Elements of monoidal topology*
Lecture 3: a generalization of the Kuratowski-Mrówka theorem

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Abstract

This lecture shows an example of the application of the theory of $(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors to general topology, i.e., a particular generalization of the Kuratowski-Mrówka theorem on the equivalence between the concepts of proper (or stably closed) and perfect map.

1. Classical Kuratowski-Mrówka theorem

1.1. Proper and perfect maps

Definition 1. Let $(X, \tau) \xrightarrow{f} (Y, \sigma)$ be a continuous map between topological spaces.

(1) f is *closed* provided that the image under f of every closed set in (X, τ) is closed in (Y, σ).

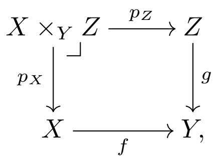
(2) f is *proper* provided that for every topological space (Z, ϱ), the map $X \times Z \xrightarrow{f \times 1_Z} Y \times Z$ is closed. ▪

Remark 2. Every proper map is closed (take Z to be a singleton topological space in Definition 1 (2)). ▪

Theorem 3. *Given a continuous map $(X, \tau) \xrightarrow{f} (Y, \sigma)$ between topological spaces, equivalent are:*

(1) *f is proper;*

(2) *for every continuous map $(Z, \varrho) \xrightarrow{g} (Y, \sigma)$, the map $X \times_Y Z \xrightarrow{p_Z} Z$, which is defined by the pullback*



is closed.

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Proof.

(1) $\Rightarrow$ (2): Given a continuous map $(Z,\varrho) \xrightarrow{g} (Y,\sigma)$ and its respective pullback

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{p_Z} & Z \\ {\scriptstyle p_X}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y, \end{array}$$

one can construct the following two diagrams (where π_X, π_Y, π_Z are product projections)

$$\begin{array}{ccccc} & & X \times_Y Z & & \\ & \swarrow^{p_X} & \vdots{\scriptstyle \langle p_X, p_Z\rangle} & \searrow^{p_Z} & \\ X & \xleftarrow[\pi_X]{} & X \times Z & \xrightarrow[\pi_Z]{} & Z \end{array} \qquad \begin{array}{ccccc} & & Z & & \\ & \swarrow^{g} & \vdots{\scriptstyle \langle g, 1_Z\rangle} & \searrow^{1_Z} & \\ Y & \xleftarrow[\pi_Y]{} & Y \times Z & \xrightarrow[\pi_Z]{} & Z, \end{array}$$

which then provide the diagram

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{p_Z} & Z \\ {\scriptstyle \langle p_X, p_Z\rangle}\downarrow & & \downarrow{\scriptstyle \langle g, 1_Z\rangle} \\ X \times Z & \xrightarrow[f \times 1_Z]{} & Y \times Z, \end{array} \tag{1.1}$$

where the two vertical arrows are given by injective maps. To show that diagram (1.1) commutes, observe that for every $(x,z) \in X \times_Y Z$, on the one hand, $(f \times 1_Z) \cdot \langle p_X, p_Z\rangle(x,z) = (f \times 1_Z)(p_X(x,z), p_Z(x,z)) = (f \times 1_Z)(x,z) = (f(x), z)$, and, on the other hand, $\langle g, 1_Z\rangle \cdot p_Z(x,z) = \langle g, 1_Z\rangle(z) = (g(z), z)$. Since $(x,z) \in X \times_Y Z$, it follows that $f(x) = g(z)$, which then implies $(f(x), z) = (g(z), z)$.

We are going to show that $X \times_Y Z$ is a closed subset of $X \times Z$. Since the map $(X,\tau) \xrightarrow{f} (Y,\sigma)$ is proper and therefore, closed by Remark 2, it follows that $f(X)$ is a closed subset of Y, and thus, $S := g^{-1}(f(X))$ is a closed subset of Z. Given $(x,z) \in (X \times Z) \backslash (X \times_Y Z)$, it follows that $z \notin S$ and therefore, $z \in Z \backslash S =: U \in \varrho$. Thus, $(x,z) \in \pi_Z^{-1}(U) =: W$, where W is an open subset of $X \times Z$. If $(x', z') \in W \bigcap (X \times_Y Z)$, then $z' \in U$ and $f(x') = g(z')$, namely, $z' \in Z \backslash S$ and $z' \in S$, which is a contradiction. As a consequence, W is an open subset of $X \times Z$ containing (x,z) and, moreover, disjoint from the set $X \times_Y Z$.

Since $X \times_Y Z$ is a closed subset of $X \times Z$, the inclusion $X \times_Y Z \overset{\langle p_X, p_Z\rangle}{\hookrightarrow} X \times Z$ is a closed map. Moreover, since the map $(X,\tau) \xrightarrow{f} (Y,\sigma)$ is proper, $X \times Z \xrightarrow{f \times 1_Z} Y \times Z$ is a closed map as well. Thus, the left-hand path in diagram (1.1) is a closed map, and thus, the right-hand path is a closed map as well. Since the inclusion $Z \overset{\langle g, 1_Z\rangle}{\hookrightarrow} Y \times Z$ is clearly injective, it follows that $X \times_Y Z \xrightarrow{p_Z} Z$ is a closed map (given a closed subset $P \subseteq X \times_Y Z$, since $\langle g, 1_Z\rangle(p_Z(P))$ is closed, $p_Z(P) = (\langle g, 1_Z\rangle)^{-1}(\langle g, 1_Z\rangle(p_Z(P)))$ is closed).

(2) $\Rightarrow$ (1): Observe that given a topological space (Z,ϱ), it follows that

$$\begin{array}{ccc} X \times Z & \xrightarrow{f \times 1_Z} & Y \times Z \\ {\scriptstyle \pi_X}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pullback. $\square$

Remark 4. Theorem 3 motivates the terminology *stably closed* w.r.t. proper maps. ▪

Definition 5.

(1) Given a topological space (X, τ), a subset $S \subseteq X$ is said to be *compact* provided that for every family $\{U_i \mid i \in I\} \subseteq \tau$ such that $S \subseteq \bigcup_{i \in I} U_i$ there exists a finite subfamily $\{U_{i_1}, \ldots, U_{i_n}\} \subseteq \{U_i \mid i \in I\}$ such that $S \subseteq \bigcup_{j=1}^{n} U_{i_j}$ (in other words, every open cover of S has a finite subcover).

(2) A topological space (X, τ) is said to be *compact* provided that its underlying set X is compact. ▪

Remark 6. Some authors call the property of Definition 5 *quasi-compactness*. A quasi-compact topological space (X, τ) is then said to be *compact* provided that it is additionally *Hausdorff* or *T_2-space*, namely, for every distinct points $x_1, x_2 \in X$ there exist $U_1, U_2 \in \tau$ such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \bigcap U_2 = \varnothing$. ▪

Definition 7. A continuous map $(X, \tau) \xrightarrow{f} (Y, \sigma)$ between topological spaces is called *perfect* provided that f is closed, and for every $y \in Y$, the fibre $f^{-1}(y)$ is a compact subset of X. ▪

1.2. Kuratowski-Mrówka theorem and its generalization

Theorem 8 (Kuratowski-Mrówka). *Given a topological space (X, τ), equivalent are:*

(1) *(X, τ) is compact;*

(2) *for every topological space (Y, σ), the projection $X \times Y \xrightarrow{\pi_Y} Y$ is closed.*

Proof. (1) $\Rightarrow$ (2) : K. Kuratowski. (2) $\Rightarrow$ (1) : S. Mrówka. □

Corollary 9. *Given a topological space (X, τ), the unique continuous map $(X, \tau) \xrightarrow{!_X} 1$ (where $1 = \{*\}$) is perfect iff it is proper.*

Proof. Observe that the map $(X, \tau) \xrightarrow{!_X} 1$ is proper iff for every topological space (Z, ϱ), the projection map $X \times Z \xrightarrow{\pi_Z} Z$, which is defined by the pullback

$$\begin{array}{ccc} X \times Z & \xrightarrow{\pi_Z} & Z \\ {\scriptstyle \pi_X}\downarrow & \lrcorner & \downarrow{\scriptstyle !_Z} \\ X & \xrightarrow[!_X]{} & 1, \end{array}$$

is closed (by Theorem 3) iff for every topological space (Z, ϱ), the projection map $X \times Z \xrightarrow{\pi_Z} Z$ is closed iff the space (X, τ) is compact (by Theorem 8) iff the map $(X, \tau) \xrightarrow{!_X} 1$ is perfect (notice that the unique map $(X, \tau) \xrightarrow{!_X} 1$ is clearly closed). □

Theorem 10 (Bourbaki). *A continuous map between topological spaces is perfect iff it is proper.*

Remark 11. Since the category **Top** of topological spaces is an instance of the categories $(\mathbb{T}, V)$-**Cat**, one could ask about the analogues of Theorems 8, 10 for the latter category. ▪

2. Proper maps in the category $(\mathbb{T}, V)$-Cat

2.1. Categorical preliminaries

Remark 12. Every category $(\mathbb{T}, V)$-**Cat** has the following two properties.

(1) The terminal object in $(\mathbb{T}, V)$-**Cat** is given by $(1, \top)$, where $\top(\mathfrak{x}, *) = \top_V$ for every $\mathfrak{x} \in T1$ (observe that one takes the initial $(\mathbb{T}, V)$-category structure on a terminal object in **Set** w.r.t. the empty source).

(2) The $(\mathbb{T}, V)$-category structure d on the pullback of $(\mathbb{T}, V)$-functors $(X, a) \xrightarrow{f} (Z, c)$ and $(Y, b) \xrightarrow{g} (Z, c)$

$$\begin{array}{ccc} (X \times_Z Y, d) & \xrightarrow{p_Y} & (Y, b) \\ {\scriptstyle p_X}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ (X, a) & \xrightarrow[f]{} & (Z, c) \end{array} \tag{2.1}$$

is given by $d = (p_X^\circ \cdot a \cdot Tp_X) \wedge (p_Y^\circ \cdot b \cdot Tp_Y)$, or, in pointwise notation, $d(\mathfrak{z}, (x, y)) = a(Tp_X(\mathfrak{z}), x) \wedge b(Tp_Y(\mathfrak{z}), y)$ for every $\mathfrak{z} \in T(X \times_Z Y)$, $x \in X$, $y \in Y$ (observe that one takes the initial $(\mathbb{T}, V)$-category structure on the set $X \times_Z Y$ w.r.t. the source $(U(X, a) \xleftarrow{p_X} X \times_Z Y \xrightarrow{p_Y} U(Y, b))$, where $(\mathbb{T}, V)$-**Cat** $\xrightarrow{U}$ **Set** is the forgetful functor). ∎

Definition 13. A lax extension $\hat{T}$ to V-**Rel** of a functor T on **Set** is called *left-whiskering* provided that $\hat{T}(f \cdot r) = Tf \cdot \hat{T}r$ for every V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} Y$ and every map $Y \xrightarrow{f} Z$. ∎

Remark 14.

(1) A lax extension $\hat{T}$ to V-**Rel** of a functor T on **Set** satisfies $Tf \cdot \hat{T}r \leqslant \hat{T}(f \cdot r)$ for every V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} Y$ and every map $Y \xrightarrow{f} Z$, since $Tf \cdot \hat{T}r \leqslant \hat{T}f \cdot \hat{T}r \leqslant \hat{T}(f \cdot r)$.

