Elements of monoidal topology
Lecture 4: properties of the category V-**Cat**

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Abstract

This lecture shows that given a commutative unital quantale V, the category V-**Cat**, first, is symmetric monoidal closed, confirming that the category **Prost** of preordered sets is cartesian closed, and the category **QPMet** of quasi-pseudo-metric spaces is monoidal closed; and, second, has a functor taking a V-category to its dual. This lecture also shows that every unital quantale V can be seen as a V-category, introduces the category V-**Mod** of V-categories and V-modules, and describes its relationships with the category V-**Cat**.

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1. Symmetric monoidal closed categories

1.1. Symmetric monoidal categories

Definition 1. A category $\mathbf{C}$ is called *monoidal* provided that it is equipped with a functor $\mathbf{C}\times\mathbf{C}\xrightarrow{\otimes}\mathbf{C}$, a distinguished $\mathbf{C}$-object E, and natural isomorphisms $A\otimes(B\otimes C)\xrightarrow{\alpha_{A,B,C}}(A\otimes B)\otimes C$, $E\otimes A\xrightarrow{\lambda_A}A$, and $A\otimes E\xrightarrow{\rho_A}A$ for every $\mathbf{C}$-objects A,B,C such that the following two diagrams commute

$$\begin{array}{ccccc}
A\otimes(B\otimes(C\otimes D)) & \xrightarrow{\alpha_{A,B,C\otimes D}} & (A\otimes B)\otimes(C\otimes D) & \xrightarrow{\alpha_{A\otimes B,C,D}} & ((A\otimes B)\otimes C)\otimes D\\
{\scriptstyle 1_A\otimes\alpha_{B,C,D}}\downarrow & & & & \uparrow{\scriptstyle \alpha_{A,B,C}\otimes 1_D}\\
A\otimes((B\otimes C)\otimes D) & & \xrightarrow[\alpha_{A,B\otimes C,D}]{} & & (A\otimes(B\otimes C))\otimes D
\end{array}$$

$$\begin{array}{ccc}
A\otimes(E\otimes B) & \xrightarrow{\alpha_{A,E,B}} & (A\otimes E)\otimes B\\
& {\scriptstyle 1_A\otimes\lambda_B}\searrow \quad \swarrow{\scriptstyle \rho_A\otimes 1_B} & \\
& A\otimes B &
\end{array}$$

for every $\mathbf{C}$-objects A,B,C,D, and, moreover, $E\otimes E\xrightarrow{\lambda_E}E=E\otimes E\xrightarrow{\rho_E}E$. ∎

Remark 2. One usually refers to the structure of a monoidal category $\mathbf{C}$ as follows: the functor $\otimes$ is the *tensor product*, the object E is the *unit*, the natural isomorphism α is the *associativity* isomorphism, and the natural isomorphism λ (resp. ρ) is the *left unit* (resp. *right unit*) isomorphism. Moreover, the two commutative diagrams of Definition 1 are called the *coherence conditions*. ∎

Remark 3. A monoidal category $\mathbf{C}$ is said to be *strict* provided that the natural isomorphisms α, λ, ρ in Definition 1 are the identity natural isomorphisms, namely, $A\otimes(B\otimes C)=(A\otimes B)\otimes C$, $E\otimes A=A$, and $A\otimes E=A$ for every $\mathbf{C}$-objects A,B,C. The coherence conditions are then trivially satisfied. ∎

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Definition 4. A monoidal category $\mathbf{C}$ is called *symmetric* provided that it is additionally equipped with natural isomorphisms $A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A$ for every $\mathbf{C}$-objects A, B such that the following diagrams commute

$$\begin{array}{ccccc}
A \otimes (B \otimes C) & \xrightarrow{\alpha_{A,B,C}} & (A \otimes B) \otimes C & \xrightarrow{\sigma_{A\otimes B,C}} & C \otimes (A \otimes B) \\
{\scriptstyle 1_A \otimes \sigma_{B,C}} \downarrow & & & & \downarrow {\scriptstyle \alpha_{C,A,B}} \\
A \otimes (C \otimes B) & \xrightarrow[\alpha_{A,C,B}]{} & (A \otimes C) \otimes B & \xrightarrow[\sigma_{A\otimes C} \otimes 1_B]{} & (C \otimes A) \otimes B
\end{array}$$

