} \bigwedge_{i\in I} (f_i^\circ \cdot a_i \cdot Tf_i \cdot m_X) \overset{(1.3)}{=} (\bigwedge_{i\in I} f_i^\circ \cdot a_i \cdot Tf_i) \cdot m_X = a \cdot m_X$, where $(\dagger)$ relies on the fact that (X_i,a_i) is a $(\mathbb{T},V)$-category for every $i \in I$, and $(\dagger\dagger)$ employs the fact that $TT \xrightarrow{m} T$ is a natural transformation.

To verify that $U(X,a) \xrightarrow{f_j} U(X_j,a_j)$ is a $(\mathbb{T},V)$-functor, i.e., $f_j \cdot a \leqslant a_j \cdot Tf_j$, for every $j \in I$, observe that given $j \in I$, $f_j \cdot a = f_j \cdot (\bigwedge_{i\in I} f_i^\circ \cdot a_i \cdot Tf_i) \leqslant f_j \cdot f_j^\circ \cdot a_j \cdot Tf_j \leqslant 1_{X_j} \cdot a_j \cdot Tf_j = a_j \cdot Tf_j$.

To check that the source $((X,a) \xrightarrow{f_i} (X_i,a_i))_I$ is initial, observe that given any other source $((Y,b) \xrightarrow{g_i} (X_i,a_i))_I$ in $(\mathbb{T},V)$-**Cat** and any map $Y \xrightarrow{h} X$ such that the triangle

$$\begin{array}{ccc} U(Y,b) & & \\ {\scriptstyle h}\downarrow & \searrow^{Ug_i} & \\ U(X,a) & \xrightarrow[Uf_i]{} & U(X_i,a_i) \end{array} \tag{1.5}$$

commutes for every $i \in I$, it follows that $U(Y,b) \xrightarrow{h} U(X,a)$ is a $(\mathbb{T},V)$-functor, i.e., $h \cdot b \leqslant a \cdot Th$, since $a \cdot Th = (\bigwedge_{i\in I} f_i^\circ \cdot a_i \cdot Tf_i) \cdot Th \overset{(1.3)}{=} \bigwedge_{i\in I} (f_i^\circ \cdot a_i \cdot Tf_i \cdot Th) = \bigwedge_{i\in I} (f_i^\circ \cdot a_i \cdot T(f_i \cdot h)) \overset{(1.5)}{=} \bigwedge_{i\in I} (f_i^\circ \cdot a_i \cdot Tg_i) \overset{(\dagger)}{\geqslant} \bigwedge_{i\in I} (f_i^\circ \cdot g_i \cdot b) \overset{(\dagger\dagger)}{\geqslant} \bigwedge_{i\in I} h \cdot b \geqslant h \cdot b$, where $(\dagger)$ relies on the fact that $(Y,b) \xrightarrow{g_i} (X_i,a_i)$ is a $(\mathbb{T},V)$-functor for every $i \in I$, and $(\dagger\dagger)$ uses the fact that for every $i \in I$, $f_i \cdot h = g_i$ implies $h = 1_X \cdot h \leqslant f_i^\circ \cdot f_i \cdot h = f_i^\circ \cdot g_i$. $\square$

Remark 26. Despite their existence, there is no convenient formula for the explicit description of final structures in the category $(\mathbb{T}, V)$-**Cat**. In particular, given a costructured sink $(U(X_i, a_i) \xrightarrow{f_i} X)_I$ in $(\mathbb{T}, V)$-**Cat**, the V-relation $TX \overset{a}{\longrightarrow\!\!\!\!\!\!+\,} X$ defined by $a = \bigvee_{i\in I} f_i \cdot a_i \cdot (Tf_i)^\circ$, in general, fails to provide a $(\mathbb{T}, V)$-category structure on X. More precisely, if $I = \varnothing$, then $a = \underline{\bot_V}$ (the constant map with value $\bot_V$), and therefore, $a \cdot e_X = \underline{\bot_V} \cdot e_X = \underline{\bot_V}$. The condition $1_X \leqslant a \cdot e_X$ holds then iff V is a singleton. ■

Example 27. Formula (1.4) provides the following examples of initial structures:

(1) $((X, \leqslant) \xrightarrow{f_i} (X_i, \leqslant_i))_I$ in **Prost**: given $x, y \in X$, $x \leqslant y$ iff $f_i(x) \leqslant_i f_i(y)$ for every $i \in I$;

(2) $((X, \rho) \xrightarrow{f_i} (X_i, \rho_i))_I$ in **QPMet**: given $x, y \in X$, $\rho(x, y) = \sup_{i\in I} \rho_i(f_i(x), f_i(y))$;

(3) $((X, a) \xrightarrow{f_i} (X_i, a_i))_I$ in **Top**: given $\mathfrak{x} \in \beta X$ and $x \in X$, $\mathfrak{x}\, a\, x$ iff $\beta f_i(\mathfrak{x})\, a_i\, f_i(x)$ for every $i \in I$;

(4) $((X, a) \xrightarrow{f_i} (X_i, a_i))_I$ in **App**: given $\mathfrak{x} \in \beta X$ and $x \in X$, $a(\mathfrak{x}, x) = \sup_{i\in I} a_i(\beta f_i(\mathfrak{x}), f_i(x))$;

(5) $((X, c) \xrightarrow{f_i} (X_i, c_i))_I$ in **Cls**: given $A \in PX$ and $x \in X$, $x \in c(A)$ iff $f_i(x) \in c_i(f_i(A))$ for every $i \in I$. ■

Corollary 28. *The category $(\mathbb{T}, V)$-**Cat** is both complete and cocomplete, well-powered and co-well-powered, and has both a separator and a coseparator. The underlying functor U has both a right- and a left-adjoint.*

Remark 29. The constructs **Prost**, **QPMet**, **Top**, **App**, and **Cls** have the properties of Corollary 28. ■

Lemma 30. *Given a lax extension $\hat{\mathbb{T}} = (\hat{T}, m, e)$ to the category V-**Rel** of a monad $\mathbb{T} = (T, m, e)$ on the category **Set**, it follows that $\hat{T}1_X = \hat{T}(e_X^\circ) \cdot m_X^\circ$ for every set X.*

Proof. First, observe that $\hat{T}1_X = \hat{T}1_X \cdot 1_{TX} = \hat{T}1_X \cdot 1_{TX}^\circ \overset{(\dagger)}{=} \hat{T}1_X \cdot (m_X \cdot Te_X)^\circ = \hat{T}1_X \cdot (Te_X)^\circ \cdot m_X^\circ \leqslant \hat{T}1_X \cdot \hat{T}(e_X^\circ) \cdot m_X^\circ \leqslant \hat{T}(1_X \cdot e_X^\circ) \cdot m_X^\circ = \hat{T}(e_X^\circ) \cdot m_X^\circ$, i.e., $\hat{T}1_X \leqslant \hat{T}(e_X^\circ) \cdot m_X^\circ$, where (†) relies on the property $m_X \cdot Te_X = 1_{TX}$ of the monad $\mathbb{T}$.

Second, observe that $\hat{T}(e_X^\circ) \cdot m_X^\circ = \hat{T}(e_X^\circ \cdot 1_{TX}) \cdot m_X^\circ = \hat{T}(e_X^\circ \cdot T1_X) \cdot m_X^\circ \leqslant \hat{T}(e_X^\circ \cdot \hat{T}1_X) \cdot m_X^\circ \overset{\text{Lemma 2}}{=} (Te_X)^\circ \cdot \hat{T}\hat{T}1_X \cdot m_X^\circ \overset{(\dagger)}{\leqslant} (Te_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X = (m_X \cdot Te_X)^\circ \cdot \hat{T}1_X \overset{(\dagger\dagger)}{=} 1_{TX}^\circ \cdot \hat{T}1_X = 1_{TX} \cdot \hat{T}1_X = \hat{T}1_X$, i.e., $\hat{T}(e_X^\circ) \cdot m_X^\circ \leqslant \hat{T}1_X$, where (†) follows from the fact that $m_X \cdot \hat{T}\hat{T}1_X \leqslant \hat{T}1_X \cdot m_X$ (since m is an oplax natural transformation) implies $\hat{T}\hat{T}1_X \cdot m_X^\circ \leqslant m_X^\circ \cdot m_X \cdot \hat{T}\hat{T}1_X \cdot m_X^\circ \leqslant m_X^\circ \cdot \hat{T}1_X \cdot m_X \cdot m_X^\circ \leqslant m_X^\circ \cdot \hat{T}1_X$, i.e., $\hat{T}\hat{T}1_X \cdot m_X^\circ \leqslant m_X^\circ \cdot \hat{T}1_X$, and (††) relies on the property $m_X \cdot Te_X = 1_{TX}$ of the monad $\mathbb{T}$. □