(2) Recall from Lecture 2 that a lax extension $\hat{T}$ to V-**Rel** of a functor T on **Set** is always *right-whiskering*, i.e., satisfies $\hat{T}(s \cdot f) = \hat{T}s \cdot Tf$ for every map $X \xrightarrow{f} Y$ and every V-relation $Y \overset{s}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} Z$. ∎

Example 15.

(1) The lax extension $\hat{P}$ to **Rel** of the powerset functor P on **Set** is left-whiskering. Observe that given a relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} Y$ and a map $Y \xrightarrow{f} Z$, for every $A \in PX$ and every $C \in PZ$, it follows that $A\,\hat{P}(f \cdot r)\,C$ iff for every $z \in C$ there exists $x \in A$ such that $x\,(f \cdot r)\,z$ iff for every $z \in C$, there exist $x \in A$ and $y \in Y$ such that $x\,r\,y$ and $y\,f_\circ\,z$ iff for every $z \in C$, there exist $x \in A$ and $y \in Y$ such that $x\,r\,y$ and $f(y) = z$ iff there exists $B \in PY$ such that for every $y \in B$ there exists $x \in A$ such that $x\,r\,y$ and $f(B) = C$ iff there exists $B \in PY$ such that $A\,\hat{P}r\,B$ and $f(B) = C$ iff there exists $B \in PY$ such that $A\,\hat{P}r\,B$ and $B\,(Pf)_\circ\,C$ iff $A\,(Pf \cdot \hat{P}r)\,C$.

(2) The lax extension $\hat{\beta}$ (resp. $\bar{\beta}$) to **Rel** (resp. P_+-**Rel**) of the ultrafilter functor β on **Set** is left-whiskering. ∎

Definition 16. A functor T on **Set** is said to be *taut* provided that it preserves pullbacks of monomorphisms along arbitrary maps, namely, if

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{p_Z} & Z \\ {\scriptstyle p_X}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y, \end{array}$$

is a pullback and $X \xrightarrow{f} Y$ is a monomorphism, then

$$\begin{array}{ccc} T(X \times_Y Z) & \xrightarrow{Tp_Z} & TZ \\ {\scriptstyle Tp_X}\downarrow & \lrcorner & \downarrow{\scriptstyle Tg} \\ TX & \xrightarrow[Tf]{} & TY \end{array}$$

is a pullback. ∎

Example 17.

(1) The powerset functor P on **Set** is taut. Observe first that monomorphisms in **Set** are precisely the injective maps, which are preserved by the powerset functor P (notice that an injective map $X \xrightarrow{f} Y$ with $X \neq \varnothing$ is a section, and sections are preserved by every functor). Given a pullback

$$\begin{array}{ccc} X\times_Y Z & \xrightarrow{p_Z} & Z \\ {\scriptstyle p_X}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y, \end{array} \tag{2.2}$$

with a monomorphism $X \xrightarrow{f} Y$, there exists a map $P(X \times_Y Z) \xrightarrow{h} PX \times_{PY} PZ$ defined by the commutative diagram

$$\begin{array}{ccccc} P(X\times_Y Z) & & & & \\ & \overset{h}{\searrow} & & \overset{Pp_Z}{\searrow} & \\ & & PX\times_{PY}PZ & \xrightarrow[p_{PZ}]{} & PZ \\ & {\scriptstyle Pp_X}\searrow & {\scriptstyle p_{PX}}\big\downarrow & & \big\downarrow{\scriptstyle Pg} \\ & & PX & \xrightarrow[Pf]{} & PY. \end{array}$$

We show that h is a bijective map. Since (2.2) is a pullback, p_Z is a monomorphism and therefore, Pp_Z is a monomorphism. Thus, h is a monomorphism, i.e., injective. To show that h is surjective, notice that given $(A, C) \in PX \times_{PY} PZ$, $f(A) = Pf(A) = Pf \cdot p_{PX}(A, C) = Pg \cdot p_{PZ}(A, C) = Pg(C) = g(C)$. Let $D = (A \times C) \bigcap (X \times_Y Z)$. To show that $h(D) = (A, C)$, observe that $h(D) = (Pp_X(D), Pp_Z(D)) = (p_X(D), p_Z(D))$. Clearly, $p_X(D) \subseteq A$ and $p_Z(D) \subseteq C$. Given $a \in A$, since $f(A) = g(C)$, there exists $c \in C$ such that $f(a) = g(c)$, which implies $(a, c) \in D$, which gives $a \in p_X(D)$. As a consequence, $A \subseteq p_X(D)$, which implies $A = p_X(D)$. Similarly, $C = p_Z(D)$. Thus, $h(D) = (p_X(D), p_Z(D)) = (A, C)$.

(2) The ultrafilter functor β on **Set** is taut. ▪

Lemma 18. *Taut functors preserve monomorphisms.*

Proof. Observe that a map $X \xrightarrow{f} Y$ is a monomorphism iff the diagram

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle 1_X}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pullback. □

Remark 19. The property of being taut can be defined for a functor T on an arbitrary category **C**. ▪

Remark 20. From now on, assume that the lax extension $\hat{\mathbb{T}}$ to V-**Rel** of a monad $\mathbb{T}$ on **Set** satisfies the following three conditions:

(T) T is taut;

(W) $\hat{T}$ is left-whiskering;

(N) $\hat{T} \xrightarrow{m^\circ} \hat{T}\hat{T}$ is natural, which means that the diagram

$$\begin{array}{ccc} TX & \xrightarrow{m_X^\circ} & TTX \\ {\scriptstyle \hat{T}r}\downarrow & & \downarrow{\scriptstyle \hat{T}\hat{T}r} \\ TY & \xrightarrow[m_Y^\circ]{} & TTY \end{array}$$

commutes for every V-relation $X \overset{r}{\longrightarrow} Y$. ∎

Remark 21. Observe that given a lax extension $\hat{\mathbb{T}}$ to V-**Rel** of a monad $\mathbb{T}$ on **Set**, since $\hat{T}\hat{T} \xrightarrow{m} \hat{T}$ is an oplax natural transformation (recall Lecture 1), it follows that

$$\begin{array}{ccc} TTX & \xrightarrow{m_X} & TX \\ {\scriptstyle \hat{T}\hat{T}r}\downarrow & \leqslant & \downarrow{\scriptstyle \hat{T}r} \\ TTY & \xrightarrow[m_Y]{} & TY \end{array}$$

for every V-relation $X \overset{r}{\longrightarrow} Y$. Thus, $m_Y \cdot \hat{T}\hat{T}r \leqslant \hat{T}r \cdot m_X$ implies $\hat{T}\hat{T}r \cdot m_X^\circ \leqslant m_Y^\circ \cdot m_Y \cdot \hat{T}\hat{T}r \cdot m_X^\circ \leqslant m_Y^\circ \cdot \hat{T}r \cdot m_X \cdot m_X^\circ \leqslant m_Y^\circ \cdot \hat{T}r$, i.e., $\hat{T}\hat{T}r \cdot m_X^\circ \leqslant m_Y^\circ \cdot \hat{T}r$. As a consequence, one obtains that $\hat{T} \xrightarrow{m^\circ} \hat{T}\hat{T}$ is a natural transformation iff

$$\begin{array}{ccc} TX & \xrightarrow{m_X^\circ} & TTX \\ {\scriptstyle \hat{T}r}\downarrow & \leqslant & \downarrow{\scriptstyle \hat{T}\hat{T}r} \\ TY & \xrightarrow[m_Y^\circ]{} & TTY \end{array}$$

for every V-relation $X \overset{r}{\longrightarrow} Y$. ∎

Example 22.

(1) The lax extension $\hat{\mathbb{P}}$ to **Rel** of the powerset monad $\mathbb{P}$ on **Set** satisfies the conditions of Remark 20. To show condition (N), observe that for every V-relation $X \overset{r}{\longrightarrow} Y$, given $A \in PX$ and $\mathcal{B} \in PPY$, on the one hand, $A\,(m_Y^\circ \cdot \hat{P}r)\,\mathcal{B}$ iff $A\,(\hat{P}r)\,m_Y(\mathcal{B})$ iff $A\,(\hat{P}r) \bigcup \mathcal{B}$ iff for every $y \in \bigcup \mathcal{B}$ there exists $x \in A$ such that $x\,r\,y$, and, on the other hand, $A\,(\hat{P}\hat{P}r \cdot m_X^\circ)\,\mathcal{B}$ iff there exists $\mathcal{A} \in PPX$ such that $A\,m_X^\circ\,\mathcal{A}$ and $\mathcal{A}\,(\hat{P}\hat{P}r)\,\mathcal{B}$ iff there exists $\mathcal{A} \in PPX$ such that $m_X(\mathcal{A}) = A$ and $\mathcal{A}\,(\hat{P}\hat{P}r)\,\mathcal{B}$ iff there exists $\mathcal{A} \in PPX$ such that $A = \bigcup \mathcal{A}$ and for every $B \in \mathcal{B}$ there exists $A' \in \mathcal{A}$ such that $A'\,(\hat{P}r)\,B$ iff there exists $\mathcal{A} \in PPX$ such that $A = \bigcup \mathcal{A}$ and for every $B \in \mathcal{B}$ there exists $A' \in \mathcal{A}$ such that for every $y \in B$ there exists $x' \in A'$ such that $x'\,r\,y$. In view of Remark 21, one has to show that $m_Y^\circ \cdot \hat{P}r \leqslant \hat{P}\hat{P}r \cdot m_X^\circ$. Observe that if $A\,(m_Y^\circ \cdot \hat{P}r)\,\mathcal{B}$, then taking $\mathcal{A} = \{A\} \in PPX$, one gets $A = \bigcup \mathcal{A}$ and for every $B \in \mathcal{B}$ there exists $A' = A \in \mathcal{A}$ such that for every $y \in B \subseteq \bigcup \mathcal{B}$ there exists $x \in A' = A$ such that $x\,r\,y$.

(2) The lax extension $\hat{\beta}$ (resp. $\bar{\beta}$) to **Rel** (resp. P_+-**Rel**) of the ultrafilter monad on **Set** satisfies the conditions of Remark 20. ∎

Theorem 23. *There exists a functor* $(\mathbb{T}, V)$-**Cat** $\xrightarrow{G}$ V-**Cat**, *which is given by* $G((X, a) \xrightarrow{f} (Y, b)) = (TX, \hat{a}) \xrightarrow{Tf} (TY, \hat{b})$, *where* $\hat{a} = \hat{T}a \cdot m_X^\circ$.

Proof. To show that $(TX, \hat{a})$ is a V-category, notice that, firstly, $1_{TX} = T1_X \leqslant \hat{T}1_X \leqslant \hat{T}(a \cdot e_X) = \hat{T}a \cdot Te_X \overset{(\dagger)}{\leqslant} \hat{T}a \cdot m_X^{\circ} = \hat{a}$, where $(\dagger)$ uses the fact that $m_X \cdot Te_X = 1_{TX}$ implies $Te_X \leqslant m_X^{\circ} \cdot m_X \cdot Te_X = m_X^{\circ}$. Secondly, $a \cdot \hat{T}a \cdot m_X^{\circ} \leqslant a \cdot m_X \cdot m_X^{\circ} \leqslant a$ gives $\hat{T}a \cdot \hat{T}\hat{T}a \cdot (Tm_X)^{\circ} \leqslant \hat{T}a \cdot \hat{T}\hat{T}a \cdot \hat{T}m_X^{\circ} \leqslant \hat{T}(a \cdot \hat{T}a \cdot m_X^{\circ}) \leqslant \hat{T}a$, and therefore, $\hat{a} \cdot \hat{a} = \hat{T}a \cdot m_X^{\circ} \cdot \hat{T}a \cdot m_X^{\circ} \overset{(\mathrm{N})}{=} \hat{T}a \cdot \hat{T}\hat{T}a \cdot m_{TX}^{\circ} \cdot m_X^{\circ} = \hat{T}a \cdot \hat{T}\hat{T}a \cdot (m_X \cdot m_{TX})^{\circ} \overset{m_X \cdot m_{TX} = m_X \cdot Tm_X}{=} \hat{T}a \cdot \hat{T}\hat{T}a \cdot (m_X \cdot Tm_X)^{\circ} = \hat{T}a \cdot \hat{T}\hat{T}a \cdot (Tm_X)^{\circ} \cdot m_X^{\circ} \leqslant \hat{T}a \cdot m_X^{\circ} = \hat{a}$.