$$\begin{array}{ccc}
E \otimes A & \xrightarrow{\sigma_{E,A}} & A \otimes E \\
& {\scriptstyle \lambda_A} \searrow \quad \swarrow {\scriptstyle \rho_A} & \\
& A &
\end{array}
\qquad
\begin{array}{ccc}
A \otimes B & \xrightarrow{\sigma_{A,B}} & B \otimes A \\
& {\scriptstyle 1_{A\otimes B}} \searrow \quad \swarrow {\scriptstyle \sigma_{B,A}} & \\
& A \otimes B &
\end{array}
\tag{1.1}$$

for every $\mathbf{C}$-objects A, B, C. Since $\lambda_E = \rho_E$, the left-hand side of diagram (1.1) gives $\sigma_{E,E} = 1_E$. ▪

Remark 5. The commutative diagrams of Definition 4 are called the *coherence conditions*, i.e., *associativity coherence*, *unit coherence*, and *symmetry axiom*, respectively. One should also observe that strict monoidal categories generally fail to be symmetric. ▪

Example 6.

(1) The category **Set** of sets and maps is symmetric monoidal w.r.t. cartesian product of sets. More generally, every category with finite products is symmetric monoidal.
(2) The category of functors on a small category with the tensor product given by the composition of functors is strict monoidal.
(3) The category **Ab** of abelian groups and group homomorphisms with the usual tensor product is symmetric monoidal. More generally, given a commutative unital ring R, the category R-**Mod** of left R-modules and left R-module homomorphisms is symmetric monoidal w.r.t. the tensor product of R-modules.
(4) The category **Sup** of $\bigvee$-semilattices and $\bigvee$-preserving maps equipped with the usual tensor product is symmetric monoidal.
(5) Every unital quantale V is a strict monoidal category w.r.t. its multiplication. Commutative quantales are additionally symmetric. More generally, a preordered set S, considered as a category, is strict monoidal precisely when it is equipped with a monoid structure $(\otimes, k)$, whose multiplication $S \times S \xrightarrow{\otimes} S$ is monotone. Commutative monoid structures provide additionally symmetric categories.
(6) Given a (symmetric) monoidal category $\mathbf{C}$, $\mathbf{C}^{op}$ is a (symmetric) monoidal category as well. ▪

1.2. Monoidal functors

Definition 7. A *morphism of monoidal categories* $\mathbf{C}$ and $\mathbf{D}$ is a functor $\mathbf{C} \xrightarrow{F} \mathbf{D}$ equipped with $\mathbf{D}$-morphisms $FA \otimes_{\mathbf{D}} FB \xrightarrow{\delta_{A,B}} F(A \otimes_{\mathbf{C}} B)$ natural in $\mathbf{C}$-objects A, B and a $\mathbf{D}$-morphism $E_{\mathbf{D}} \xrightarrow{\varepsilon} FE_{\mathbf{C}}$ natural in $E_{\mathbf{D}}$, $E_{\mathbf{C}}$ such that the following three diagrams commute

$$\begin{array}{ccc}
FA \otimes (FB \otimes FC) & \xrightarrow{\alpha_{FA,FB,FC}} & (FA \otimes FB) \otimes FC \\
{\scriptstyle 1_{FA} \otimes \delta_{B,C}} \downarrow & & \downarrow {\scriptstyle \delta_{A,B} \otimes 1_{FC}} \\
FA \otimes F(B \otimes C) & & F(A \otimes B) \otimes FC \\
{\scriptstyle \delta_{A,B\otimes C}} \downarrow & & \downarrow {\scriptstyle \delta_{A\otimes B,C}} \\
F(A \otimes (B \otimes C)) & \xrightarrow[F\alpha_{A,B,C}]{} & F((A \otimes B) \otimes C)
\end{array}$$

$$\begin{array}{ccc} E \otimes FA & \xrightarrow{\lambda_{FA}} & FA \\ {\scriptstyle \varepsilon\otimes 1_{FA}}\downarrow & & \uparrow{\scriptstyle F\lambda_A} \\ FE \otimes FA & \xrightarrow[\delta_{E,A}]{} & F(E\otimes A) \end{array} \qquad \begin{array}{ccc} FA \otimes E & \xrightarrow{\rho_{FA}} & FA \\ {\scriptstyle 1_{FA}\otimes\varepsilon}\downarrow & & \uparrow{\scriptstyle F\rho_A} \\ FA \otimes FE & \xrightarrow[\delta_{A,E}]{} & F(A\otimes E). \end{array}$$