Lemma 31. *Given a $(\mathbb{T}, V)$-category (X, a), it follows that $e_X^\circ \leqslant a$.*

Proof. Observe that $1_X \leqslant a \cdot e_X$ implies $e_X^\circ \leqslant a \cdot e_X \cdot e_X^\circ \leqslant a$, i.e., $e_X^\circ \leqslant a$. □

Lemma 32. *Given a $(\mathbb{T}, V)$-category (X, a), it follows that $a \cdot \hat{T}1_X = a$.*

Proof. First, observe that $a = a \cdot 1_{TX} = a \cdot T1_X \leqslant a \cdot \hat{T}1_X$, i.e., $a \leqslant a \cdot \hat{T}1_X$. Second, observe that $a \cdot \hat{T}1_X \overset{\text{Lemma 30}}{=} a \cdot \hat{T}(e_X^\circ) \cdot m_X^\circ \overset{\text{Lemma 31}}{\leqslant} a \cdot \hat{T}a \cdot m_X^\circ \overset{(\dagger)}{\leqslant} a \cdot m_X \cdot m_X^\circ \leqslant a$, i.e., $a \cdot \hat{T}1_X \leqslant a$, where (†) relies on the transitivity property of the $(\mathbb{T}, V)$-category (X, a). □

Theorem 33. *Let X be a set.*

(1) *The discrete $(\mathbb{T}, V)$-category structure on X is given by $1_X^\sharp = e_X^\circ \cdot \hat{T}1_X$. The functor **Set** $\xrightarrow{D}$ $(\mathbb{T}, V)$-**Cat**, defined by $D(X \xrightarrow{f} Y) = (X, 1_X^\sharp) \xrightarrow{f} (Y, 1_Y^\sharp)$, is a left adjoint to the forgetful functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{U}$ **Set**.*

(2) *The indiscrete $(\mathbb{T}, V)$-category structure on X is given by the constant map $TX \times X \xrightarrow{\underline{\top_V}} V$ with value $\top_V$. The functor **Set** $\xrightarrow{I}$ $(\mathbb{T}, V)$-**Cat**, defined by $I(X \xrightarrow{f} Y) = (X, \underline{\top_V}) \xrightarrow{f} (Y, \underline{\top_V})$, is a right adjoint to the forgetful functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{U}$ **Set**.*

Proof. For item (1): To show that $(X, 1_X^\sharp)$ is a $(\mathbb{T}, V)$-category, one has to check that $1_X^\sharp$ is both reflexive and transitive. To verify $1_X \leqslant 1_X^\sharp \cdot e_X$ (reflexivity), observe that $1_X^\sharp \cdot e_X = e_X^\circ \cdot \hat{T}1_X \cdot e_X \geqslant e_X^\circ \cdot T1_X \cdot e_X \geqslant e_X^\circ \cdot e_X \geqslant 1_X$. To verify $1_X^\sharp \cdot \hat{T}1_X^\sharp \leqslant 1_X^\sharp \cdot m_X$ (transitivity), observe that $1_X^\sharp \cdot \hat{T}1_X^\sharp = e_X^\circ \cdot \hat{T}1_X \cdot \hat{T}(e_X^\circ \cdot \hat{T}1_X) \leqslant e_X^\circ \cdot \hat{T}(1_X \cdot e_X^\circ \cdot \hat{T}1_X) = e_X^\circ \cdot \hat{T}(e_X^\circ \cdot \hat{T}1_X) \overset{\text{Lemma 2}}{=} e_X^\circ \cdot (Te_X)^\circ \cdot \hat{T}\hat{T}1_X \overset{(\dagger)}{\leqslant} e_X^\circ \cdot (Te_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X \cdot m_X = e_X^\circ \cdot (m_X \cdot Te_X)^\circ \cdot \hat{T}1_X \cdot m_X \overset{(\dagger\dagger)}{=} e_X^\circ \cdot (1_{TX})^\circ \cdot \hat{T}1_X \cdot m_X = e_X^\circ \cdot \hat{T}1_X \cdot m_X = 1_X^\sharp \cdot m_X$, where $(\dagger)$ follows from the fact that $m_X \cdot \hat{T}\hat{T}1_X \leqslant \hat{T}1_X \cdot m_X$ (since m is oplax) implies $\hat{T}\hat{T}1_X \leqslant m_X^\circ \cdot m_X \cdot \hat{T}\hat{T}1_X \leqslant m_X^\circ \cdot \hat{T}1_X \cdot m_X$, i.e., $\hat{T}\hat{T}1_X \leqslant m_X^\circ \cdot \hat{T}1_X \cdot m_X$, and $(\dagger\dagger)$ relies on the property $m_X \cdot Te_X = 1_{TX}$ of the monad $\mathbb{T}$.

To show that $1_X^\sharp$ is the discrete structure on X, one has to check that given a $(\mathbb{T}, V)$-category (Y, b), every map $U(X, 1_X^\sharp) \xrightarrow{f} U(Y, b)$ provides a $(\mathbb{T}, V)$-functor $(X, 1_X^\sharp) \xrightarrow{f} (Y, b)$, i.e., $f \cdot 1_X^\sharp \leqslant b \cdot Tf$. Observe that $f \cdot 1_X^\sharp = f \cdot e_X^\circ \cdot \hat{T}1_X \overset{(\dagger)}{\leqslant} e_Y^\circ \cdot Tf \cdot e_X \cdot e_X^\circ \cdot \hat{T}1_X \leqslant e_Y^\circ \cdot Tf \cdot \hat{T}1_X \overset{\text{Lemma 31}}{\leqslant} b \cdot Tf \cdot \hat{T}1_X \leqslant b \cdot \hat{T}f \cdot \hat{T}1_X \leqslant b \cdot \hat{T}(f \cdot 1_X) = b \cdot \hat{T}f = b \cdot \hat{T}(1_Y \cdot f) \overset{\text{Lemma 2}}{=} b \cdot \hat{T}1_Y \cdot Tf \overset{\text{Lemma 32}}{=} b \cdot Tf$, where $(\dagger)$ relies on the fact that $e_Y \cdot f = Tf \cdot e_X$ implies $f \leqslant e_Y^\circ \cdot e_Y \cdot f = e_Y^\circ \cdot Tf \cdot e_X$, i.e., $f \leqslant e_Y^\circ \cdot Tf \cdot e_X$.

As a consequence of the above paragraph, $(X, 1_X^\sharp) \xrightarrow{f} (Y, 1_Y^\sharp)$ is a $(\mathbb{T}, V)$-functor for every map $X \xrightarrow{f} Y$.

For item (2): To show that $(X, \underline{\top_V})$ is a $(\mathbb{T}, V)$-category, observe that $1_X \leqslant \underline{\top_V} \cdot e_X$ (reflexivity) is implied by $\underline{\top_V} \cdot e_X(x, x) = \underline{\top_V}(e_X(x), x) = \top_V \geqslant k$ for every $x \in X$, and $\underline{\top_V} \cdot \hat{T}\underline{\top_V} \leqslant \underline{\top_V} \cdot m_X$ (transitivity) is implied by $\underline{\top_V} \cdot m_X(\mathfrak{X}, x) = \underline{\top_V}(m_X(\mathfrak{X}), x) = \top_V \geq \underline{\top_V} \cdot \hat{T}\underline{\top_V}(\mathfrak{X}, x)$ for every $\mathfrak{X} \in TTX$ and every $x \in X$.

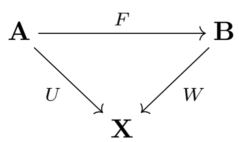
To show that $\underline{\top_V}$ is the indiscrete structure on X, one has to check that given a $(\mathbb{T}, V)$-category (Y, b), every map $U(Y, b) \xrightarrow{f} U(X, \underline{\top_V})$ provides a $(\mathbb{T}, V)$-functor $(Y, b) \xrightarrow{f} (X, \underline{\top_V})$, i.e., $f \cdot b \leqslant \underline{\top_V} \cdot Tf$. Observe that for every $\mathfrak{y} \in TY$ and every $x \in X$, it follows that $\underline{\top_V} \cdot Tf(\mathfrak{y}, x) = \underline{\top_V}(Tf(\mathfrak{y}), x) = \top_V \geqslant f \cdot b(\mathfrak{y}, x)$.