To show that $(TX, \hat{a}) \xrightarrow{Tf} (TY, \hat{b})$ is a V-functor, notice that $f \cdot a \leqslant b \cdot Tf$ gives $a \leqslant f^{\circ} \cdot b \cdot Tf$, and therefore, $\hat{T}a \leqslant \hat{T}(f^{\circ} \cdot b \cdot Tf) = (Tf)^{\circ} \cdot \hat{T}b \cdot TTf$, which then yields $Tf \cdot \hat{a} = Tf \cdot \hat{T}a \cdot m_X^{\circ} \leqslant Tf \cdot (Tf)^{\circ} \cdot \hat{T}b \cdot TTf \cdot m_X^{\circ} \overset{Tf \cdot (Tf)^{\circ} \leqslant 1_{TY}}{\leqslant} \hat{T}b \cdot TTf \cdot m_X^{\circ} \overset{TTf \cdot m_X^{\circ} \leqslant m_Y^{\circ} \cdot Tf}{\leqslant} \hat{T}b \cdot m_Y^{\circ} \cdot Tf = \hat{b} \cdot Tf$. □

Proposition 24. *Given a lax extension $\hat{\mathbb{T}}$ to V-**Rel** of a monad $\mathbb{T} = (T, m, e)$ on **Set**, the natural transformation $1_{\mathbf{Set}} \xrightarrow{e} T$ provides a natural transformation $Ind \xrightarrow{e} G$, where $(\mathbb{T}, V)$-**Cat** $\xrightarrow{Ind}$ **Prost**, $Ind((X, a) \xrightarrow{f} (Y, b)) = (X, \leqslant_a) \xrightarrow{f} (Y, \leqslant_b)$ (with $x \leqslant_a x'$ iff $k \leqslant a(e_X(x), x')$) is the induced preorder functor, and **Prost** is considered as a full subcategory of V-**Cat** w.r.t. the full embedding **Prost** $\xhookrightarrow{B_\iota}$ V-**Cat** (cf. Lecture 2).*

Proof. It will be enough to show that given a $(\mathbb{T}, V)$-category (X, a), $(Ind(X, a) = (X, \leqslant_a)) \xrightarrow{e_X} (G(X, a) = (TX, \hat{a}))$ is a $(\mathbb{T}, V)$-functor, namely,

$$\begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ {\scriptstyle \leqslant_a}\big\downarrow & \leqslant & \big\downarrow{\scriptstyle \hat{a}} \\ X & \xrightarrow[e_X]{} & TX. \end{array}$$

Given $x \in X$, $\mathfrak{x} \in TX$, on the one hand, $e_X \cdot \leqslant_a (x, \mathfrak{x}) = \bigvee_{x' \in X} \leqslant_a (x, x') \otimes (e_X)_{\circ}(x', \mathfrak{x}) = \bigvee \{k \mid x' \in X$ such that $k \leqslant a(e_X(x), x')$ and $e_X(x') = \mathfrak{x}\} = \begin{cases} k, & \text{there exists } x' \in X \text{ with } k \leqslant a(e_X(x), x') \text{ and } e_X(x') = \mathfrak{x} \\ \bot_V, & \text{otherwise,} \end{cases}$

and, on the other hand, $\hat{a} \cdot e_X(x, \mathfrak{x}) = \hat{a}(e_X(x), \mathfrak{x}) = \hat{T}a \cdot m_X^{\circ}(e_X(x), \mathfrak{x}) = \bigvee_{\mathfrak{X} \in TTX} m_X^{\circ}(e_X(x), \mathfrak{X}) \otimes \hat{T}a(\mathfrak{X}, \mathfrak{x}) = \bigvee_{\mathfrak{X} \in TTX} (m_X)_{\circ}(\mathfrak{X}, e_X(x)) \otimes \hat{T}a(\mathfrak{X}, \mathfrak{x}) = \bigvee_{m_X(\mathfrak{X}) = e_X(x)} \hat{T}a(\mathfrak{X}, \mathfrak{x}) \geqslant \hat{T}a(e_{TX} \cdot e_X(x), \mathfrak{x})$, since $m_X(e_{TX} \cdot e_X(x)) = (m_X \cdot e_{TX}) \cdot e_X(x) \overset{m_X \cdot e_{TX} = 1_{TX}}{=} 1_{TX} \cdot e_X(x) = e_X(x)$. If $e_X \cdot \leqslant_a (x, \mathfrak{x}) = k$, then there exists $x' \in X$ such that $k \leqslant a(e_X(x), x')$ and $e_X(x') = \mathfrak{x}$, which implies $\hat{T}a(e_{TX} \cdot e_X(x), \mathfrak{x}) = \hat{T}a(e_{TX} \cdot e_X(x), e_X(x')) = \hat{T}a \cdot e_{TX}(e_X(x), e_X(x')) \overset{e_X \cdot a \leqslant \hat{T}a \cdot e_{TX}}{\geqslant} e_X \cdot a(e_X(x), e_X(x')) = e_X^{\circ} \cdot e_X \cdot a(e_X(x), x') \overset{1_X \leqslant e_X^{\circ} \cdot e_X}{\geqslant} a(e_X(x), x') \geqslant k$. □

Remark 25. Notice that if $X \xrightarrow{e_X} TX$ is injective for every set X, then *Ind* is a subfunctor of G, i.e., G is an extension of the induced preorder from the underlying set X of a $(\mathbb{T}, V)$-category (X, a) to the set TX. ▪

Example 26.

(1) If $\mathbb{T}$ is the identity monad on **Set**, then V-**Cat** $\xrightarrow{G}$ V-**Cat** is the identity functor.

(2) For the lax extension $\hat{\beta}$ to **Rel** of the ultrafilter monad β on **Set**, the functor **Top** $\xrightarrow{G}$ **Prost** is defined by $G((X, \tau) \xrightarrow{f} (Y, \sigma)) = (\beta X, \leqslant) \xrightarrow{\beta f} (\beta Y, \leqslant)$, where for every $\mathfrak{x}, \mathfrak{z} \in \beta X$, $\mathfrak{x} \leqslant \mathfrak{z}$ iff $\mathfrak{z} \bigcap \tau \subseteq \mathfrak{x}$. In particular, given principal ultrafilters $\dot{x}, \dot{y} \in \beta X$, it follows that $\dot{x} \leqslant \dot{y}$ iff $y \in cl\{x\}$. In other words, since the principal ultrafilter natural transformation $1_{\mathbf{Set}} \xrightarrow{e} \beta$ has injective components $X \xrightarrow{e} \beta X$ for every set X, one obtains that G is an extension of the induced preorder from the underlying set X of a topological space (X, τ) to the set of ultrafilters on X (cf. Remark 25). ▪

2.2. Algebraic preliminaries

Definition 27. Let $(V, \bigvee, \otimes)$ be a quantale.

(1) V is called *strictly two-sided* provided that $(V, \otimes, \top_V)$ is a monoid.

(2) V is called *cartesian closed* provided that $a \wedge (\bigvee B) = \bigvee_{b \in B}(a \wedge b)$ for every $a \in V$, $B \subseteq V$. ■

Remark 28. Observe that a quantale V is cartesian closed iff its underlying partially ordered set is a *frame*, namely, a complete lattice, in which finite meets distribute over arbitrary joins. ■

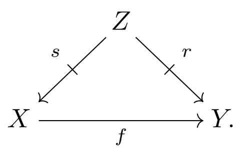
Theorem 29. *Given a unital quantale V, equivalent are:*

(1) *V is cartesian closed;*

(2) *the* left Frobenius law

$$f \cdot ((f^\circ \cdot r) \wedge s) = r \wedge (f \cdot s) \tag{F}$$

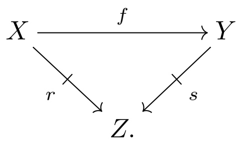
*holds in V-**Rel** for every triangle of the form*



(3) *the* right Frobenius law

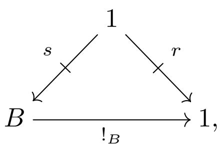
$$(r \wedge (s \cdot f)) \cdot f^\circ = (r \cdot f^\circ) \wedge s$$

*holds in V-**Rel** for every triangle of the form*



Proof.
(1) ⇒ (2): Given $y \in Y$ and $z \in Z$, on the one hand, $f \cdot ((f^\circ \cdot r) \wedge s)(z, y) = \bigvee_{f(x)=y}((f^\circ \cdot r) \wedge s)(z, x) = \bigvee_{f(x)=y}((f^\circ \cdot r)(z, x) \wedge s(z, x)) = \bigvee_{f(x)=y}(r(z, f(x)) \wedge s(z, x)) = \bigvee_{f(x)=y}(r(z, y) \wedge s(z, x))$; on the other hand, $(r \wedge (f \cdot s))(z, y) = r(z, y) \wedge (\bigvee_{f(x)=y} s(z, x)) \overset{(1)}{=} \bigvee_{f(x)=y}(r(z, y) \wedge s(z, x))$.