Notice that since δ is natural, the following diagram

$$\begin{array}{ccc} FA \otimes FB & \xrightarrow{Ff\otimes Fg} & FC \otimes FD \\ {\scriptstyle \delta_{A,B}}\downarrow & & \downarrow{\scriptstyle \delta_{C,D}} \\ F(A\otimes B) & \xrightarrow[F(f\otimes g)]{} & F(C\otimes D) \end{array}$$

commutes for every **C**-morphisms $A \xrightarrow{f} C$, $B \xrightarrow{g} D$. ■

Remark 8. The first three commutative diagrams of Definition 7 are called the *coherence conditions*, i.e., the *associativity condition*, and the two *unit conditions*, respectively. ■

Definition 9. A *morphism of symmetric monoidal categories* **C** and **D** is a functor $\mathbf{C} \xrightarrow{F} \mathbf{D}$, which is a morphism of monoidal categories **C** and **D**, making additionally the following diagram commute

$$\begin{array}{ccc} FA \otimes FB & \xrightarrow{\sigma_{FA,FB}} & FB \otimes FA \\ {\scriptstyle \delta_{A,B}}\downarrow & & \downarrow{\scriptstyle \delta_{B,A}} \\ F(A\otimes B) & \xrightarrow[F\sigma_{A,B}]{} & F(B\otimes A) \end{array}$$

for every **C**-objects A, B. ■

Remark 10.

(1) The commutative diagram of Definition 9 is called the *symmetry condition.*
(2) Morphisms of (symmetric) monoidal categories are sometimes called *monoidal functors.* ■

Definition 11. A morphism of (symmetric) monoidal categories is called *strong* provided that the morphisms δ and ε of Definition 7 are isomorphisms. It is called *strict* provided that δ and ε are identities. It should be clear that strictness implies strongness, but not vice versa. ■

Example 12.

(1) The forgetful functor $(\mathbf{Ab}, \otimes, \mathbb{Z}) \xrightarrow{U} (\mathbf{Set}, \times, \{*\})$ is monoidal. Given two abelian groups A, B, one defines $UA \times UB \xrightarrow{\delta_{A,B}} U(A \otimes B)$ by $\delta_{A,B}(a, b) = a \otimes b$, and $\{*\} \xrightarrow{\varepsilon} U\mathbb{Z}$ by $\varepsilon(*) = 1$. The monoidal functor U is not strong.
(2) Every homomorphism of (commutative) unital quantales is a strict monoidal functor between (symmetric) monoidal categories.
(3) Given two categories with finite products, every functor between them, which preserves finite products, is strong monoidal. ■

Definition 13. An object A of a monoidal category $\mathbf{C}$ is called $\otimes$-*exponentiable* provided that the functors $\mathbf{C} \xrightarrow{A\otimes(-)} \mathbf{C}$ and $\mathbf{C} \xrightarrow{(-)\otimes A} \mathbf{C}$ have right adjoints $\mathbf{C} \xrightarrow{A \multimap (-)} \mathbf{C}$ and $\mathbf{C} \xrightarrow{(-) \multimapinv A} \mathbf{C}$, respectively. ▪

Remark 14.

(1) The functors $\mathbf{C} \xrightarrow{A \multimap (-)} \mathbf{C}$ and $\mathbf{C} \xrightarrow{(-) \multimapinv A} \mathbf{C}$ of Definition 13 are called the *right internal hom-functor of* A and the *left internal hom-functor of* A, respectively. One also sometimes uses the notation $[A, -]$ instead of $(-) \multimapinv A$, and $[\![A, -]\!]$ instead of $A \multimap (-)$.