As a result of the above paragraph, $(X, \underline{\top_V}) \xrightarrow{f} (Y, \underline{\top_V})$ is a $(\mathbb{T}, V)$-functor for every map $X \xrightarrow{f} Y$. □

Example 34. In the construct **Top** of topological spaces, the discrete (resp. indiscrete) structure on a set X is given by the powerset PX (resp. the topology $\{\varnothing, X\}$). ▪

2. Preordered sets induced by $(\mathbb{T}, V)$-categories

Definition 35. Given concrete categories $(\mathbf{A}, U)$ and $(\mathbf{B}, W)$ over $\mathbf{X}$, a *concrete functor from* $(\mathbf{A}, U)$ *to* $(\mathbf{B}, W)$ is a functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$ such that the triangle



commutes. ▪

Remark 36. Every topological space (X, τ) gives rise to the *underlying* (or *induced*) *preorder* on X, which is given by $x \leqslant_\tau y$ iff $y \in cl(\{x\})$ (equivalently, for every $A \in \tau$, if $y \in A$, then $x \in A$). Observe that the dual of this preorder is called the *specialization preorder* of a topological space (the specialization preorder is a partial order iff (X, τ) is a T_0*-space*, i.e., for every pair of distinct points $x, y \in X$ there exists $A \in \tau$ containing exactly one of them). A continuous map $(X, \tau) \xrightarrow{f} (Y, \sigma)$ provides an order-preserving map $(X, \leqslant_\tau) \xrightarrow{f} (Y, \leqslant_\sigma)$. Thus, one gets a concrete functor $\mathbf{Top} \xrightarrow{Ind} \mathbf{Prost}$. Since **Top** is an instance of $(\mathbb{T}, V)$-**Cat**, one could ask for a similar functor in case of the latter category. ▪

Theorem 37. *Given a* $(\mathbb{T}, V)$*-category* (X, a)*, the binary relation* $\leqslant_a$ *on* X*, which is defined by* $x \leqslant_a y$ *iff* $k \leqslant a(e_X(x), y)$*, provides a preordered set* $(X, \leqslant_a)$*.*

Proof. Reflexivity follows directly from the property $1_X \leqslant a \cdot e_X$ of the $(\mathbb{T}, V)$-category (X, a). To show transitivity, notice that since e is an oplax natural transformation, the $(\mathbb{T}, V)$-category (X, a) has the property

$$a \leqslant e_X^\circ \cdot \hat{T}a \cdot e_{TX}. \tag{2.1}$$

Given $x, y, z \in X$ such that $x \leqslant_a y$ and $y \leqslant_a z$, it follows that $k = k \otimes k \leqslant a(e_X(x), y) \otimes a(e_X(y), z) \overset{(2.1)}{\leqslant} e_X^\circ \cdot \hat{T}a \cdot e_{TX}(e_X(x), y) \otimes a(e_X(y), z) \overset{(1.2)}{=} \hat{T}a(e_{TX}(e_X(x)), e_X(y)) \otimes a(e_X(y), z) \leqslant a \cdot \hat{T}a(e_{TX}(e_X(x)), z) \overset{a \cdot \hat{T}a \leqslant a \cdot m_X}{\leqslant} a \cdot m_X(e_{TX}(e_X(x)), z) \overset{(1.2)}{=} a(m_X \cdot e_{TX}(e_X(x)), z) \overset{m_X \cdot e_{TX} = 1_{TX}}{=} a(e_X(x), z)$, and therefore, $x \leqslant_a z$. □

Remark 38. Given a $(\mathbb{T}, V)$-category (X, a), the preorder $\leqslant_a$ on the set X defined in Theorem 37 is called the *underlying preorder* induced by a (or the *induced preorder* for short). ▪

Theorem 39. *Every $(\mathbb{T}, V)$-functor $(X, a) \xrightarrow{f} (Y, b)$ provides an order-preserving map $(X, \leqslant_a) \xrightarrow{f} (Y, \leqslant_b)$.*

Proof. The $(\mathbb{T}, V)$-functor condition $f \cdot a \leqslant b \cdot Tf$ can be rewritten as $a \leqslant f^\circ \cdot b \cdot Tf$. Given $x, z \in X$ such that $x \leqslant_a z$, it follow that $k \leqslant a(e_X(x), z) \leqslant f^\circ \cdot b \cdot Tf(e_X(x), z) \overset{(1.2)}{=} b(Tf \cdot e_X(x), f(z)) \overset{Tf \cdot e_X = e_Y \cdot f}{=} b(e_Y(f(x)), f(z))$, and therefore, $f(x) \leqslant_b f(z)$. □

Corollary 40. *There exists a concrete functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{Ind}$ **Prost**, which is defined by $Ind((X, a) \xrightarrow{f} (Y, b)) = (X, \leqslant_a) \xrightarrow{f} (Y, \leqslant_b)$.*

Proof. Follows from Theorems 37, 39. □

Example 41. Corollary 40 provides the following functors:

(1) the identity functor **Prost** $\xrightarrow{1_{\mathbf{Prost}}}$ **Prost**;

(2) the functor **QPMet** $\xrightarrow{Ind}$ **Prost**, which is given by $Ind(X, \rho) = (X \leqslant_\rho)$, where $x \leqslant_\rho y$ iff $\rho(x, y) = 0$;

(3) the functor **Top** $\xrightarrow{Ind}$ **Prost**, which is given by $Ind(X, a) = (X, \leqslant_a)$, where $x \leqslant_a y$ iff $\dot{x}\, a\, y$ iff $y \in cl(\{x\})$, which is the induced preorder of Remark 36;

(4) the functor **App** $\xrightarrow{Ind}$ **Prost**, which is given by $Ind(X, \delta) = (X, \leqslant_\delta)$, where $x \leqslant_\delta y$ iff $\delta(y, \{x\}) = 0$;

(5) the functor **Cls** $\xrightarrow{Ind}$ **Prost**, which is given by $Ind(X, c) = (X, \leqslant_c)$, where $x \leqslant_c y$ iff $y \in c(\{x\})$. ▪

3. Algebraic functors

Definition 42. Given monads $\mathbb{T} = (T, m, e)$ and $\mathbb{S} = (S, n, d)$ on a category $\mathbf{X}$, a *morphism of monads* $\mathbb{S} \xrightarrow{\alpha} \mathbb{T}$ is a natural transformation $S \xrightarrow{\alpha} T$, which makes the diagrams

$$\begin{array}{ccc} SS & \xrightarrow{\alpha \circ \alpha} & TT \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle m} \\ S & \xrightarrow[\alpha]{} & T \end{array} \qquad\qquad \begin{array}{ccc} & 1_{\mathbf{X}} & \\ {\scriptstyle d}\swarrow & & \searrow{\scriptstyle e} \\ S & \xrightarrow[\alpha]{} & T \end{array} \tag{3.1}$$

commute, where $\alpha \circ \alpha$ is defined by the diagonal of the commutative diagram

$$\begin{array}{ccc} SS & \xrightarrow{S\alpha} & ST \\ {\scriptstyle \alpha S}\downarrow & \searrow & \downarrow{\scriptstyle \alpha T} \\ TS & \xrightarrow[T\alpha]{} & TT. \end{array}$$

▪

Example 43. There exists a unique monad morphism from the identity monad $\mathbb{I}$ on a category $\mathbf{X}$ to every monad $\mathbb{T} = (T, m, e)$ on $\mathbf{X}$, which is given by $1_{\mathbf{X}} \xrightarrow{e} T$. ▪

Definition 44. Let $\hat{S}$ and $\hat{T}$ be lax extensions to V-**Rel** of functors S and T on **Set**, respectively. A *morphism of lax extensions of functors* $(S, \hat{S}) \xrightarrow{\alpha} (T, \hat{T})$ is an oplax natural transformation $\hat{S} \xrightarrow{\alpha} \hat{T}$, which means

$$\begin{array}{ccc} SX & \xrightarrow{\alpha_X} & TX \\ {\scriptstyle \hat{S}r} \big\downarrow & \leqslant & \big\downarrow {\scriptstyle \hat{T}r} \\ SY & \xrightarrow[\alpha_Y]{} & TY \end{array} \tag{3.2}$$

for every V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$. ▪

Definition 45. Let $\hat{\mathbb{S}}$ and $\hat{\mathbb{T}}$ be lax extensions to V-**Rel** of monads $\mathbb{S} = (S, n, d)$ and $\mathbb{T} = (T, m, e)$ on **Set**, respectively. A *morphism of lax extensions of monads* $\hat{\mathbb{S}} \xrightarrow{\alpha} \hat{\mathbb{T}}$ is a monad morphism $\mathbb{S} \xrightarrow{\alpha} \mathbb{T}$, which, additionally, is a morphism of lax extensions $(S, \hat{S}) \xrightarrow{\alpha} (T, \hat{T})$. ▪