(2) ⇒ (1): Given $a \in V$ and $B \subseteq V$, consider the triangle



where $s(*, b) = b$ for every $b \in B$, and $r(*, *) = a$. It then follows that $\bigvee_{b \in B}(a \wedge b) = \bigvee_{b \in B}((a \otimes k) \wedge b) = \bigvee_{b \in B}((r(*, *) \otimes !_B^\circ(*, b)) \wedge b) = \bigvee_{b \in B}((!_B^\circ \cdot r)(*, b) \wedge s(*, b)) \otimes k = \bigvee_{b \in B}((!_B^\circ \cdot r) \wedge s)(*, b) \otimes (!_B)_\circ(b, *) = !_B \cdot ((!_B^\circ \cdot r) \wedge s)(*, *) \overset{(2)}{=} (r \wedge (!_B \cdot s))(*, *) = r(*, *) \wedge (!_B \cdot s)(*, *) = a \wedge (\bigvee_{b \in B} s(*, b) \otimes (!_B)_\circ(b, *)) = a \wedge (\bigvee_{b \in B} b \otimes k) = a \wedge (\bigvee B)$, i.e., $a \wedge (\bigvee B) = \bigvee_{b \in B}(a \wedge b)$. □

Remark 30. From now on, V stands for a cartesian closed, strictly two-sided quantale. ■

Example 31. The quantales 2 and P_+ satisfy the conditions of Remark 30. ■

Definition 32. Given a topological space (X,τ) and an ultrafilter $\mathfrak{x}\in\beta X$, an element $x\in X$ is a *limit* of $\mathfrak{x}$ ($\mathfrak{x}$ *converges to* x) provided that $\mathfrak{x}$ contains every $U\in\tau$ such that $x\in U$. $\lim\mathfrak{x}$ is the set of limits of $\mathfrak{x}$. ▪

Theorem 33. *Given topological spaces (X,τ) and (Y,σ), a continuous map $X\xrightarrow{f}Y$ is proper iff for every ultrafilter $\mathfrak{x}\in\beta X$ and every $y\in\lim\beta f(\mathfrak{x})$, there exists $x\in\lim\mathfrak{x}$ such that $f(x)=y$.*

Remark 34. Representing the category **Top** as $(\beta,2)$-**Cat**, one gets that a $(\beta,2)$-functor $(X,a)\xrightarrow{f}(Y,b)$ is proper iff $b(\beta f(\mathfrak{x}),y)\leqslant\bigvee_{f(x)=y}a(\mathfrak{x},x)$ for every $\mathfrak{x}\in\beta X$, $y\in Y$ iff $b\cdot\beta f(\mathfrak{x},y)\leqslant f\cdot a(\mathfrak{x},y)$ for every $\mathfrak{x}\in\beta X$, $y\in Y$ iff $b\cdot\beta f\leqslant f\cdot a$ in **Rel** iff $f\cdot a=b\cdot\beta f$ in **Rel** ($f\cdot a\leqslant b\cdot\beta f$ is the definition of $(\beta,2)$-functors). ▪

Definition 35. A $(\mathbb{T},V)$-functor $(X,a)\xrightarrow{f}(Y,b)$ is *proper* provided that the diagram

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutes, i.e., $f\cdot a=b\cdot Tf$. ▪

Example 36.

(1) **Prost**: an order-preserving map $(X,\leqslant_X)\xrightarrow{f}(Y,\leqslant_Y)$ is proper iff $f\cdot\leqslant_X\;=\;\leqslant_Y\cdot f$ iff for every $x\in X$ and every $y\in Y$ such that $f(x)\leqslant y$, there exists $z\in X$ such that $x\leqslant z$ and $f(z)=y$.
(2) **QPMet**: a non-expansive map $(X,\rho)\xrightarrow{f}(Y,\varrho)$ is proper iff $\varrho(f(x),y)=\inf\{\rho(x,z)\,|\,z\in X \text{ and } f(z)=y\}$ for every $x\in X$, $y\in Y$.
(3) **Top**: Definition 35 gives precisely the proper maps of Definition 1 (2).
(4) **App**: a non-expansive map $(X,\delta)\xrightarrow{f}(Y,\sigma)$ is proper iff $\sup_{f^{-1}(B)\in\mathfrak{x}}\sigma(y,B)=\inf_{f(x)=y}\sup_{A\in\mathfrak{x}}\delta(x,A)$ for every $\mathfrak{x}\in\beta X$, $y\in Y$.
(5) **Cls**: a continuous map $(X,c)\xrightarrow{f}(Y,d)$ is proper iff for every $A\in PX$, $y\in Y$ such that $y\in d(f(A))$, there exists $x\in X$ such that $x\in c(A)$ and $f(x)=y$ iff $d(f(A))\subseteq f(c(A))$ for every $A\in PX$. ▪

Theorem 37. *Proper maps are stable under pullbacks in* $(\mathbb{T},V)$-**Cat**.

Proof. Notice that given a pullback diagram

$$\begin{array}{ccc} X\times_Z Y & \xrightarrow{p_Y} & Y \\ {\scriptstyle p_X}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

in **Set**, it follows that

$$g^\circ\cdot f=p_Y\cdot p_X^\circ, \qquad (2.3)$$

since given $x\in X$ and $y\in Y$, $g^\circ\cdot f(x,y)=f_\circ(x,g(y))=\begin{cases}k, & f(x)=g(y)\\ \bot_V, & \text{otherwise}\end{cases}=\begin{cases}k, & (x,y)\in X\times_Z Y\\ \bot_V, & \text{otherwise}\end{cases}=$ $\bigvee_{(x',y')\in X\times_Z Y}p_X((x',y'),x)\otimes p_Y((x',y'),y)=\bigvee_{(x',y')\in X\times_Z Y}p_X^\circ(x,(x',y'))\otimes p_Y((x',y'),y)=p_Y\cdot p_X^\circ(x,y)$.

Consider now diagram (2.1), in which f is proper. To show that p_Y is proper, notice that $b\cdot Tp_Y=(b\wedge b)\cdot Tp_Y\overset{b\leqslant g^\circ\cdot c\cdot Tg}{\leqslant}((g^\circ\cdot c\cdot Tg)\wedge b)\cdot Tp_Y=(g^\circ\cdot c\cdot Tg\cdot Tp_Y)\wedge(b\cdot Tp_Y)=(g^\circ\cdot c\cdot T(g\cdot p_Y))\wedge(b\cdot Tp_Y)\overset{g\cdot p_Y=f\cdot p_X}{=}(g^\circ\cdot c\cdot T(f\cdot p_X))\wedge(b\cdot Tp_Y)=(g^\circ\cdot c\cdot Tf\cdot Tp_X)\wedge(b\cdot Tp_Y)\overset{c\cdot Tf=f\cdot a}{=}(g^\circ\cdot f\cdot a\cdot Tp_X)\wedge(b\cdot Tp_Y)\overset{(2.3)}{=}(p_Y\cdot p_X^\circ\cdot a\cdot Tp_X)\wedge(b\cdot Tp_Y)\overset{(F)}{=}p_Y\cdot((p_X^\circ\cdot a\cdot Tp_X)\wedge(p_Y^\circ\cdot b\cdot Tp_Y))=p_Y\cdot d$. □

Definition 38. Given a $(\mathbb{T},V)$-functor $(X,a)\xrightarrow{f}(Y,b)$, the *fibre of f on y* is the pullback $(f^{-1}(y),\tilde{a})\xrightarrow{!_{f^{-1}(y)}}(1,1^\sharp)$ of f along the $(\mathbb{T},V)$-functor $(1,1^\sharp)\xrightarrow{y}(Y,b)$, where $1^\sharp=e_1^\circ\cdot\hat{T}1_1$ is the discrete structure on 1, i.e.,

$$\begin{array}{ccc} (f^{-1}(y),\tilde{a}) & \xrightarrow{\;!_{f^{-1}(y)}\;} & (1,1^\sharp) \\ {\scriptstyle i_{f^{-1}(y)}}\big\downarrow & & \big\downarrow{\scriptstyle y} \\ (X,a) & \xrightarrow[\;f\;]{} & (Y,b), \end{array} \tag{2.4}$$

where $\tilde{a}=(i^\circ_{f^{-1}(y)}\cdot a\cdot Ti_{f^{-1}(y)})\wedge(!^\circ_{f^{-1}(y)}\cdot e_1^\circ\cdot\hat{T}1_1\cdot T!_{f^{-1}(y)})$, or, in pointwise notation, $\tilde{a}(\mathfrak{x},x)=a(\mathfrak{x},x)\wedge\hat{T}1_1(T!_{f^{-1}(y)}(\mathfrak{x}),e_1(*))$ for every $\mathfrak{x}\in T(f^{-1}(y))$ and every $x\in f^{-1}(y)$. ▪

Theorem 39. *A $(\mathbb{T},V)$-functor $(X,a)\xrightarrow{f}(Y,b)$ is proper iff all its fibres are proper, and the V-functor $(TX,\hat{a})\xrightarrow{Tf}(TY,\hat{b})$ is proper.*

Proof. For the necessity, notice that Theorem 37 provides properness of fibres. To show the second claim, notice first that for every lax extension $\hat{\mathbb{T}}$ to V-**Rel** of a monad $\mathbb{T}$ on **Set**, and every set X, one can obtain

$$\hat{T}1_X=\hat{T}(e_X^\circ)\cdot m_X^\circ \tag{2.5}$$

(see Lecture 2 for more detail). Then $\hat{b}\cdot Tf=\hat{T}b\cdot m_Y^\circ\cdot Tf\leqslant\hat{T}b\cdot m_Y^\circ\cdot\hat{T}f\overset{(\mathrm{N})}{=}\hat{T}b\cdot\hat{T}\hat{T}f\cdot m_X^\circ\leqslant\hat{T}(b\cdot\hat{T}f)\cdot m_X^\circ\overset{(\dagger)}{\leqslant}\hat{T}(b\cdot Tf)\cdot m_X^\circ\overset{b\cdot Tf=f\cdot a}{=}\hat{T}(f\cdot a)\cdot m_X^\circ\overset{(\mathrm{W})}{=}Tf\cdot\hat{T}a\cdot m_X^\circ=Tf\cdot\hat{a}$, where $(\dagger)$ uses the fact that $b\cdot\hat{T}f=b\cdot\hat{T}(1_Y\cdot f)=b\cdot\hat{T}1_Y\cdot Tf\overset{(2.5)}{=}b\cdot\hat{T}(e_Y^\circ)\cdot m_Y^\circ\cdot Tf\overset{e_Y^\circ\leqslant b}{\leqslant}b\cdot\hat{T}b\cdot m_Y^\circ\cdot Tf\overset{b\cdot\hat{T}b\leqslant b\cdot m_Y}{\leqslant}b\cdot m_Y\cdot m_Y^\circ\cdot Tf\overset{m_Y\cdot m_Y^\circ\leqslant 1_{TY}}{\leqslant}b\cdot Tf$.

The sufficiency can be shown as follows. Firstly, notice that a sink $(X_i\xrightarrow{f_i}X)_I$ in **Set** is an epi-sink iff

$$\bigvee_{i\in I}f_i\cdot f_i^\circ=1_X. \tag{2.6}$$

Secondly, notice that $b=b\cdot 1^\circ_{TY}=b\cdot(m_Y\cdot e_{TY})^\circ=b\cdot e^\circ_{TY}\cdot m_Y^\circ\overset{b\cdot e^\circ_{TY}\leqslant e^\circ_Y\cdot\hat{T}b}{\leqslant}e_Y^\circ\cdot\hat{T}b\cdot m_Y^\circ=e_Y^\circ\cdot\hat{b}$. It follows then that $b\cdot Tf\leqslant e_Y^\circ\cdot\hat{b}\cdot Tf\overset{\hat{b}\cdot Tf=Tf\cdot\hat{a}}{=}e_Y^\circ\cdot Tf\cdot\hat{a}=e_Y^\circ\cdot Tf\cdot\hat{T}a\cdot m_X^\circ=1_Y\cdot e_Y^\circ\cdot Tf\cdot\hat{T}a\cdot m_X^\circ\overset{(\dagger)}{=}(\bigvee_{y\in Y}y\cdot y^\circ)\cdot e_Y^\circ\cdot Tf\cdot\hat{T}a\cdot m_X^\circ=(\bigvee_{y\in Y}y\cdot y^\circ\cdot e_Y^\circ\cdot Tf)\cdot\hat{T}a\cdot m_X^\circ=(\bigvee_{y\in Y}y\cdot(e_Y\cdot y)^\circ\cdot Tf)\cdot\hat{T}a\cdot m_X^\circ\overset{e_Y\cdot y=Ty\cdot e_1}{=}(\bigvee_{y\in Y}y\cdot(Ty\cdot e_1)^\circ\cdot Tf)\cdot\hat{T}a\cdot m_X^\circ=(\bigvee_{y\in Y}y\cdot e_1^\circ\cdot(Ty)^\circ\cdot Tf)\cdot\hat{T}a\cdot m_X^\circ=r_1$, where $(\dagger)$ uses that fact that $(1\xrightarrow{y}Y)_Y$ is an epi-sink in **Set**. Since the underlying **Set**-diagram of (2.4) is a pullback along the monomorphism $1\xrightarrow{y}Y$, by (T),