(2) The existence of the internal hom-functors implies, in particular, that every $\mathbf{C}$-object B has an $A \otimes (-)$-co-universal arrow $A \otimes (A \multimap B) \xrightarrow{\mathrm{ev}_{\multimap B}} B$ and a $(-) \otimes A$-co-universal arrow $(B \multimapinv A) \otimes A \xrightarrow{\mathrm{ev}_{B \multimapinv}} B$, i.e., for every $\mathbf{C}$-morphisms $A \otimes C \xrightarrow{f} B$ and $C \otimes A \xrightarrow{g} B$, there exist unique $\mathbf{C}$-morphisms $C \xrightarrow{\hat{f}} A \multimap B$ and $C \xrightarrow{\hat{g}} B \multimapinv A$ such that the following two diagrams commute

$$\begin{array}{ccc} A \otimes C & & \\ {\scriptstyle 1_A \otimes \hat{f}} \downarrow & \searrow^{f} & \\ A \otimes (A \multimap B) & \xrightarrow[\mathrm{ev}_{\multimap B}]{} & B \end{array} \qquad \begin{array}{ccc} C \otimes A & & \\ {\scriptstyle \hat{g} \otimes 1_A} \downarrow & \searrow^{g} & \\ (B \multimapinv A) \otimes A & \xrightarrow[\mathrm{ev}_{B \multimapinv}]{} & B. \end{array} \tag{1.2}$$

(3) Since in a symmetric monoidal category $\mathbf{C}$ both functors $A \otimes (-)$ and $(-) \otimes A$ are naturally isomorphic, their right adjoint functors $A \multimap (-)$ and $(-) \multimapinv A$ are naturally isomorphic as well, i.e., there is no need to distinguish between right- and left-hom functors, and one can use any of the notations $\multimap$ and $\multimapinv$. The respective co-universal arrows are then called *evaluation morphisms*.

(4) For every $\otimes$-exponentiable $\mathbf{C}$-object A, since the functor $A \otimes (-)$ (resp. $(-) \otimes A$) has a right adjoint $A \multimap (-)$ (resp. $(-) \multimapinv A$), then $A \otimes (-)$ (resp. $(-) \otimes A$) preserves the existing colimits, and $A \multimap (-)$ (resp. $(-) \multimapinv A$) preserves the existing limits. ▪

Definition 15. A monoidal category $\mathbf{C}$ is called *closed* provided that every its object is $\otimes$-exponentiable. ▪

Remark 16.

(1) A monoidal category $\mathbf{C}$ is sometimes called *right closed* (resp. *left closed*, *biclosed*) provided that for every its object A, the functor $A \otimes (-)$ has a right adjoint (resp. the functor $(-) \otimes A$ has right adjoint, both functors $A \otimes (-)$ and $(-) \otimes A$ have a right adjoint).

(2) Given a category $\mathbf{C}$ with the monoidal structure defined by finite products, one says *exponentiable* instead of $\times$-*exponentiable* and generally writes B^A for the *internal hom-object* $A \multimap B \cong B \multimapinv A$, which is also called an *exponential* or *power object*. The morphism $\hat{f}$ of diagram (1.2) is then called *exponential morphism for* f. The category $\mathbf{C}$ itself is then said to be *cartesian closed* provided that it is closed. ▪

Example 17.

(1) The category $\mathbf{Set}$ is cartesian closed. Given two sets A, B, the respective power object B^A is the set of all maps $B \xrightarrow{\alpha} A$, and the respective evaluation morphism $A \times B^A \xrightarrow{\mathrm{ev}_B} B$ is the usual evaluation map given by $\mathrm{ev}(a, \alpha) = \alpha(a)$. Given a morphism $A \times C \xrightarrow{f} B$, the exponential morphism $C \xrightarrow{\hat{f}} B^A$ for f is defined by $\hat{f}(c) = f(-, c)$. It is easy to see that $\hat{f}$ is a unique map making the next diagram commute

$$\begin{array}{ccc} A \times C & & \\ {\scriptstyle 1_A \times \hat{f}} \downarrow & \searrow^{f} & \\ A \times B^A & \xrightarrow[\mathrm{ev}_B]{} & B. \end{array}$$