Theorem 46. *Every morphism of lax extensions of monads $\hat{\mathbb{S}} \xrightarrow{\alpha} \hat{\mathbb{T}}$ gives rise to a concrete functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{A_\alpha}$ $(\mathbb{S}, V)$-**Cat**, which is defined by $A_\alpha((X, a) \xrightarrow{f} (Y, b)) = (X, a \cdot \alpha_X) \xrightarrow{f} (Y, b \cdot \alpha_Y)$, and which, additionally, preserves initial sources.*

Proof. To show that $(X, a \cdot \alpha_X)$ provides an $(\mathbb{S}, V)$-category, notice that $1_X \leqslant a \cdot e_X \overset{(3.1)}{=} a \cdot \alpha_X \cdot d_X$, and, additionally, $a \cdot \alpha_X \cdot \hat{S}(a \cdot \alpha_X) \overset{(1.1)}{=} a \cdot \alpha_X \cdot \hat{S}a \cdot S\alpha_X \overset{(3.2)}{\leqslant} a \cdot \hat{T}a \cdot \alpha_{TX} \cdot S\alpha_X \overset{a \cdot \hat{T}a \leqslant a \cdot m_X}{\leqslant} a \cdot m_X \cdot \alpha_{TX} \cdot S\alpha_X \overset{(3.1)}{=} a \cdot \alpha_X \cdot n_X$. To show that $U(X, a \cdot \alpha_X) \xrightarrow{f} U(Y, b \cdot \alpha_Y)$ provides an $(\mathbb{S}, V)$-functor, notice that $f \cdot a \cdot \alpha_X \leqslant b \cdot Tf \cdot \alpha_X \overset{Tf \cdot \alpha_X = \alpha_Y \cdot Sf}{=} b \cdot \alpha_Y \cdot Sf$.

To show the second statement, notice that given an initial source $((X, a) \xrightarrow{f_i} (X_i, a_i))_{i \in I}$ in $(\mathbb{T}, V)$-**Cat**, by Theorem 25, it follows that $a = \bigwedge_{i \in I} f_i^\circ \cdot a_i \cdot Tf_i$. Applying the functor A_α, one gets an initial source $((X, a \cdot \alpha_X) \xrightarrow{f_i} (X_i, a_i \cdot \alpha_{X_i}))_{i \in I}$ in $(\mathbb{S}, V)$-**Cat**, since $\bigwedge_{i \in I} f_i^\circ \cdot (a_i \cdot \alpha_{X_i}) \cdot Sf_i = \bigwedge_{i \in I} f_i^\circ \cdot a_i \cdot (\alpha_{X_i} \cdot Sf_i) \overset{\alpha_{X_i} \cdot Sf_i = Tf_i \cdot \alpha_X}{=} \bigwedge_{i \in I} f_i^\circ \cdot a_i \cdot (Tf_i \cdot \alpha_X) \overset{(1.3)}{=} (\bigwedge_{i \in I} f_i^\circ \cdot a_i \cdot Tf_i) \cdot \alpha_X = a \cdot \alpha_X$. □

Remark 47. A_α is called the *algebraic functor* associated with α. ▪

Theorem 48. *Every lax extension $\hat{\mathbb{T}}$ of a monad $\mathbb{T} = (T, m, e)$ on **Set** provides a morphism of lax extensions of monads $\mathbb{I} \xrightarrow{e} \hat{\mathbb{T}}$, and therefore, there exists a concrete functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{A_e}$ V-**Cat**, which is given by $A_e((X, a) \xrightarrow{f} (Y, b)) = (X, a \cdot e_X) \xrightarrow{f} (Y, b \cdot e_Y)$.*

Remark 49. Given a $(\mathbb{T}, V)$-category (X, a), $(X, a \cdot e_X)$ is called the *underlying V-category* of (X, a). ▪

Definition 50. Let $(\mathbf{A}, U)$ and $(\mathbf{B}, W)$ be concrete categories over $\mathbf{X}$. If $\mathbf{A} \xrightarrow{G} \mathbf{B}$ and $\mathbf{B} \xrightarrow{F} \mathbf{A}$ are concrete functors, then the pair (F, G) is a *Galois correspondence* (between $\mathbf{A}$ and $\mathbf{B}$) provided that $UFGA \xrightarrow{1_{UA}} UA$ is an $\mathbf{A}$-morphism for every $\mathbf{A}$-object A (namely, there exists an $\mathbf{A}$-morphism $FGA \xrightarrow{f} A$ such that $Uf = 1_{UA}$), and $WB \xrightarrow{1_{WB}} WGFB$ is a $\mathbf{B}$-morphism for every $\mathbf{B}$-object B. ▪

Remark 51. If $(\mathbf{A}, U)$, $(\mathbf{B}, W)$ are concrete categories over $\mathbf{X}$, and $\mathbf{A} \xrightarrow{G} \mathbf{B}$, $\mathbf{B} \xrightarrow{F} \mathbf{A}$ are concrete functors, then (F, G) is a Galois correspondence iff there exist concrete (i.e., given by the identity $\mathbf{X}$-morphisms) natural transformations η and ε such that $(\eta, \varepsilon) : F \dashv G : \mathbf{A} \to \mathbf{B}$ is an adjoint situation. ▪

Theorem 52. *If (F, G) is a Galois correspondence, then G preserves initial sources.*

Theorem 53. *The algebraic functor A_e has a concrete left adjoint functor V-**Cat** $\xrightarrow{A^\circ}$ $(\mathbb{T},V)$-**Cat** defined by $A^\circ((X,a)\xrightarrow{f}(Y,b)) = (X, e_X^\circ\cdot\hat{T}a)\xrightarrow{f}(Y, e_Y^\circ\cdot\hat{T}b)$. The adjoint situation $A^\circ \dashv A_e$ is concrete (both its unit and co-unit are given by the identity maps), i.e., provides a Galois correspondence (A°, A_e), and therefore, the functor A° preserves final sinks. If the lax extension $\hat{\mathbb{T}}$ of $\mathbb{T}$ satisfies the condition*

$$e_Y^\circ\cdot\hat{T}r\cdot e_X\leqslant r \text{ for every } V\text{-relation } X\overset{r}{\longrightarrow\!\!\!\!\!\!\!+}\,Y, \tag{3.3}$$

then A° is a full embedding.