$$\begin{array}{ccc} T(f^{-1}(y)) & \xrightarrow{\;T!_{f^{-1}(y)}\;} & T1 \\ {\scriptstyle Ti_{f^{-1}(y)}}\big\downarrow & & \big\downarrow{\scriptstyle Ty} \\ TX & \xrightarrow[\;Tf\;]{} & TY \end{array}$$

is a pullback as well. Similar to (2.3), $(Ty)^\circ\cdot Tf=T!_{f^{-1}(y)}\cdot(Ti_{f^{-1}(y)})^\circ$, and then $r_1=(\bigvee_{y\in Y}y\cdot e_1^\circ\cdot T!_{f^{-1}(y)}\cdot(Ti_{f^{-1}(y)})^\circ)\cdot\hat{T}a\cdot m_X^\circ=(\bigvee_{y\in Y}y\cdot e_1^\circ\cdot T1_1\cdot T!_{f^{-1}(y)}\cdot(Ti_{f^{-1}(y)})^\circ)\cdot\hat{T}a\cdot m_X^\circ\leqslant(\bigvee_{y\in Y}y\cdot e_1^\circ\cdot\hat{T}1_1\cdot T!_{f^{-1}(y)}\cdot(Ti_{f^{-1}(y)})^\circ)\cdot\hat{T}a\cdot m_X^\circ=r_2$. Given $y\in Y$, properness of the fibres of f implies that the $(\mathbb{T},V)$-functor $(f^{-1}(y),\tilde{a})\xrightarrow{!_{f^{-1}(y)}}(1,1^\sharp)$ is proper, i.e., $1^\sharp\cdot T!_{f^{-1}(y)}=!_{f^{-1}(y)}\cdot\tilde{a}$. Definition 38 implies that $e_1^\circ\cdot\hat{T}1_1\cdot T!_{f^{-1}(y)}=1^\sharp\cdot T!_{f^{-1}(y)}=!_{f^{-1}(y)}\cdot\tilde{a}=!_{f^{-1}(y)}\cdot((i^\circ_{f^{-1}(y)}\cdot a\cdot Ti_{f^{-1}(y)})\wedge(!^\circ_{f^{-1}(y)}\cdot e_1^\circ\cdot\hat{T}1_1\cdot T!_{f^{-1}(y)}))\leqslant!_{f^{-1}(y)}\cdot i^\circ_{f^{-1}(y)}\cdot a\cdot$

$Ti_{f^{-1}(y)}$, and thus, $r_2 \leqslant (\bigvee_{y\in Y} y\cdot !_{f^{-1}(y)} \cdot i^{\circ}_{f^{-1}(y)} \cdot a \cdot Ti_{f^{-1}(y)} \cdot (Ti_{f^{-1}(y)})^{\circ}) \cdot \hat{T}a \cdot m_X^{\circ} \overset{Ti_{f^{-1}(y)}\cdot (Ti_{f^{-1}(y)})^{\circ} \leqslant 1_{TX}}{\leqslant}$ $(\bigvee_{y\in Y} y\cdot !_{f^{-1}(y)} \cdot i^{\circ}_{f^{-1}(y)} \cdot a) \cdot \hat{T}a \cdot m_X^{\circ} \overset{y\cdot !_{f^{-1}(y)} = f\cdot i_{f^{-1}(y)}}{=} (\bigvee_{y\in Y} f \cdot i_{f^{-1}(y)} \cdot i^{\circ}_{f^{-1}(y)} \cdot a) \cdot \hat{T}a \cdot m_X^{\circ} \overset{i_{f^{-1}(y)}\cdot i^{\circ}_{f^{-1}(y)} \leqslant 1_X}{\leqslant}$ $(\bigvee_{y\in Y} f \cdot a) \cdot \hat{T}a \cdot m_X^{\circ} \leqslant f \cdot a \cdot \hat{T}a \cdot m_X^{\circ} \overset{a\cdot \hat{T}a \leqslant a\cdot m_X}{\leqslant} f \cdot a \cdot m_X \cdot m_X^{\circ} \overset{m_X\cdot m_X^{\circ} \leqslant 1_{TX}}{\leqslant} f \cdot a.$ □

Theorem 40. *If $\hat{T} \xrightarrow{e^{\circ}} 1_{V\text{-}\mathbf{Rel}}$ is a natural transformation, then every $(\mathbb{T}, V)$-functor has proper fibres.*

Proof. Given a $(\mathbb{T}, V)$-functor $(X, a) \xrightarrow{f} (Y, b)$, one has to show that the diagram

$$\begin{array}{ccc} T(f^{-1}(y)) & \xrightarrow{T!_{f^{-1}(y)}} & T1 \\ {\scriptstyle \tilde{a}}\big\downarrow & & \big\downarrow{\scriptstyle 1^{\sharp}} \\ f^{-1}(y) & \xrightarrow[!_{f^{-1}(y)}]{} & 1 \end{array}$$

commutes. The condition of the theorem implies commutativity of the diagram

$$\begin{array}{ccccc} T(f^{-1}(y)) & \xrightarrow{\hat{T}!_{f^{-1}(y)}} & T1 & \xrightarrow{\hat{T}1_1} & TX \\ {\scriptstyle e^{\circ}_{f^{-1}(y)}}\big\downarrow & & {\scriptstyle e^{\circ}_1}\big\downarrow & & \big\downarrow{\scriptstyle e^{\circ}_1} \\ f^{-1}(y) & \xrightarrow[!_{f^{-1}(y)}]{} & 1 & \xrightarrow[1_1]{} & 1. \end{array}$$

Given $\mathfrak{x} \in T(f^{-1}(y))$, it follows then that $1^{\sharp} \cdot T!_{f^{-1}(y)}(\mathfrak{x}, *) = e^{\circ}_1 \cdot \hat{T}1_1 \cdot T!_{f^{-1}(y)}(\mathfrak{x}, *) \leqslant e^{\circ}_1 \cdot \hat{T}1_1 \cdot \hat{T}!_{f^{-1}(y)}(\mathfrak{x}, *) =$ $!_{f^{-1}(y)} \cdot e^{\circ}_{f^{-1}(y)}(\mathfrak{x}, *) = \bigvee_{f(x)=y} e^{\circ}_{f^{-1}(y)}(\mathfrak{x}, x) \overset{e^{\circ}_{f^{-1}(y)} \leqslant \tilde{a}}{\leqslant} \bigvee_{f(x)=y} \tilde{a}(\mathfrak{x}, x) = !_{f^{-1}(y)} \cdot \tilde{a}(\mathfrak{x}, *)$. □

Corollary 41. *If $\hat{T} \xrightarrow{e^{\circ}} 1_{V\text{-}\mathbf{Rel}}$ is a natural transformation, then a $(\mathbb{T}, V)$-functor $(X, a) \xrightarrow{f} (Y, b)$ is proper iff the V-functor $(TX, \hat{a}) \xrightarrow{Tf} (TY, \hat{b})$ is proper.*

Proof. Follows from Theorems 39, 40. □

Remark 42. Observe that given a lax extension $\hat{\mathbb{T}}$ to V-**Rel** of a monad $\mathbb{T}$ on **Set**, since $1_{V\text{-}\mathbf{Rel}} \xrightarrow{e} \hat{T}$ is an oplax natural transformation (recall Lecture 1), it follows that

$$\begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ {\scriptstyle r}\big\downarrow & \leqslant & \big\downarrow{\scriptstyle \hat{T}r} \\ Y & \xrightarrow[e_Y]{} & TY \end{array}$$

for every V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$. Thus, $e_Y \cdot r \leqslant \hat{T}r \cdot e_X$ implies $r \cdot e^{\circ}_X \leqslant e^{\circ}_Y \cdot e_Y \cdot r \cdot e^{\circ}_X \leqslant e^{\circ}_Y \cdot \hat{T}r \cdot e_X \cdot e^{\circ}_X \leqslant e^{\circ}_Y \cdot \hat{T}r$, i.e., $r \cdot e^{\circ}_X \leqslant e^{\circ}_Y \cdot \hat{T}r$. As a consequence, one obtains that $\hat{T} \xrightarrow{e^{\circ}} 1_{V\text{-}\mathbf{Rel}}$ is a natural transformation iff

$$\begin{array}{ccc} TX & \xrightarrow{e^{\circ}_X} & X \\ {\scriptstyle \hat{T}r}\big\downarrow & \leqslant & \big\downarrow{\scriptstyle r} \\ TY & \xrightarrow[e^{\circ}_Y]{} & Y \end{array}$$

for every V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$. ▪

Remark 43.

(1) The lax extension $\hat{\mathbb{P}}$ of the powerset monad $\mathbb{P}$ on **Set** to **Rel** fails to satisfy the condition of Corollary 41. Observe that following Remark 42, it is enough to a find a relation $X \stackrel{r}{\relbar\joinrel\mapstochar\joinrel\rightarrow} Y$ such that $e_Y^\circ \cdot \hat{P}r \nleqslant r \cdot e_X^\circ$. Given $A \in PX$ and $y \in Y$, on the one hand, $A\,(e_Y^\circ \cdot \hat{P}r)\,y$ iff $A\,(\hat{P}r)\,e_Y(y)$ iff $A\,\hat{P}r\,\{y\}$ iff there exists $x \in A$ such that $x\,r\,y$ (recall Lecture 1), and, on the other hand, $A\,(r \cdot e_X^\circ)\,y$ iff there exists $x \in X$ such that $A\,e_X^\circ\,x$ and $x\,r\,y$ iff there exists $x \in X$ such that $e_X(x) = A$ and $x\,r\,y$ iff there exists $x \in X$ such that $A = \{x\}$ and $x r y$. If $X = Y = \{0,1\}$ and $r = \{(0,0),(1,1)\} \subseteq X \times Y$, then $X\,(e_Y^\circ \cdot \hat{P}r)\,0$ (since $0\,r\,0$), but X and 0 fail to be in relation $r \cdot e_X^\circ$ since $\{0\} \neq \{0,1\}$.

(2) The lax extension $\hat{\beta}$ (resp. $\bar{\beta}$) of the ultrafilter monad β on **Set** to **Rel** (resp. P_+-**Rel**) fails to satisfy the condition of Corollary 41.

(3) There exist monads on **Set**, whose lax extensions satisfy the condition of Corollary 41. ▪

2.4. Compact $(\mathbb{T}, V)$-categories

Definition 44. A $(\mathbb{T}, V)$-category (X, a) is said to be *compact* provided that the unique $(\mathbb{T}, V)$-functor $(X, a) \xrightarrow{!_X} (1, \top)$ is proper. ▪

Example 45. Given a compact $(\mathbb{T}, V)$-category (X, a), it follows that $!_X \cdot a = \top \cdot T!_X$, or, in pointwise notation, $\bigvee_{x \in X} a(\mathfrak{x}, x) = \top_V$ for every $\mathfrak{x} \in TX$.