(2) The category **Prost** of preordered sets and monotone maps is cartesian closed. Given two preordered sets A, B, the respective power object B^A is the set of all monotone maps $B \xrightarrow{\alpha} A$, and the respective evaluation morphism $A \times B^A \xrightarrow{\mathrm{ev}_B} B$ is the usual evaluation map given by $\mathrm{ev}(a, \alpha) = \alpha(a)$.
(3) The category **Cat** of *small* categories (the class of objects of a category is a set) and functors is cartesian closed. Given two categories $\mathbf{A}$, $\mathbf{B}$, the respective power object is the functor category $\mathbf{B}^{\mathbf{A}}$, and the respective evaluation morphism $\mathbf{A} \times \mathbf{B}^{\mathbf{A}} \xrightarrow{\mathrm{ev}_{\mathbf{B}}} \mathbf{B}$ is defined on objects by $\mathrm{ev}_{\mathbf{B}}(A, F) = FA$ and on morphisms by $\mathrm{ev}_{\mathbf{B}}(h, \tau) = \tau_{A'} \cdot Fh$, where $A \xrightarrow{h} A'$ is an $\mathbf{A}$-morphism and $F \xrightarrow{\tau} F'$ is a natural transformation. Given a functor $\mathbf{A} \times \mathbf{C} \xrightarrow{F} \mathbf{B}$, the exponential morphism $\mathbf{C} \xrightarrow{\hat{F}} \mathbf{B}^{\mathbf{A}}$ is defined by $\mathbf{A} \xrightarrow{\hat{F}C} \mathbf{B}$, $\hat{F}C(A \xrightarrow{h} A') = F(A, C) \xrightarrow{F(h, 1_C)} F(A', C)$.
(4) The category **Top** of topological spaces and continuous maps is not cartesian closed, since the functor $\mathbf{Top} \xrightarrow{\mathbb{Q} \times (-)} \mathbf{Top}$ (where $\mathbb{Q}$ is the space of rational numbers, with the topology induced by that of the real line $\mathbb{R}$) does not preserve quotients, and, thus, does not preserve coequalizers (notice that left adjoint functors preserve the existing colimits).
(5) The category **Ab** of abelian groups is monoidal closed, where $A \multimap B = \mathbf{Ab}(A, B)$ with the pointwise structure of an abelian group. More generally, the category R-**Mod** of left R-modules is monoidal closed, where $A \multimap B = R\text{-}\mathbf{Mod}(A, B)$ with the pointwise structure of an R-module (recall from Example 6 (3) that both categories **Ab** and R-**Mod** are symmetric).
(6) The category **Sup** of $\bigvee$-semilattices is monoidal closed, where $A \multimap B = \mathbf{Sup}(A, B)$ with the pointwise structure of a $\bigvee$-semilattice (recall from Example 6 (4) that the category **Sup** is symmetric).
(7) A unital quantale V is monoidal closed. Given $a \in V$, the maps $V \xrightarrow{a \multimap (-)} V$ and $V \xrightarrow{(-) \multimapinv a} V$ are defined by

$$a \otimes c \leqslant b \quad \text{iff} \quad c \leqslant a \multimap b \qquad\qquad c \otimes a \leqslant b \quad \text{iff} \quad c \leqslant b \multimapinv a$$

for every $c, b \in V$. In particular, $a \multimap b = \bigvee\{c \in V \mid a \otimes c \leqslant b\}$ and $b \multimapinv a = \bigvee\{c \in V \mid c \otimes a \leqslant b\}$. If the quantale V is commutative, then the maps $a \multimap (-)$ and $(-) \multimapinv a$ coincide (recall from Example 6 (5) that V is then a symmetric monoidal category). Additionally, by Remark 14 (4), it follows that both maps $a \multimap (-)$ and $(-) \multimapinv a$ are $\bigwedge$-preserving. In particular, they preserve the largest element $\top_V$. ■

Proposition 18. *Let* $\mathbf{C}$ *be a monoidal closed category, and let* A, B, C *be* $\mathbf{C}$*-objects. There exist the following natural isomorphisms:*

(1) $(A \otimes B) \multimap C \cong B \multimap (A \multimap C)$;
(2) $C \multimapinv (A \otimes B) \cong (C \multimapinv B) \multimapinv A$;
(3) $(A \multimap C) \multimapinv B \cong A \multimap (C \multimapinv B)$.