Proof. To show that $(X, e_X^\circ\cdot\hat{T}a)$ is a $(\mathbb{T},V)$-category, notice that $1_X\leqslant e_X^\circ\cdot e_X \overset{T1_X\leqslant\hat{T}1_X}{\leqslant} e_X^\circ\cdot\hat{T}1_X\cdot e_X \overset{1_X\leqslant a}{\leqslant} e_X^\circ\cdot\hat{T}a\cdot e_X$, and, moreover, $e_X^\circ\cdot\hat{T}a\cdot\hat{T}(e_X^\circ\cdot\hat{T}a) \overset{(1.1)}{=} e_X^\circ\cdot\hat{T}a\cdot(Te_X)^\circ\cdot\hat{T}\hat{T}a \overset{1_{TTX}\leqslant m_X^\circ\cdot m_X}{\leqslant} e_X^\circ\cdot\hat{T}a\cdot(Te_X)^\circ\cdot\hat{T}\hat{T}a\cdot m_X^\circ\cdot m_X \overset{\hat{T}\hat{T}a\cdot m_X^\circ\leqslant m_X^\circ\cdot\hat{T}a}{\leqslant} e_X^\circ\cdot\hat{T}a\cdot(Te_X)^\circ\cdot m_X^\circ\cdot\hat{T}a\cdot m_X = e_X^\circ\cdot\hat{T}a\cdot(m_X\cdot Te_X)^\circ\cdot\hat{T}a\cdot m_X \overset{m_X\cdot Te_X=1_X}{=} e_X^\circ\cdot\hat{T}a\cdot\hat{T}a\cdot m_X \overset{\hat{T}a\cdot\hat{T}a\leqslant\hat{T}(a\cdot a)}{\leqslant} e_X^\circ\cdot\hat{T}(a\cdot a)\cdot m_X \overset{a\cdot a\leqslant a}{\leqslant} e_X^\circ\cdot\hat{T}a\cdot m_X$. To show that $U(X, e_X^\circ\cdot\hat{T}a)\xrightarrow{f} U(Y, e_Y^\circ\cdot\hat{T}b)$ is a $(\mathbb{T},V)$-functor, notice that $f\cdot e_X^\circ\cdot\hat{T}a \overset{f\cdot e_X^\circ\leqslant e_Y^\circ\cdot Tf}{\leqslant} e_Y^\circ\cdot Tf\cdot\hat{T}a \overset{Tf\leqslant\hat{T}f}{\leqslant} e_Y^\circ\cdot\hat{T}f\cdot\hat{T}a \overset{\hat{T}f\cdot\hat{T}a\leqslant\hat{T}(f\cdot a)}{\leqslant} e_Y^\circ\cdot\hat{T}(f\cdot a) \overset{f\cdot a\leqslant b\cdot f}{\leqslant} e_Y^\circ\cdot\hat{T}(b\cdot f) \overset{(1.1)}{=} e_Y^\circ\cdot\hat{T}b\cdot Tf$. For the last statement, notice first that given a V-relation $X\overset{r}{\longrightarrow\!\!\!\!\!\!\!+}\,Y$, it follows that $r \overset{1_Y\leqslant e_Y^\circ\cdot e_Y}{\leqslant} e_Y^\circ\cdot e_Y\cdot r \overset{e_Y\cdot r\leqslant\hat{T}r\cdot e_X}{\leqslant} e_Y^\circ\cdot\hat{T}r\cdot e_X$, which, together with (3.3), implies $r = e_Y^\circ\cdot\hat{T}r\cdot e_X$. To show that A° is an embedding, notice that given a V-category (X,a), $A_eA^\circ(X,a) = A_e(X, e_X^\circ\cdot\hat{T}a) = (X, e_X^\circ\cdot\hat{T}a\cdot e_X) = (X,a)$. To show that A° is full, notice that given a $(\mathbb{T},V)$-functor $(X, e_X^\circ\cdot\hat{T}a)\xrightarrow{f}(Y, e_Y^\circ\cdot\hat{T}b)$, $f\cdot e_X^\circ\cdot\hat{T}a\leqslant e_Y^\circ\cdot\hat{T}b\cdot Tf$ implies $f\cdot e_X^\circ\cdot\hat{T}a\cdot e_X\leqslant e_Y^\circ\cdot\hat{T}b\cdot Tf\cdot e_X = e_Y^\circ\cdot\hat{T}b\cdot e_Y\cdot f$ implies $f\cdot a\leqslant b\cdot f$. □

Remark 54. By Lemma 23, (3.3) is equivalent to $\hat{T}r(e_X(x), e_Y(y))\leqslant r(x,y)$ for every V-relation $X\overset{r}{\longrightarrow\!\!\!\!\!\!\!+}\,Y$.

(1) The lax extension of the identity monad $\mathbb{I}$ on **Set** to the identity monad $\mathbb{I}$ on V-**Rel** satisfies (3.3).
(2) The extension $\hat{\beta}$ to **Rel** (resp. $\bar{\beta}$ to P_+-**Rel**) of the ultrafilter monad β on **Set** satisfies (3.3). Observe that $e_X(x)\,(\hat{\beta}r)\,e_Y(y)$ iff $\dot{x}\,\hat{\beta}r\,\dot{y}$ iff for every $A\in\dot{x}$ and every $B\in\dot{y}$, there exist $x'\in A$ and $y'\in B$ such that $x'\,r\,y'$ iff $x\,r\,y$, since $\{x\}\in\dot{x}$ and $\{y\}\in\dot{y}$.
(3) The extension $\hat{\mathbb{P}}$ to **Rel** of the powerset monad $\mathbb{P}$ on **Set** satisfies (3.3). Observe that $e_X(x)\,(\hat{P}r)\,e_Y(y)$ iff $\{x\}\,\hat{P}r\,\{y\}$ iff for every $y'\in\{y\}$ there exists $x'\in\{x\}$ such that $x'\,r\,y'$ iff $x\,r\,y$.
(4) The largest lax extension $\mathbb{T}^\top$ of a monad $\mathbb{T}$ on **Set** does not satisfy (3.3). Observe that for the V-relation $\{*\}\overset{r}{\longrightarrow\!\!\!\!\!\!\!+}\,\{*\}$ with $r(*,*)=\bot_V$, it follows that $\hat{T}^\top r(e_{\{*\}}(*), e_{\{*\}}(*)) = \top_V > \bot_V = r(*,*)$, since the quantale V is assumed to have at least two elements. ▪

Example 55. Theorems 48, 53 and Remark 54 (2) give the next adjoint situation $A^\circ\dashv A_e:$ **Top** $\to$ **Prost**.

(1) A_e is the induced preorder functor (Remark 36).
(2) The full embedding **Prost** $\overset{A^\circ}{\hookrightarrow}$ **Top** is the *Alexandroff topology* functor, i.e., $A^\circ(X,\leqslant) = (X,\tau)$, where $\tau = \{B\in PX\mid {\downarrow}B = B\}$ with ${\downarrow}B = \{x\in X\mid x\leqslant y \text{ for some } y\in B\}$. Observe that given a preordered set $(X,\leqslant)$, $A^\circ(X,\leqslant) = (X, e_X^\circ\cdot\hat{\beta}\leqslant)$, where for every $\mathfrak{x}\in\beta X$ and every $x\in X$, $\mathfrak{x}\,(e_X^\circ\cdot\hat{\beta}\leqslant)\,x$ iff $\mathfrak{x}\,(\hat{\beta}\leqslant)\,e_X(x)$ (by (1.2)) iff $\mathfrak{x}\,(\hat{\beta}\leqslant)\,\dot{x}$ iff for every $B\in\mathfrak{x}$ and every $C\in\dot{x}$, there exist $y\in B$ and $z\in C$ such that $y\leqslant z$. Since $\{x\}\in\dot{x}$, it follows that for every $B\in\mathfrak{x}$, there exists $y\in B$ such that $y\leqslant x$. Thus, given a subset $S\subseteq X$, for every $x\in X$, $x\in cl(S)$ (where $cl(S)$ is the closure of the set S w.r.t. the topology on X) iff there exists $\mathfrak{x}\in\beta X$ such that $S\in\mathfrak{x}$ and $\mathfrak{x}\,(e_X^\circ\cdot\hat{\beta}\leqslant)\,x$ (see Lecture 1) iff $s\leqslant x$ for some $s\in S$, where given $s\in S$ such that $s\leqslant x$, one defines $\mathfrak{x}=\dot{s}$. As a consequence, S is closed iff $cl(S)=S$ iff $S={\uparrow}S$ with ${\uparrow}S = \{x\in X\mid s\leqslant x \text{ for some } s\in S\}$. The open sets are then exactly the sets of the form $B = X\setminus{\uparrow}S = {\downarrow}D$ for some $D\subseteq X$, i.e., $B={\downarrow}B$ as defined above.

Observe that the Alexandroff topology has the property that arbitrary intersections of open sets are open. ▪

4. Change-of-base functors

Definition 56. A *homomorphism of unital quantales* $(V, \otimes, k) \xrightarrow{\varphi} (W, \otimes, l)$ is a map $V \xrightarrow{\varphi} W$ such that

(1) $\varphi(\bigvee A) = \bigvee \varphi(A)$ for every $A \subseteq V$;
(2) $\varphi(a \otimes b) = \varphi(a) \otimes \varphi(b)$ for every $a, b \in V$;
(3) $\varphi(k) = l$. ▪

Definition 57. A *lax homomorphism of unital quantales* $(V, \otimes, k) \xrightarrow{\varphi} (W, \otimes, l)$ is a map $V \xrightarrow{\varphi} W$ such that

(1) $\bigvee \varphi(A) \leqslant \varphi(\bigvee A)$ for every $A \subseteq V$;
(2) $\varphi(a) \otimes \varphi(b) \leqslant \varphi(a \otimes b)$ for every $a, b \in V$;
(3) $l \leqslant \varphi(k)$. ▪

Remark 58. The first condition of the above definition is equivalent to φ being order-preserving. ▪

Definition 59. A *quantic nucleus* on a quantale $(Q, \otimes)$ is a map $Q \xrightarrow{j} Q$ such that

(1) j is order-preserving;
(2) j is *increasing*, i.e., $a \leqslant j(a)$ for every $a \in Q$;
(3) j is *idempotent*, i.e., $j(j(a)) = j(a)$ for every $a \in Q$;
(4) $j(a) \otimes j(b) \leqslant j(a \otimes b)$ for every $a, b \in Q$. ▪

Example 60. A quantic nucleus on a unital quantale V is a lax homomorphism of V. ▪

Theorem 61. *Every lax homomorphism of unital quantales $V \xrightarrow{\varphi} W$ gives a lax functor V-**Rel** $\xrightarrow{\varphi}$ W-**Rel** defined by $\varphi(X \xrightarrow{r} Y) = X \xrightarrow{\varphi r} Y$, where φr is the composition of the maps $X \times Y \xrightarrow{r} V$ and $V \xrightarrow{\varphi} W$.*

Proof. By the definition of lax functor, φ should satisfy the following:

(1) $\varphi r \leqslant \varphi s$ for every V-relations $X \overset{r}{\underset{s}{\rightrightarrows}} Y$ such that $r \leqslant s$;

(2) $\varphi s \cdot \varphi r \leqslant \varphi(s \cdot r)$ for every V-relations $X \xrightarrow{r} Y$ and $Y \xrightarrow{s} Z$;
(3) $1_X \leqslant \varphi 1_X$ for every set X.