(1) **Prost**: a preordered set $(X, \leqslant)$ is compact provided that for every $y \in X$, there exists $x \in X$ such that $y \leqslant x$, which is always true (choose $x = y$).
(2) **QPMet**: a quasi-pseudo-metric space (X, ρ) is compact provided that $\inf_{x \in X} \rho(y, x) = 0$ for every $y \in X$, which is always true (choose $x = y$).
(3) **Top**: a topological space (X, τ) is compact provided that every ultrafilter on X has a limit point, which is precisely the standard definition of compactness of topological spaces.
(4) **App**: an approach space (X, δ) is compact provided that $\inf_{x \in X} \sup_{A \in \mathfrak{x}} \delta(x, A) = 0$ for every $\mathfrak{x} \in \beta X$.
(5) **Cls**: a closure space (X, c) is compact provided that $c(A) \neq \varnothing$ for every $A \in PX$. Observe that given $A \in PX$, it follows that $A \subseteq c(A)$, which implies that $c(A) \neq \varnothing$ provided that $A \neq \varnothing$. Thus, a closure space (X, c) is compact iff $c(\varnothing) \neq \varnothing$. It then follows that a closure space induced by a topological space is never compact, since $\varnothing$ is closed ($c(\varnothing) = \varnothing$) in every topological space. ▪

Remark 46. If $T1 \cong 1$, then the $(\mathbb{T}, V)$-category $(1, 1^\sharp)$ (which is additionally a separator in $(\mathbb{T}, V)$-**Cat**) coincides with the terminal object $(1, \top)$ (since $1^\sharp(*, *) = e_1^\circ \cdot \hat{T}1_1(*, *) = \hat{T}1_1(*, e_1(*)) = \hat{T}1_1(*, *) \geqslant 1_1(*, *) = \top_V$), and therefore, it follows that (X, a) is compact iff the only fibre of $(X, a) \xrightarrow{!_X} (1, \top)$ is proper, since the respective fibre is then given by the pullback

$$\begin{array}{ccc} (X, a) & \xrightarrow{!_X} & (1, \top) \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle 1_1} \\ (X, a) & \xrightarrow[!_X]{} & (1, \top). \end{array}$$

▪

Example 47.

(1) For the powerset functor P on **Set**, $P1 = \{\varnothing, \{*\}\} \ncong \{*\} = 1$.
(2) For the ultrafilter functor β on **Set**, $\beta 1 \cong 1$. ▪

Theorem 48. *If (X, a) is a compact $(\mathbb{T}, V)$-category, then the fibre of the $(\mathbb{T}, V)$-functor $(X, a) \xrightarrow{!_X} (1, \top)$ is proper. If the two structures $1^\sharp$ and $\top$ on 1 coincide, then the converse is true.*

Proof. Follows from Theorem 37 and the arguments of Remark 46. ☐

Corollary 49. *Suppose that $\top$ is the discrete structure on* 1. *Given a* $(\mathbb{T},V)$*-functor* $(X,a)\xrightarrow{f}(Y,b)$, *equivalent are:*

(1) f *is proper;*

(2) *the* V*-functor* $(TX,\hat{a})\xrightarrow{Tf}(TY,\hat{b})$ *is proper and* f *has compact fibres.*

Proof. Follows from Theorems 39, 48. ☐

Corollary 50. *If $\top$ is the discrete structure on* 1, *and* $\hat{T}\xrightarrow{e^\circ}1_{V\text{-}\mathbf{Rel}}$ *is a natural transformation, then every* $(\mathbb{T},V)$*-category is compact.*

Proof. Recall Theorem 40. ☐

Theorem 51. *If the lax extension $\hat{T}$ to V-**Rel** of a functor T on **Set** is flat, then $\top=1^\sharp$ iff $T1\cong 1$.*

Proof. The sufficiency is clear. For the necessity, notice that given $\mathfrak{x}\in T1$, it follows that $\top_V=\top(\mathfrak{x},*)=1^\sharp(\mathfrak{x},*)=e_1^\circ\cdot\hat{T}1_1(\mathfrak{x},*)\overset{\hat{T}\text{ is flat}}{=}e_1^\circ\cdot T1_1(\mathfrak{x},*)=e_1^\circ(\mathfrak{x},*)$, and therefore, $\mathfrak{x}=e_1(*)$. ☐

3. Closed maps in the category $(\mathbb{T},V)$-Cat

Lemma 52. *A continuous map* $(X,\tau)\xrightarrow{f}(Y,\sigma)$ *between topological spaces is closed iff* $cl(f(A))\subseteq f(cl(A))$ *for every* $A\subseteq X$.

Proof.

$\Rightarrow$: Given a subset $A\subseteq X$, if f is closed, then $f(cl(A))$ is closed. Thus, $A\subseteq cl(A)$ implies $f(A)\subseteq f(cl(A))$ implies $cl(f(A))\subseteq cl(f(cl(A)))=f(cl(A))$, i.e., $cl(f(A))\subseteq f(cl(A))$.

$\Leftarrow$: Given a subset $A\subseteq X$, since $A\subseteq f^{-1}(f(A))\subseteq f^{-1}(cl(f(A)))$ and $f^{-1}(cl(f(A)))$ is closed, $cl(A)\subseteq f^{-1}(cl(f(A)))$ and then $f(cl(A))\subseteq f(f^{-1}(cl(f(A))))\subseteq cl(f(A))$, i.e., $f(cl(A))\subseteq cl(f(A))$. Thus, $f(cl(A))=cl(f(A))$ by the assumption of the lemma. If A is closed, then $f(A)=f(cl(A))=cl(f(A))$. ☐

Lemma 53. *Given a topological space* (X,τ) *and* $A\subseteq X$, *it follows that* $cl(A)=\bigcup_{\mathfrak{x}\in\beta X\text{ and }A\in\mathfrak{x}}\lim\mathfrak{x}$.

Definition 54. A $(\mathbb{T},V)$-functor $(X,a)\xrightarrow{f}(Y,b)$ is *closed* provided that for every $A\subseteq X$,

$$f\cdot a\cdot Ti_A\cdot !_{TA}^\circ=b\cdot Tf\cdot Ti_A\cdot !_{TA}^\circ, \tag{3.1}$$

where $A\xhookrightarrow{i_A}X$ is the inclusion map and $TA\xrightarrow{!_{TA}}1$ is the unique map. ▪

Remark 55. Observe that given a V-relation $TX\overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;}X$, for every subset $A\subseteq X$, the composite V-relation $1\overset{!_{TA}^\circ}{\longrightarrow\!\!\!\!\!\!\!+\;\;}TA\xhookrightarrow{Ti_A}TX\overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;}X$ in pointwise notation provides $r\cdot Ti_A\cdot !_{TA}^\circ(*,x)=\bigvee_{\mathfrak{y}\in TA}!_{TA}^\circ(*,\mathfrak{y})\otimes(r\cdot Ti_A)(\mathfrak{y},x)=\bigvee_{\mathfrak{y}\in TA}(!_{TA})_\circ(\mathfrak{y},*)\otimes(\bigvee_{\mathfrak{x}\in TX}(Ti_A)_\circ(\mathfrak{y},\mathfrak{x})\otimes r(\mathfrak{x},x))=\bigvee_{\mathfrak{y}\in TA}r(Ti_A(\mathfrak{y}),x)$ for every $x\in X$. ▪

Lemma 56. *Given a* $(\mathbb{T},V)$*-functor* $(X,a)\xrightarrow{f}(Y,b)$, *the following are equivalent:*

(1) f *is closed;*
(2) $b\cdot Tf\cdot Ti_A\cdot !_{TA}^\circ\leqslant f\cdot a\cdot Ti_A\cdot !_{TA}^\circ$ *for every* $A\subseteq X$.

Proof. Recall that since $(X,a)\xrightarrow{f}(Y,b)$ is a $(\mathbb{T},V)$-functor, it follows that $f\cdot a\leqslant b\cdot Tf$. ☐

Example 57. Let $(X,a) \xrightarrow{f} (Y,b)$ be a $(\mathbb{T},V)$-functor. Given $A \subseteq X$, denote by $A \xrightarrow{\overline{f}} f(A)$ the restriction of f to A and $f(A)$, respectively. Commutativity of the diagram

$$\begin{array}{ccccc} 1 & \xrightarrow[\;\;+\;\;]{!^{\circ}_{TA}} & TA & \xrightarrow{Ti_A} & TX \\ & \searrow^{\!\!+}_{!^{\circ}_{T(f(A))}} & \downarrow{\scriptstyle T\overline{f}} & & \downarrow{\scriptstyle Tf} \\ & & T(f(A)) & \xrightarrow[Ti_{f(A)}]{} & TY \end{array}$$

and Lemma 56 replace (3.1) with $b \cdot Ti_{f(A)} \cdot !^{\circ}_{T(f(A))} \leqslant f \cdot a \cdot Ti_A \cdot !^{\circ}_{TA}$, which, in pointwise notation, provides

$$\bigvee_{\mathfrak{y} \in T(f(A))} b(Ti_{f(A)}(\mathfrak{y}), y) \leqslant \bigvee_{\mathfrak{x} \in TA} \bigvee_{f(x)=y} a(Ti_A(\mathfrak{x}), x) \tag{3.2}$$

for every $y \in Y$. In some particular cases, (3.2) can be rewritten as follows.

(1) **Prost**: an order-preserving map $(X,\leqslant) \xrightarrow{f} (Y,\leqslant)$ is closed iff for every $x \in X$ and every $y \in Y$ such that $f(x) \leqslant y$, there exists $z \in X$ such that $x \leqslant z$ and $f(z) = y$.

(2) **QPMet**: a non-expansive map $(X,\rho) \xrightarrow{f} (Y,\varrho)$ is closed iff $\inf\{\rho(x,z) \mid z \in X \text{ and } f(z) = y\} \leqslant \varrho(f(x),y)$ for every $x \in X$, $y \in Y$.

(3) **Top**: one gets precisely the result of Lemma 52.

(4) **App**: a non-expansive map $(X,\delta) \xrightarrow{f} (Y,\sigma)$ is closed iff $\inf_{f(x)=y} \delta(x,A) \leqslant \sigma(y,f(A))$ for every $A \subseteq X$.

(5) **Cls**: a continuous map $(X,c) \xrightarrow{f} (Y,d)$ is closed iff $\bigcup\{d(C) \mid C \subseteq f(A)\} \subseteq \bigcup\{f(c(B)) \mid B \subseteq A\}$ for every $A \in PX$ iff $d(f(A)) \subseteq f(c(A))$. ▪

Theorem 58. *Every proper $(\mathbb{T},V)$-functor is closed.*

Proof. Follows directly from the definition of the two concepts. □

Theorem 59. *Suppose every $(\mathbb{T},V)$-category (X,a) has the property that given $\mathfrak{x} \in TX$, there exists $A \subseteq X$ such that*

$$\mathfrak{x} \in Ti_A(TA) \quad \text{and} \quad a \cdot Ti_A \cdot !^{\circ}_{TA} \leqslant a \cdot \mathfrak{x}, \text{ where } \mathfrak{x} \text{ is considered as a map } 1 \xrightarrow{\mathfrak{x}} TX. \tag{3.3}$$

Then every closed $(\mathbb{T},V)$-functor $(X,a) \xrightarrow{f} (Y,b)$ is proper.