If $\mathbf{C}$ *is cartesian closed, then* $C^{A \times B} \cong (C^A)^B \cong (C^B)^A$.

Proof. As an illustration, we show the proof of items (1) – (3) in case of $\mathbf{C}$ being a unital quantale V.

(1) Given $v \in V$, in view of Example 17 (7), it follows that $v \leqslant (a \otimes b) \multimap c$ iff $(a \otimes b) \otimes v \leqslant c$ iff $a \otimes (b \otimes v) \leqslant c$ iff $b \otimes v \leqslant a \multimap c$ iff $v \leqslant b \multimap (a \multimap c)$. As a result, one gets $(a \otimes b) \multimap c = b \multimap (a \multimap c)$.
(2) Given $v \in V$, in view of Example 17 (7), it follows that $v \leqslant c \multimapinv (a \otimes b)$ iff $v \otimes (a \otimes b) \leqslant c$ iff $(v \otimes a) \otimes b \leqslant c$ iff $v \otimes a \leqslant c \multimapinv b$ iff $v \leqslant (c \multimapinv b) \multimapinv a$. As a result, one gets $c \multimapinv (a \otimes b) = (c \multimapinv b) \multimapinv a$.
(3) Given $v \in V$, in view of Example 17 (7), it follows that $v \leqslant (a \multimap c) \multimapinv b$ iff $v \otimes b \leqslant a \multimap c$ iff $a \otimes (v \otimes b) \leqslant c$ iff $(a \otimes v) \otimes b \leqslant c$ iff $a \otimes v \leqslant c \multimapinv b$ iff $v \leqslant a \multimap (c \multimapinv b)$. Thus, one gets $(a \multimap c) \multimapinv b = a \multimap (c \multimapinv b)$. □

Remark 19. Given a monoidal closed category $\mathbf{C}$ and a $\mathbf{C}$-object C, there exists a functor $\mathbf{C}^{op} \xrightarrow{(-) \multimap C} \mathbf{C}$ defined for a $\mathbf{C}$-morphism $A \xrightarrow{f} B$ by $(-) \multimap C(A \xrightarrow{f} B) = B \multimap C \xrightarrow{f \multimap C} A \multimap C$, where $B \multimap C \xrightarrow{f \multimap C}$

$A \multimap C$ is the unique **C**-morphism making the diagram

$$\begin{array}{ccc} A \otimes (B \multimap C) & \xrightarrow{f \otimes 1_{B \multimap C}} & B \otimes (B \multimap C) \\ {\scriptstyle 1_A \otimes (f \multimap C)} \downarrow & & \downarrow {\scriptstyle \mathrm{ev}^B_{\multimap C}} \\ A \otimes (A \multimap C) & \xrightarrow[\mathrm{ev}^A_{\multimap C}]{} & C \end{array}$$

commute. ▪

Proposition 20. *Given a symmetric monoidal closed category* **C**, *for every* **C***-object* C, *it follows that the functor* $\mathbf{C}^{op} \xrightarrow{(-) \multimap C} \mathbf{C}$ *has a left adjoint, which is given by the dual functor* $\mathbf{C} \xrightarrow{((-) \multimap C)^{op}} \mathbf{C}^{op}$.

Remark 21.