Item (1) (resp. (3)) follows from item (1) (resp. (3)) of Definition 57. To show item (2), notice that $\varphi s \cdot \varphi r(x, z) = \bigvee_{y \in Y} \varphi r(x, y) \otimes \varphi s(y, z) = \bigvee_{y \in Y} \varphi(r(x, y)) \otimes \varphi(s(y, z)) \leqslant \bigvee_{y \in Y} \varphi(r(x, y) \otimes s(y, z)) \leqslant \varphi(\bigvee_{y \in Y} r(x, y) \otimes s(y, z)) = \varphi(s \cdot r(x, z)) = \varphi(s \cdot r)(x, z)$ for every $x \in X$, $z \in Z$. □

Lemma 62. *Given a lax homomorphism of unital quantales $V \xrightarrow{\varphi} W$, maps $X \xrightarrow{f} Y$, $S \xrightarrow{g} Z$, and V-relations $Y \xrightarrow{r} Z$, $U \xrightarrow{s} X$, it follows that*

$$f \leqslant \varphi f, \quad f^\circ \leqslant \varphi(f^\circ), \quad g^\circ \cdot \varphi r \cdot f = \varphi(g^\circ \cdot r \cdot f), \quad f \cdot \varphi s \leqslant \varphi(f \cdot s), \tag{4.1}$$

and, moreover, if φ is $\bigvee$-preserving, then $f \cdot \varphi s = \varphi(f \cdot s)$, where f, f°, and g° are considered as W-relations when appearing on the left-hand side of the above (in)equalities, and as V-relations on the right-hand side.

Proof. Given $x \in X$ and $w \in W$, it follows that $(\varphi(g^\circ \cdot r \cdot f))(x, w) = \varphi(g^\circ \cdot r \cdot f(x, w)) \overset{(1.2)}{=} \varphi(r(f(x), g(w))) = \varphi r(f(x), g(w)) = g^\circ \cdot \varphi r \cdot f(x, w)$. Moreover, given $u \in U$ and $y \in Y$, it follows that $f \cdot \varphi s(u, y) = \bigvee_{f(x)=y} \varphi s(u, x) = \bigvee_{f(x)=y} \varphi(s(u, x)) \overset{(\dagger)}{\leqslant} \varphi(\bigvee_{f(x)=y} s(u, x)) = \varphi(f \cdot s(u, y)) = \varphi(f \cdot s)(u, y)$, in which (†) turns into "=" provided that φ is $\bigvee$-preserving, i.e., $f \cdot \varphi s(u, y) = \varphi(f \cdot s)(u, y)$. □

Definition 63. Given lax extensions $\hat{T}$ and $\check{T}$ of a functor T on **Set** to the categories V-**Rel** and W-**Rel**, respectively, a lax homomorphism of unital quantales $V \xrightarrow{\varphi} W$ is said to be *compatible* with the structure of the lax extensions $\hat{T}$ and $\check{T}$ provided that $\check{T}(\varphi r) \leqslant \varphi(\hat{T} r)$ for every V-relation r, which means

$$\begin{array}{ccc} V\text{-}\mathbf{Rel} & \xrightarrow{\hat{T}} & V\text{-}\mathbf{Rel} \\ {\scriptstyle\varphi}\downarrow & \leqslant & \downarrow{\scriptstyle\varphi} \\ W\text{-}\mathbf{Rel} & \xrightarrow[\check{T}]{} & W\text{-}\mathbf{Rel}. \end{array} \tag{4.2}$$

■

Theorem 64. *Given lax extensions $\hat{\mathbb{T}}$ and $\check{\mathbb{T}}$ of a monad $\mathbb{T}$ on* **Set** *to the categories V-***Rel** *and W-***Rel**, *respectively, every lax homomorphism of unital quantales $V \xrightarrow{\varphi} W$, which is compatible with the structure of the lax extensions, induces a concrete functor $(\mathbb{T}, V)$-***Cat** $\xrightarrow{B_\varphi}$ *$(\mathbb{T}, W)$-***Cat** *defined by $B_\varphi((X,a) \xrightarrow{f} (Y,b)) = (X, \varphi a) \xrightarrow{f} (Y, \varphi b)$. If φ is injective (resp. a $\bigvee$-preserving order-embedding), then B_φ is a (resp. full) embedding.*

Proof. To show that $(X, \varphi a)$ is a $(\mathbb{T}, W)$-category, notice that $e_X^\circ \overset{(4.1)}{\leqslant} \varphi e_X^\circ \overset{\text{Lemma 31}}{\leqslant} \varphi a$, i.e., $1_X \leqslant \varphi a \cdot e_X$, and, moreover, $\varphi a \cdot \check{T}(\varphi a) \overset{(4.2)}{\leqslant} \varphi a \cdot \varphi(\hat{T} a) \leqslant \varphi(a \cdot \hat{T} a) \overset{a \cdot \hat{T} a \leqslant a \cdot m_X}{\leqslant} \varphi(a \cdot m_X) \overset{(4.1)}{=} \varphi a \cdot m_X$. To show that $U(X, \varphi a) \xrightarrow{f} U(Y, \varphi b)$ is a $(\mathbb{T}, W)$-functor, notice that $f \cdot \varphi a \overset{(4.1)}{\leqslant} \varphi(f \cdot a) \overset{a \cdot f \leqslant b \cdot Tf}{\leqslant} \varphi(b \cdot Tf) \overset{(4.1)}{=} \varphi b \cdot Tf$. To show fullness of B_φ, notice that given a $(\mathbb{T}, W)$-functor $B_\varphi(X,a) \xrightarrow{f} B_\varphi(Y,b)$, $\bigvee$-preservation of φ and the last statement of Lemma 62 imply that $\varphi(f \cdot a) = f \cdot \varphi a \leqslant \varphi b \cdot Tf \overset{(4.1)}{=} \varphi(b \cdot Tf)$, and thus, $f \cdot a \leqslant b \cdot Tf$. □

Remark 65. B_φ is called the *change-of-base functor* associated to φ. ■

Definition 66. Given partially ordered sets $(X, \leqslant)$, $(Y, \leqslant)$ and order-preserving maps $(X, \leqslant) \underset{g}{\overset{f}{\rightleftarrows}} (Y, \leqslant)$, f is *left adjoint* to g and g is *right adjoint* to f (denoted $f \dashv g$) provided that $1_X \leqslant g \cdot f$ and $f \cdot g \leqslant 1_Y$. ■

Remark 67. A right adjoin map preserves all existing $\bigwedge$, and a left adjoint map preserves all existing $\bigvee$. ■

Theorem 68. *Let $\hat{\mathbb{T}}$ and $\check{\mathbb{T}}$ be lax extensions of a monad $\mathbb{T}$ on* **Set** *to the categories V-***Rel** *and W-***Rel**, *respectively, and let $V \underset{\psi}{\overset{\varphi}{\rightleftarrows}} W$ be lax homomorphisms of unital quantales compatible with the structure of the lax extensions. If $\varphi \dashv \psi$, then $B_\varphi \dashv B_\psi : (\mathbb{T}, W)$-***Cat** $\to$ *$(\mathbb{T}, V)$-***Cat** *(B_φ is a left adjoint to B_ψ), and, moreover, the latter adjoint situation is concrete (both its unit and co-unit are given by the identity maps).*