Proof. To show that $b \cdot Tf \leqslant f \cdot a$, notice that given $\mathfrak{x} \in TX$ and $y \in Y$, $b \cdot Tf(\mathfrak{x},y) \overset{(\dagger)}{=} b \cdot Tf(Ti_A(\mathfrak{z}),y) = b \cdot Tf \cdot Ti_A(\mathfrak{z},y) \leqslant \bigvee_{\mathfrak{w} \in TA} b \cdot Tf \cdot Ti_A(\mathfrak{w},y) = b \cdot Tf \cdot Ti_A \cdot !^{\circ}_{TA}(*,y) \overset{(\dagger\dagger)}{=} f \cdot a \cdot Ti_A \cdot !^{\circ}_{TA}(*,y) \overset{(\dagger\dagger\dagger)}{\leqslant} f \cdot a \cdot \mathfrak{x}(*,y) = f \cdot a(\mathfrak{x},y)$, where (†) (resp. († † †)) relies on the left-hand (resp. right-hand) side of (3.3), and (††) uses the closedness of the $(\mathbb{T},V)$-functor $(X,a) \xrightarrow{f} (Y,b)$. □

Lemma 60. *The categories V-**Cat** and $(\mathbb{P},2)$-**Cat** (for the lax extension $\hat{\mathbb{P}}$ to **Rel** of the powerset monad $\mathbb{P}$ on **Set**) satisfy condition* (3.3).

Proof. The case of V-**Cat** is clear (given $y \in X$, take the singleton set $A = \{y\}$). To show condition (3.3) for the category $(\mathbb{P},2)$-**Cat**, recall that every $(\mathbb{P},2)$-category (X,a) can be equivalently described as a closure space (X,c), in which, given $A \in PX$ and $x \in X$, $x \in c(A)$ iff $A\,a\,x$. Therefore, if $B \subseteq A \in PX$, then $B\,a\,x$ implies $x \in c(B)$ implies $x \in c(A)$ implies $A\,a\,x$. As a result, given $A \in PX$, for every $x \in X$, it follows that $a \cdot Pi_A \cdot !^{\circ}_{PA}(*,x) = \bigvee_{B \in PA} a \cdot Pi_A(B,x) = \bigvee_{B \in PA} a(Pi_A(B),x) = \bigvee_{B \in PA} a(B,x) \leqslant a(A,x) = a \cdot A(*,x)$. □

Corollary 61. *The concepts of proper and closed map are equivalent in the categories V-**Cat**, $(\mathbb{P},2)$-**Cat**. Thus, for every $(\mathbb{T},V)$-functor $(X,a)\xrightarrow{f}(Y,b)$, the V-functor $(TX,\hat{a})\xrightarrow{Tf}(TY,\hat{b})$ is proper iff it is closed.*

Definition 62. A monad $\mathbb{T}$ on the category **Set** is said to be *non-trivial* provided that it admits Eilenberg-Moore algebras, whose underlying sets have more than one element. ▪

Proposition 63. *Let $\mathbb{T}$ be non-trivial, let $T\varnothing=\varnothing$, and let $\hat{T}$ be flat. If every $(\mathbb{T},V)$-category (X,a) satisfies condition (3.3), then T is isomorphic to the identity functor on* **Set**.

Proof. Given a set X, the assumption on non-triviality of $\mathbb{T}$ and [7, Subsection 3.1] together imply that the map $X\xrightarrow{e_X}TX$ is injective. We show that the map is also surjective.

Since $\hat{T}$ is flat, the discrete $(\mathbb{T},V)$-category structure on X is provided by $1_X^{\sharp}=e_X^{\circ}\cdot\hat{T}1_X=e_X^{\circ}$. Given $\mathfrak{x}\in TX$, there exists $A\subseteq X$, which satisfies condition (3.3) w.r.t. e_X°. Since $\mathfrak{x}\in Ti_A(TA)$, $A\neq\varnothing$ (by the assumption of the proposition), and therefore, there exists $x\in A$. One gets then that $k\leqslant e_X^{\circ}(e_X(x),x)\leqslant\bigvee_{\mathfrak{y}\in TA}e_X^{\circ}\cdot Ti_A(\mathfrak{y},x)=e_X^{\circ}\cdot Ti_A\cdot !_{TA}^{\circ}(*,x)\overset{(3.3)}{\leqslant}e_X^{\circ}\cdot\mathfrak{x}(*,x)=e_X^{\circ}(\mathfrak{x},x)$, which yields the desired $e_X(x)=\mathfrak{x}$. □

Remark 64. Notice that while the ultrafilter monad β on **Set** has the property $\beta\varnothing=\varnothing$, the powerset monad $\mathbb{P}$ on **Set** satisfies the converse condition $P\varnothing\neq\varnothing$. In particular, the category $(\beta,2)$-**Cat** (for the lax extension $\hat{\beta}$ of the ultrafilter monad β) does not satisfy condition (3.3). ▪

Definition 65. Given a topological category $(\mathbf{A},U)$ over $\mathbf{X}$, an $\mathbf{A}$-morphism $A\xrightarrow{f}B$ is said to be an *embedding* provided that f is initial, and its underlying $\mathbf{X}$-morphism $UA\xrightarrow{Uf}UB$ is a monomorphism. ▪

Remark 66. A $(\mathbb{T},V)$-functor $(X,a)\xrightarrow{f}(Y,b)$ is an embedding provided that the map $X\xrightarrow{f}Y$ is injective and $a=f^{\circ}\cdot b\cdot Tf$. ▪

Theorem 67. *If $(X,a)\xrightarrow{f}(Y,b)$ is an embedding $(\mathbb{T},V)$-functor, then f is closed iff f is proper.*

Proof. The sufficiency follows from Theorem 58. For the necessity, we notice that $f\cdot a\cdot !_{TX}^{\circ}=b\cdot Tf\cdot !_{TX}^{\circ}$ since f is closed (take $A=X$ in Definition 54), and also fix $\mathfrak{x}_0\in TX$. For every $y\in f(X)$, it follows that $f\cdot a(\mathfrak{x}_0,y)=\bigvee_{f(x)=y}a(\mathfrak{x}_0,x)\overset{(\dagger)}{=}a(\mathfrak{x}_0,f^{-1}(y))\overset{(\dagger)}{=}f^{\circ}\cdot b\cdot Tf(\mathfrak{x}_0,f^{-1}(y))=b\cdot Tf(\mathfrak{x}_0,f(f^{-1}(y)))=b\cdot Tf(\mathfrak{x}_0,y)$, where $(\dagger)$ relies on the embedding assumption. For every $y\notin f(X)$, it follows that $b\cdot Tf(\mathfrak{x}_0,y)\leqslant\bigvee_{\mathfrak{x}\in TX}b(Tf(\mathfrak{x}),y)=b\cdot Tf\cdot !_{TX}^{\circ}(*,y)\overset{(\dagger\dagger)}{=}f\cdot a\cdot !_{TX}^{\circ}(*,y)=\bigvee_{\mathfrak{x}\in TX}\bigvee_{f(x)=y}a(\mathfrak{x},x)=\bot_V$, which yields the desired $b\cdot Tf(\mathfrak{x}_0,y)=\bot_V=f\cdot a(\mathfrak{x}_0,y)$, where $(\dagger\dagger)$ relies on the above property of closed maps. □

Remark 68. The result of Theorem 67 extends the classical one in the category **Top**, which states that the embedding assumption makes the concepts of closedness and properness equivalent. ▪

4. Generalized Kuratowski-Mrówka theorem

Remark 69. Given a $(\mathbb{T},V)$-category (X,a) and $\mathfrak{x}\in TX$, define $Y=X\biguplus\{w\}$, and let a V-relation $TY\overset{b}{\longrightarrow\!\!\!\!\!+}Y$ be given by

$$b(\mathfrak{y},y)=\begin{cases}\top_V, & \mathfrak{y}=e_Y(y)\text{ or }(\mathfrak{y}=Ti_X(\mathfrak{x})\text{ and }y=w)\\ \bot_V, & \text{otherwise.}\end{cases}$$

Below, sufficient conditions are provided for the above construction to define a $(\mathbb{T},V)$-category (Y,b). ▪

Definition 70.

(1) A V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$ is said to *have finite fibres* provided that the set $r^\circ(y) = \{x \in X \mid \bot_V < r(x,y)\}$ is finite for every $y \in Y$.

(2) A lax natural transformation $1_{V\text{-}\mathbf{Rel}} \overset{e}{\dot{\to}} \hat{T}$ is said to be *finitely* $(-)^\circ$*-strict* provided that the diagram

$$\begin{array}{ccc} TX & \overset{e_X^\circ}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & X \\ {\scriptstyle \hat{T}r}\downarrow & & \downarrow{\scriptstyle r} \\ TY & \underset{e_Y^\circ}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & Y \end{array}$$

commutes for every V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$ with finite fibres. ∎

Example 71.

(1) The lax natural transformation $1_{\mathbf{Rel}} \overset{e}{\dot{\to}} \hat{\beta}$ (resp. $1_{V\text{-}\mathbf{Rel}} \overset{e}{\dot{\to}} \bar{\beta}$) of the extension $\hat{\mathbb{\beta}}$ (resp. $\bar{\mathbb{\beta}}$) to **Rel** (resp. P_+-**Rel**) of the ultrafilter monad $\mathbb{\beta}$ on **Set** is finitely $(-)^\circ$-strict.

(2) The lax natural transformation $1_{\mathbf{Rel}} \overset{e}{\dot{\to}} \hat{P}$ of the extension $\hat{\mathbb{P}}$ to **Rel** of the powerset monad $\mathbb{P}$ on **Set** fails to be finitely $(-)^\circ$-strict (cf. Remark 43 (1)). ∎

Remark 72. The V-relation $TY \overset{b}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$ of Remark 69 has finite fibres. ∎

Theorem 73. *If $\hat{T}$ is flat and $1_{V\text{-}\mathbf{Rel}} \overset{e}{\dot{\to}} \hat{T}$ is finitely $(-)^\circ$-strict, then (Y,b) is a $(\mathbb{T},V)$-category.*

Proof. The definition of the map b gives $1_Y \leqslant b \cdot e_Y$. The condition $b \cdot \hat{T}b \leqslant b \cdot m_Y$ can be shown as follows.

Given $\mathfrak{Y} \in TTY$ and $y \in Y$, one gets that $b \cdot \hat{T}b(\mathfrak{Y}, y) = \bigvee_{\mathfrak{y} \in TY} \hat{T}b(\mathfrak{Y}, \mathfrak{y}) \otimes b(\mathfrak{y}, y)$ and $b \cdot m_Y(\mathfrak{Y}, y) = b(m_Y(\mathfrak{Y}), y)$. If there exists $\mathfrak{y} \in TY$ such that $\bot_V < \hat{T}b(\mathfrak{Y}, \mathfrak{y}) \otimes b(\mathfrak{y}, y)$ (otherwise, the claim is clear), then $b(\mathfrak{y}, y) = \top_V$, and therefore, $\mathfrak{y} = e_Y(y)$ or ($\mathfrak{y} = Ti_X(\mathfrak{x})$ and $y = w$).

If $\mathfrak{y} = e_Y(y)$, then $\hat{T}b(\mathfrak{Y}, \mathfrak{y}) \otimes b(\mathfrak{y}, y) = \hat{T}b(\mathfrak{Y}, e_Y(y)) = e_Y^\circ \cdot \hat{T}b(\mathfrak{Y}, y)$. Since the V-relation b has finite fibres, apply finite $(-)^\circ$-strictness of e and get $e_Y^\circ \cdot \hat{T}b = b \cdot e_{TY}^\circ$. As a consequence, $e_Y^\circ \cdot \hat{T}b(\mathfrak{Y}, y) = b \cdot e_{TY}^\circ(\mathfrak{Y}, y) \overset{(\dagger)}{\leqslant} b \cdot m_Y(\mathfrak{Y}, y) = b(m_Y(\mathfrak{Y}), y)$, where (†) uses the fact that $m_Y \cdot e_{TY} = 1_{TY}$ implies $e_{TY}^\circ \leqslant m_Y$.