(1) For every object C of a symmetric monoidal closed category **C**, since the functor $\mathbf{C}^{op} \xrightarrow{(-) \multimap C} \mathbf{C}$ has a left adjoint, it preserves the existing limits, i.e., it takes colimits to limits. In particular, for every element c of a commutative unital quantale V, the map $(-) \multimap c$ takes $\bigvee$ to $\bigwedge$.
(2) Given a (not necessarily unital) quantale V, the map $(-) \multimap c$ (defined as in Example 17 (7)) takes $\bigvee$ to $\bigwedge$, which can be seen as follows. Given a subset $S \subseteq V$, for every $v \in V$, $v \leqslant (\bigvee S) \multimap c$ iff $(\bigvee S) \otimes v \leqslant c$ iff $\bigvee_{s \in S}(s \otimes v) \leqslant c$ iff $s \otimes v \leqslant c$ for every $s \in S$ iff $v \leqslant s \multimap c$ for every $s \in S$ iff $v \leqslant \bigwedge_{s \in S}(s \multimap c)$. As a consequence, one gets that $(\bigvee S) \multimap c = \bigwedge_{s \in S}(s \multimap c)$. Similar result holds for the map $c \multimapinv (-)$. ▪

Example 22. If **C** is the cartesian closed category **Set** and C is the two-element set 2, then, for every set A, the internal hom-object $2^A = A \multimap 2$ is the powerset PA of A. The functor $2^{(-)} = (-) \multimap 2$ is then precisely the *contravariant powerset functor* $\mathbf{Set}^{op} \xrightarrow{Q} \mathbf{Set}$ defined on a map $B \xrightarrow{f} A$ by $Q(A \xrightarrow{f^{op}} B) = PA \xrightarrow{f^{-1}} PB$, where $f^{-1}(S)$ is the *preimage* of a subset $S \subseteq A$, i.e., $f^{-1}(S) = \{b \in B \mid f(b) \in S\}$. ▪

2. Properties of the category V-Cat

*2.1. Symmetric monoidal closed structure on the category V-***Cat**

Definition 23. Let V be a unital quantale, and let (X, a), (Y, b) be V-categories.

(1) Define a V-relation $[-,-]$ on the set $V\text{-}\mathbf{Cat}((X,a),(Y,b)) = \{X \xrightarrow{f} Y \mid f \text{ is a } V\text{-functor}\}$ by

$$[f,g] = \bigwedge_{x \in X} b(f(x), g(x)),$$

and let $[(X,a),(Y,b)]$ stand for the pair $(V\text{-}\mathbf{Cat}((X,a),(Y,b)), [-,-])$.
(2) Define a V-relation $a \otimes b$ on the set $X \times Y$ (the cartesian product of the sets X and Y) by

$$(a \otimes b)((x_1, y_1), (x_2, y_2)) = a(x_1, x_2) \otimes b(y_1, y_2)$$

for every $x_1, x_2 \in X$ and every $y_1, y_2 \in Y$, and let $(X,a) \otimes (Y,b)$ stand for the pair $(X \times Y, a \otimes b)$. ▪

Proposition 24. *Give a unital quantale* V *and* V*-categories* (X,a), (Y,b), *the following holds:*

(1) $[(X,a),(Y,b)]$ *is a* V*-category;*
(2) *If the quantale* V *is commutative, then* $(X,a) \otimes (Y,b)$ *is a* V*-category.*

Proof. In both cases, one shows the two required properties of a V-category (see Lecture 1).

(1) First, given $f \in V\text{-}\mathbf{Cat}((X,a),(Y,b))$, it follows that $[f,f] = \bigwedge_{x\in X} b(f(x),f(x)) \geqslant \bigwedge_{x\in X} k \geqslant k$, namely, $k \leqslant [f,f]$. Second, given $f,g,h \in V\text{-}\mathbf{Cat}((X,a),(Y,b))$, it follows that $[f,g]\otimes[g,h] = (\bigwedge_{x\in X} b(f(x),g(x))) \otimes (\bigwedge_{x\in X} b(g(x),h(x))) \leqslant \bigwedge_{x\in X}(b(f(x),g(x)) \otimes b(g(x),h(x))) \leqslant$ (since (Y,b) is a V-category) $\leqslant \bigwedge_{x\in X} b(f(x),h(x)) = [f,h]$, namely, $[f,g]\otimes[g,h] \leqslant [f,h]$.
(2) First, given $x \in X$ and $y \in Y$, it follows that $(a\otimes b)((x,y),(x,y)) = a(x,x)\otimes b(y,y) \geqslant k \otimes k = k$, namely, $k \leqslant (a\otimes b)((x,y),(x,y))$. Second, given $x_1,x_2,x_3 \in X$ and $y_1,y_2,y_3 \in Y$, it follows that $(a\otimes b)((x_1,y_1),(x_2,y_2)) \otimes (a\otimes b)((x_2,y_2),(x_3,y_3)) = (a(x_1,x_2)\otimes b(y_1,y_2)) \otimes (a(x_2,x_3)\otimes b(y_2,y_3)) =$ (since the quantale V is commutative) $= (a(x_1,x_2)\otimes a(x_2,x_3)) \otimes (b(y_1,y_2)\otimes b(y_2,y_3)) \leqslant$ (since both (X,a) and (Y,b) are V-categories) $\leqslant a(x_1,x_3)\otimes b(y_1,y_3) = (a\otimes b)((x_1,y_1),(x_3,y_3))$, namely, $(a\otimes b)((x_1,y_1),(x_2,y_2)) \otimes (a\otimes b)((x_2,y_2),(x_3,y_3)) \leqslant (a\otimes b)((x_1,y_1),(x_3,y_3))$. □