Proof. Given a $(\mathbb{T}, V)$-category (X, a), $B_\psi B_\varphi(X, a) = (X, \psi\varphi a)$. Since $\varphi \dashv \psi$, it follows that $1_V \leqslant \psi \cdot \varphi$, and therefore, $a \leqslant \psi\varphi a$. As a consequence, $(X, a) \xrightarrow{1_X} (X, \psi\varphi a)$ is a $(\mathbb{T}, V)$-functor. Dually, given a $(\mathbb{T}, W)$-category (X, b), $(X, \varphi\psi b) \xrightarrow{1_X} (X, b)$ is a $(\mathbb{T}, W)$-functor. The above two maps provide the unit and the co-unit of the adjunction $B_\varphi \dashv B_\psi$, respectively. For example, for the former statement, observe that given a $(\mathbb{T}, V)$-functor $(X, a) \xrightarrow{f} B_\psi(Y, b)$ (which implies $f \cdot a \leqslant \psi b \cdot Tf$), it follows that $B_\varphi(X, a) \xrightarrow{f} (Y, b)$ is a $(\mathbb{T}, W)$-functor, since $f \cdot \varphi a \overset{(4.1)}{\leqslant} \varphi(f \cdot a) \leqslant \varphi(\psi b \cdot Tf) \overset{(4.1)}{=} \varphi\psi b \cdot Tf \leqslant b \cdot Tf$, which makes the triangle

$$\begin{array}{ccc} (X, a) & \xrightarrow{1_X} & B_\psi B_\varphi(X, a) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle B_\psi f} \\ & & B_\psi(Y, b) \end{array}$$

commute, and which, moreover, is uniquely determined by the above commutativity property. □

Remark 69. One could generalize Theorem 68 in the following way. First, given a monad $\mathbb{T}$ on the category **Set**, there exists the quasicategory $\mathbf{Quant}(\mathbb{T})$, the objects of which are pairs $(V, \hat{\mathbb{T}})$ comprising a unital quantale V (with at least two elements) and a lax extension $\hat{\mathbb{T}}$ of the monad $\mathbb{T}$ to the category V-**Rel**, and whose morphisms $(V, \hat{\mathbb{T}}) \xrightarrow{\varphi} (W, \check{\mathbb{T}})$ are lax homomorphisms of unital quantales $V \xrightarrow{\varphi} W$ compatible with the lax extensions $\hat{\mathbb{T}}$ and $\check{\mathbb{T}}$. Since $\mathbf{Quant}(\mathbb{T})$ is a partially ordered quasicategory (given $\mathbf{Quant}(\mathbb{T})$-morphisms $(V, \hat{\mathbb{T}}) \overset{\varphi}{\underset{\psi}{\rightrightarrows}} (W, \check{\mathbb{T}})$, one defines $\varphi \leqslant \psi$ iff $\varphi(a) \leqslant \psi(a)$ for every $a \in V$), it is a 2-quasicategory with thin 2-cells. Second, let **CAT** be the 2-quasicategory of categories and functors. Third, there exists a 2-functor $\mathbf{Quant}(\mathbb{T}) \xrightarrow{B} \mathbf{CAT}$ given by $B((V, \hat{\mathbb{T}}) \xrightarrow{\varphi} (W, \check{\mathbb{T}})) = V\text{-}\mathbf{Cat} \xrightarrow{B_\varphi} W\text{-}\mathbf{Cat}$, where B_φ is the functor of Theorem 64. Observe that given $\mathbf{Quant}(\mathbb{T})$-morphisms $(V, \hat{\mathbb{T}}) \overset{\varphi}{\underset{\psi}{\rightrightarrows}} (W, \check{\mathbb{T}})$ such that $\varphi \leqslant \psi$, for every $(\mathbb{T}, V)$-category (X, a), it follows that $B_\varphi(X, a) \xrightarrow{1_X} B_\psi(X, a) = (X, \varphi a) \xrightarrow{1_X} (X, \psi a)$ is a $(\mathbb{T}, W)$-functor (since $1_X \cdot \varphi a = \varphi a \leqslant \psi a = \psi a \cdot 1_{TX} = \psi a \cdot T1_X$), which is a part of a natural transformation $B_\varphi \xrightarrow{\alpha^{\leqslant}} B_\psi$ given by the identity maps. As a consequence, the functor B preserves adjunctions, i.e., if $\varphi \dashv \psi$ in terms of partially ordered sets, then $B_\varphi \dashv B_\psi$ in terms of categories and functors, which then implies Theorem 68. ∎

Example 70. Given a unital quantale V with at least two elements, there exists the (unique) unital quantale embedding $2 \overset{\iota}{\hookrightarrow} V$ given by

$$\iota(a) = \begin{cases} k, & a = \top \\ \bot_V, & a = \bot, \end{cases}$$

which has a right adjoint $V \xrightarrow{p} 2$ given by

$$p(a) = \begin{cases} \top, & k \leqslant a \\ \bot, & \text{otherwise}, \end{cases}$$

and which is a lax homomorphism of unital quantales (for example, to show Definition 57 (1) for p, observe that $\bigvee p(A) = \top$ iff there exists $a \in A$ such that $p(a) = \top$ iff there exists $a \in A$ such that $k \leqslant a$, which implies $k \leqslant a \leqslant \bigvee A$, which gives $p(\bigvee A) = \top$). ι has a left adjoint $V \xrightarrow{o} 2$ iff $k = \top_V$ (for the necessity, notice that since ι is $\bigwedge$-preserving by Remark 67, $k = \iota(\top) = \iota(\bigwedge \varnothing) = \bigwedge \iota(\varnothing) = \top_V$), which is given by

$$o(a) = \begin{cases} \bot, & a = \bot_V \\ \top, & \text{otherwise}. \end{cases}$$

Observe that o is $\bigvee$-preserving (as a left-adjoint map) and $o(k) = \top$. Thus, o is a lax homomorphism of unital quantales iff o is a homomorphism of unital quantales (given $a, b \in V$, it follows that $o(a \otimes b) \leqslant o(a) \otimes o(b)$, since $o(a \otimes b) = \top$ iff $a \otimes b \neq \bot_V$, which implies $a \neq \bot_V$ and $b \neq \bot_V$, which gives $o(a) = \top$ and $o(b) = \top$, which finally provides $o(a) \otimes o(b) = \top$) iff $a \otimes b = \bot_V$ implies $a = \bot_V$ or $b = \bot_V$ for every $a, b \in V$ (observe that then $o(a \otimes b) = \bot$ iff $a \otimes b = \bot_V$ iff $a = \bot_V$ or $b = \bot_V$ iff $o(a) = \bot$ or $o(b) = \bot$ iff $o(a) \otimes o(b) = \bot$).

The above maps are compatible with the lax extensions of the identity functor on **Set** to **Rel** and V-**Rel**, respectively. Theorem 68 provides then the adjunctions

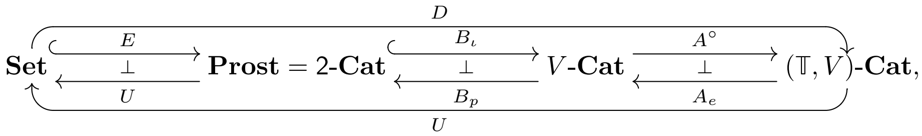
$$\mathbf{Prost} = 2\text{-}\mathbf{Cat} \underset{\underset{B_p}{\longleftarrow}}{\overset{\overset{B_o}{\longleftarrow}}{\underset{\bot}{\overset{\bot}{\xhookrightarrow{B_\iota}}}}} V\text{-}\mathbf{Cat},$$

where B_p is the induced preorder functor of Corollary 40, and $B_\iota(X, \leqslant) = (X, a)$, where for every $x, y \in X$,

$$a(x, y) = \begin{cases} k, & x \leqslant y \\ \bot_V, & \text{otherwise}. \end{cases}$$

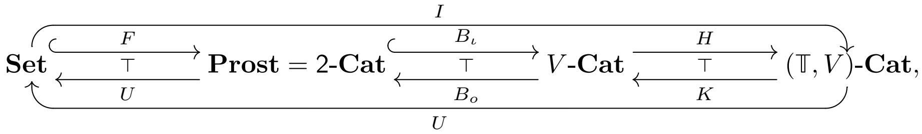
Moreover, $B_o(X, a) = (X, \leqslant)$, where for every $x, y \in X$, $x \leqslant y$ iff $\bot_V < a(x, y)$. ■

Remark 71. The adjunction of item (1) of Theorem 33 can be decomposed now as follows:

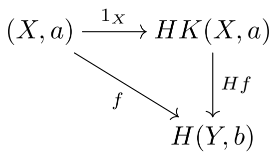


where $EX = (X, \Delta) = (X, \{(x, x) \,|\, x \in X\})$ (*discrete* preorder). Observe that given a set X, it follows that $A^\circ B_\iota EX = A^\circ B_\iota (X, \Delta) = A^\circ (X, \iota\Delta) = A^\circ (X, 1_X) = (X, e_X^\circ \cdot \hat{T} 1_X) = (X, 1_X^\sharp) = DX$. ■

Problem 72. The adjunction of item (2) of Theorem 33 can be decomposed as follows:

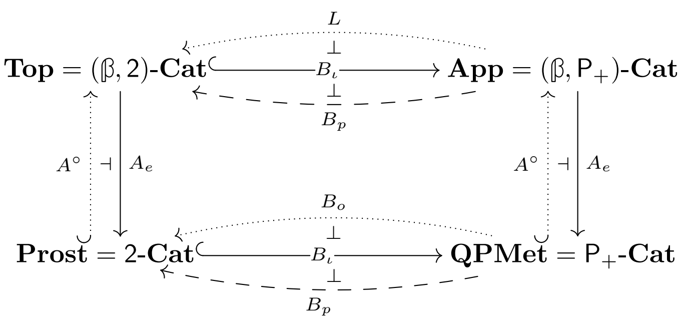


where $FX = (X, X \times X)$ (*indiscrete* preorder), $H(X, a) = (X, \underline{\top_V})$ (indiscrete $(\mathbb{T}, V)$-category structure), in which $TX \times X \xrightarrow{\top_V} V$ is the constant map with value $\top_V$, and $K(X, a) = (X, 1_X)$ (identity V-relation). Observe that given a set X, it follows that $HB_\iota FX = HB_\iota(X, X \times X) = H(X, \iota(X \times X)) \overset{(\dagger)}{=} H(X, \underline{\top_V}) = (X, \underline{\top_V}) = IX$, where (†) relies on the fact that the existence of a left adjoint map $V \xrightarrow{o} 2$ to $2 \xhookrightarrow{\iota} V$ implies $k = \top_V$. To show that K is left adjoint to H, notice that given a $(\mathbb{T}, V)$-category (X, a), the identity map $X \xrightarrow{1_X} X$ provides an H-universal arrow $(X, a) \xrightarrow{1_X} HK(X, a)$ for (X, a), i.e., first, $(X, a) \xrightarrow{1_X} HK(X, a) = (X, \underline{\top_V})$ is a $(\mathbb{T}, V)$-functor (since $a \leqslant \underline{\top_V}$), and, second, given a $(\mathbb{T}, V)$-functor $(X, a) \xrightarrow{f} H(Y, b)$, there is a unique V-functor $K(X, a) = (X, 1_X) \xrightarrow{f} (Y, b)$ (since $f \cdot 1_X = f = 1_Y \cdot f \leqslant b \cdot f$), which makes the triangle



commute, and which, moreover, is uniquely determined by the above commutativity property. ■

Remark 73. The diagram



(4.3)

commutes w.r.t. both the solid and the dotted arrows (excluding the dashed ones). The functors of (4.3) can be described explicitly as follows.

(1) The adjunction $A^\circ \dashv A_e : \mathbf{Top} \to \mathbf{Prost}$ is that of Example 55.

(2) The full embedding $\mathbf{Prost} \overset{B_\iota}{\hookrightarrow} \mathbf{QPMet}$ is given by $B_\iota(X, \leqslant) = (X, \rho)$, where for every $x, y \in X$,

$$\rho(x,y) = \begin{cases} 0, & x \leqslant y \\ \infty, & \text{otherwise.} \end{cases}$$

The functors $\mathbf{QPMet} \underset{B_p}{\overset{B_o}{\rightrightarrows}} \mathbf{Prost}$ are given by $B_{o,p}(X,\rho) = (X, \leqslant_{o,p})$, where for every $x, y \in X$, $x \leqslant_o y$ iff $\rho(x,y) < \infty$, and $x \leqslant_p y$ iff $\rho(x,y) = 0$, respectively.

(3) The full embedding $\mathbf{QPMet} \overset{A^\circ}{\hookrightarrow} \mathbf{App}$ is given by $A^\circ(X,\rho) = (X,\delta)$, where for every $x \in X$, $A \in PX$, $\delta(x,A) = \inf\{\rho(y,x) \mid y \in A\}$. The functor $\mathbf{App} \xrightarrow{A_e} \mathbf{QPMet}$ is given by $A_e(X,\delta) = (X,\rho)$, where for every $x, y \in X$, $\rho(x,y) = \sup\{\delta(y,A) \mid x \in A\}$.

(4) The full embedding $\mathbf{Top} \overset{B_\iota}{\hookrightarrow} \mathbf{App}$ is given by $B_\iota(X,\tau) = (X,\delta)$, where for every $x \in X$, $A \in PX$,

$$\delta(x,A) = \begin{cases} 0, & x \in cl(A) \\ \infty, & \text{otherwise.} \end{cases}$$

The functor $\mathbf{App} \xrightarrow{B_p} \mathbf{Top}$ sends an approach space (X,a) (represented as a (β, P_+)-category) to a topological space, in which an ultrafilter $\mathfrak{x}$ converges to a point x iff $a(\mathfrak{x},x) = 0$. The unital quantale homomorphism $\mathsf{P}_+ \xrightarrow{o} 2$ is incompatible with the lax extensions of the ultrafilter monad β to P_+-**Rel** and **Rel**, respectively, but still provides a left adjoint functor L to B_ι.

Observe that o is compatible with the lax extensions of the ultrafilter monad β to P_+-**Rel** and **Rel**, respectively, provided that $\hat{\beta}(or) \leqslant o(\bar{\beta}r)$ for every V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\ } Y$. Also notice that $\hat{\beta}(or)(\mathfrak{x},\mathfrak{y}) = \bigwedge_{A\in\mathfrak{x},B\in\mathfrak{y}} \bigvee_{x\in A,y\in B} or(x,y)$ and $o(\bar{\beta}r)(\mathfrak{x},\mathfrak{y}) = o(\sup_{A\in\mathfrak{x},B\in\mathfrak{y}} \inf_{x\in A,y\in B} r(x,y))$ for every $\mathfrak{x} \in \beta X$ and every $\mathfrak{y} \in \beta Y$ (see Lecture 1 for more detail). Take a set X such that there exists a non-principal ultrafilter $\mathfrak{x}$ on X, and consider the identity V-relation 1_X on X. On the one hand, $o(\bar{\beta}1_X)(\mathfrak{x},\mathfrak{x}) = o(\sup_{A,B\in\mathfrak{x}} \inf_{x\in A,y\in B}(1_X)_\circ(x,y)) \overset{(\dagger)}{=} o(\sup_{A,B\in\mathfrak{x}} \bot_V) = o(\bot_V) = \bot$, where (†) uses the fact that for every $A, B \in \mathfrak{x}$, it follows that $\inf_{x\in A,y\in B}(1_X)_\circ(x,y) \leqslant \inf_{x,y\in A\bigcap B, x\neq y}(1_X)_\circ(x,y) = \inf_{x,y\in A\bigcap B,x\neq y} \bot_V = \bot_V$, since $A \bigcap B \in \mathfrak{x}$ and therefore, $A \bigcap B$ has at least two elements (recall that the ultrafilter $\mathfrak{x}$ is non-principal). On the other hand, $\hat{\beta}(o1_X)(\mathfrak{x},\mathfrak{x}) = \bigwedge_{A,B\in\mathfrak{x}} \bigvee_{x\in A,y\in B} o1_X(x,y) \geqslant \bigwedge_{A,B\in\mathfrak{x}} \bigvee_{x\in A\bigcap B} o1_X(x,x) = \bigwedge_{A,B\in\mathfrak{x}} \bigvee_{x\in A\bigcap B} o(k) = \bigwedge_{A,B\in\mathfrak{x}} \bigvee_{x\in A\bigcap B} \top \overset{(\dagger)}{\geqslant} \bigwedge_{A,B\in\mathfrak{x}} \top \geqslant \top$, where (†) uses the fact that for every $A, B \in \mathfrak{x}$, it follows that $A \bigcap B \in \mathfrak{x}$ and thus, $A \bigcap B \neq \varnothing$. As a consequence, $\hat{\beta}(o1_X)(\mathfrak{x},\mathfrak{x}) = \top > \bot = o(\bar{\beta}1_X)(\mathfrak{x},\mathfrak{x})$, violating the condition of compatibility with the lax extensions. ▪

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