If $\mathfrak{y} = Ti_X(\mathfrak{x})$ and $y = w$, then $\hat{T}b(\mathfrak{Y}, \mathfrak{y}) \otimes b(\mathfrak{y}, y) = \hat{T}b(\mathfrak{Y}, Ti_X(\mathfrak{x})) = (Ti_X)^\circ \cdot \hat{T}b(\mathfrak{Y}, \mathfrak{x}) = \hat{T}(i_X^\circ \cdot b)(\mathfrak{Y}, \mathfrak{x})$. Since for every $\mathfrak{z} \in TY$ and every $x \in X$,

$$i_X^\circ \cdot b(\mathfrak{z}, x) = b(\mathfrak{z}, i_X(x)) = \begin{cases} \top_V, & \mathfrak{z} = e_Y \cdot i_X(x) \\ \bot_V, & \text{otherwise} \end{cases} = (e_Y \cdot i_X)^\circ(\mathfrak{z}, x),$$

it follows that $\hat{T}(i_X^\circ \cdot b) = \hat{T}(e_Y \cdot i_X)^\circ = (Te_Y \cdot Ti_X)^\circ$, since $\hat{T}$ is flat. Moreover, $\bot_V < (Te_Y \cdot Ti_X)^\circ(\mathfrak{Y}, \mathfrak{x})$ implies $Te_Y \cdot Ti_X(\mathfrak{x}) = \mathfrak{Y}$. As a result, $b(m_Y(\mathfrak{Y}), y) = b(m_Y \cdot Te_Y \cdot Ti_X(\mathfrak{x}), w) = b(Ti_X(\mathfrak{x}), w) = \top_V$. □

Remark 74. The $(\mathbb{T},V)$-category (Y,b) constructed in Remark 69 is called the *test structure for* $\mathfrak{x}$. ∎

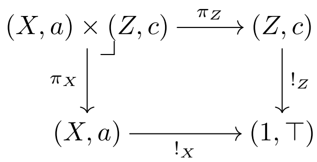
Theorem 75 (Generalized Kuratowski-Mrówka theorem). *Let $\hat{T}$ be flat and let $1_{V\text{-}\mathbf{Rel}} \overset{e}{\dot{\to}} \hat{T}$ be finitely $(-)^\circ$-strict. Given a $(\mathbb{T},V)$-category (X,a), the following are equivalent:*

(1) *(X,a) is compact;*

(2) *for every $(\mathbb{T},V)$-category (Z,c), the projection $(X,a) \times (Z,c) \xrightarrow{\pi_Z} (Z,c)$ is closed.*

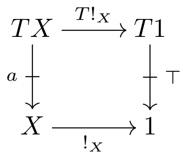
Proof.

(1) ⇒ (2): Since (X,a) is compact, then $(X,a) \xrightarrow{!_X} (1,\top)$ is proper, and therefore, its pullback along every $(\mathbb{T},V)$-functor is proper by Theorem 37. In particular, the pullback

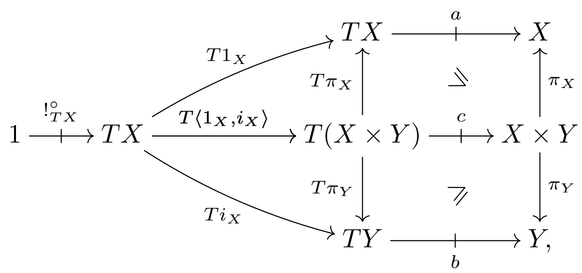


provides the proper map $(X,a)\times(Z,c) \xrightarrow{\pi_Z} (Z,c)$, which is then necessarily closed by Theorem 58.

(2) ⇒ (1): One has to show that the diagram



commutes. Given $\mathfrak{x} \in TX$, there exists the respective test structure (Y,b), constructed in Theorem 73. Moreover, one has the following diagram



where the triangles are commutative, whereas the rectangles are lax commutative. It follows then that

$$\top \cdot T!_X(\mathfrak{x},*) = \top_V = b(Ti_X(\mathfrak{x}),w) \leqslant \bigvee_{\mathfrak{z}\in TX} b(Ti_X(\mathfrak{z}),w) = b\cdot Ti_X \cdot !^\circ_{TX}(*,w) =$$

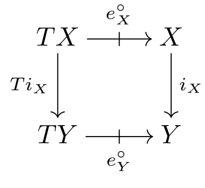
$$b\cdot T\pi_Y \cdot T\langle 1_X, i_X\rangle \cdot !^\circ_{TX}(*,w) \overset{(\dagger)}{=} \pi_Y \cdot c \cdot T\langle 1_X, i_X\rangle \cdot !^\circ_{TX}(*,w) = \bigvee_{x\in X}\bigvee_{\mathfrak{z}\in TX} c(T\langle 1_X,i_X\rangle(\mathfrak{z}),(x,w)) \overset{(\dagger\dagger)}{=}$$

$$\bigvee_{x\in X}\bigvee_{\mathfrak{z}\in TX} ((\pi_X^\circ\cdot a\cdot T\pi_X)\wedge(\pi_Y^\circ\cdot b\cdot T\pi_Y))(T\langle 1_X,i_X\rangle(\mathfrak{z}),(x,w)) =$$

$$\bigvee_{x\in X}\bigvee_{\mathfrak{z}\in TX} a(T\pi_X\cdot T\langle 1_X,i_X\rangle(\mathfrak{z}),\pi_X(x,w))\wedge b(T\pi_Y\cdot T\langle 1_X,i_X\rangle(\mathfrak{z}),\pi_Y(x,w)) =$$

$$\bigvee_{x\in X}\bigvee_{\mathfrak{z}\in TX} a(\mathfrak{z},x)\wedge b(Ti_X(\mathfrak{z}),w) \overset{(\dagger\dagger\dagger)}{=} \bigvee_{x\in X} a(\mathfrak{x},x)\wedge b(Ti_X(\mathfrak{x}),w) = \bigvee_{x\in X} a(\mathfrak{x},x) = !_X\cdot a(\mathfrak{x},*),$$

where (†) uses the assumption on closedness, (††) relies on the construction of pullbacks in the category $(\mathbb{T},V)$-**Cat** given in Remark 12 (2), whereas († † †) uses the fact that if $Ti_X(\mathfrak{z}) = e_Y(w)$ for some $\mathfrak{z}\in TX$, then, since $X \overset{i_X}{\hookrightarrow} Y$ has finite fibres, and, moreover, $\hat{T}$ is flat, the diagram



commutes, which gives $\top_V = e_Y^\circ \cdot Ti_X(\mathfrak{z}, w) = i_X \cdot e_X^\circ(\mathfrak{z}, w)$, and therefore, there exists $x \in X$ such that $i_X(x) = w$, which is a contradiction. □

5. Generalized Bourbaki theorem

Theorem 76 (Generalized Bourbaki theorem). *Let $T1 \cong 1$, let $\hat{T}$ be flat, and let $1_{V\text{-}\mathbf{Rel}} \xrightarrow{e} \hat{T}$ be finitely $(-)^\circ$-strict. Given a $(\mathbb{T}, V)$-functor $(X, a) \xrightarrow{f} (Y, b)$, the following are equivalent:*

(1) *f is proper;*

(2) *every pullback of f is closed, and $(TX, \hat{a}) \xrightarrow{Tf} (TY, \hat{b})$ is closed;*

(3) *all fibres of f are compact, and $(TX, \hat{a}) \xrightarrow{Tf} (TY, \hat{b})$ is closed.*

Proof.

(1) ⇒ (2): Follows from Theorems 39, 37 and 58.

(2) ⇒ (3): Follows from Theorem 75 and the assumption $T1 \cong 1$ (and therefore, $1^\sharp = \top$), through the composition of the pullbacks

$$\begin{array}{ccc}
(f^{-1}(y), \tilde{a}) \times (Z, c) & \xrightarrow{\pi_Z} & (Z, c) \\
\downarrow{\scriptstyle \pi_{f^{-1}(y)}} & & \downarrow{\scriptstyle !_Z} \\
(f^{-1}(y), \tilde{a}) & \xrightarrow{!_{f^{-1}(y)}} & (1, \top) \\
\downarrow{\scriptstyle i_{f^{-1}(y)}} & & \downarrow{\scriptstyle y} \\
(X, a) & \xrightarrow[f]{} & (Y, b)
\end{array}$$

for every $(\mathbb{T}, V)$-category (Z, c).

(3) ⇒ (1): Follows from Corollaries 49, 61. □

Remark 77.

(1) Without the assumption $T1 \cong 1$, stably closed maps need not be proper.

(2) It is unclear, whether the condition "$(TX, \hat{a}) \xrightarrow{Tf} (TY, \hat{b})$ is closed" can be removed from Theorem 76 (2), and also, whether it can be replaced by the condition "$(X, a) \xrightarrow{f} (Y, b)$ is closed" in Theorem 76 (3). ▪

Example 78.

(1) By Corollary 50, every object of the category **Prost** (resp. **QPMet**) is compact. By Corollary 61, proper and closed maps in the category **Prost** (resp. **QPMet**) are equivalent concepts.
(2) In **Top**, one gets the above-mentioned Kuratowski-Mrówka and Bourbaki theorems.
(3) In **App**, one gets the results from the theory of approach spaces of [3].
(4) In case of the powerset functor P on **Set**, it follows that $P1 \ncong 1$, $1_{\mathbf{Rel}} \xrightarrow{e} \hat{P}$ is not finitely $(-)^\circ$-strict (Example 71), and $\hat{P}$ is not flat (Lecture 2). Thus, Theorems 75, 76 are not applicable to the category **Cls**. Corollary 61 though shows that the concepts of proper and closed map in **Cls** are equivalent. ▪

References

[1] N. Bourbaki, *Elements of mathematics. General topology. Part 1*, Addison-Wesley Publishing Company, 1966.
[2] M. M. Clementino and W. Tholen, *Proper maps for lax algebras and the Kuratowski-Mrówka theorem*, Theory Appl. Categ. **27** (2013), no. 14, 327–346.
[3] E. Colebunders, R. Lowen, and P. Wuyts, *A Kuratowski-Mrówka theorem in approach theory*, Topology Appl. **153** (2005), no. 5-6, 756–766.
[4] R. Engelking, *General Topology*, Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, 1989.

[5] D. Hofmann, G. J. Seal, and W. Tholen (eds.), *Monoidal Topology: A Categorical Approach to Order, Metric and Topology*, Cambridge University Press, 2014.

[6] K. Kuratowski, *Topology II. New Edition, Revised and Augmented*, New York, NY: Academic Press; Warszawa: PWN-Polish Scientific Publishers, 1968.

[7] J. MacDonald and M. Sobral, *Aspects of Monads*, Categorical Foundations: Special Topics in Order, Topology, Algebra, and Sheaf Theory (M. C. Pedicchio and W. Tholen, eds.), Cambridge University Press, 2004, pp. 213–268.

[8] E. G. Manes, *Taut monads and* $T0$*-spaces*, Theoretical Computer Science **275** (2002), no. 1–2, 79 – 109.

[9] S. Mrówka, *Compactness and product spaces*, Colloq. Math. **7** (1959), 19–22.