Remark 25. Notice that given V-categories (X,a), (Y,b) over a commutative unital quantale V, $(X,a)\otimes(Y,b)$ is *not* the product V-category $(X,a)\times(Y,b)$. Indeed, following the results of Lecture 2, V-**Cat** is a topological category over **Set**, and, therefore, the limits in V-**Cat** are lifted from those in **Set** by the forgetful functor. In particular, the product V-category of (X,a) and (Y,b) is V-category $(X\times Y, c)$, where V-relation c is given by $c((x_1,y_1),(x_2,y_2)) = a(x_1,x_2)\wedge b(y_1,y_2)$ for every $x_1,x_2 \in X$ and every $y_1,y_2 \in Y$. ▪

Example 26. If $V = \mathsf{P}_+$, then P_+-categories (X,a), (Y,b) are quasi-pseudo-metric spaces (generalized metric spaces in the sense of F. W. Lawvere). It then follows that $[f,g] = \sup_{x\in X} b(f(x),g(x))$ is the usual "sup-metric" on the function space $[X,Y]$, and, moreover, $(a\otimes b)((x_1,y_1),(x_2,y_2)) = a(x_1,x_2)+b(y_1,y_2)$ equips $X\times Y$ with the usual "+-metric". ▪

Proposition 27. *Given a V-category $[(X,a),(Y,b)]$, $[f,g] = \bigwedge_{x_1,x_2\in X} a(x_1,x_2) \multimap b(f(x_1),g(x_2))$ for every $f,g \in V\text{-}\mathbf{Cat}((X,a),(Y,b))$.*

Proof. Given $x_1,x_2 \in X$, it follows that $a(x_1,x_2)\otimes b(f(x_2),g(x_2)) \leqslant$ (f is a V-functor) $\leqslant b(f(x_1),f(x_2))\otimes b(f(x_2),g(x_2)) \leqslant$ ((Y,b) is a V-category) $\leqslant b(f(x_1),g(x_2))$, i.e., $b(f(x_2),g(x_2)) \leqslant a(x_1,x_2) \multimap b(f(x_1),g(x_2))$. Thus, $[f,g] = \bigwedge_{x\in X} b(f(x),g(x)) = \bigwedge_{x_1,x_2\in X} b(f(x_2),g(x_2)) \leqslant \bigwedge_{x_1,x_2\in X} a(x_1,x_2) \multimap b(f(x_1),g(x_2))$.

Given $x\in X$, $k \leqslant a(x,x)$ and Remark 21 (2) together imply $a(x,x) \multimap b(f(x),g(x)) \leqslant k \multimap b(f(x),g(x)) \leqslant b(f(x),g(x))$. Thus, it follows that $\bigwedge_{x_1,x_2\in X} a(x_1,x_2) \multimap b(f(x_1),g(x_2)) \leqslant \bigwedge_{x\in X} a(x,x) \multimap b(f(x),g(x)) \leqslant \bigwedge_{x\in X} b(f(x),g(x)) = [f,g]$, which finishes the proof. □

Remark 28. By analogy with Proposition 27, one can show that given a V-category $[(X,a),(Y,b)]$, $[f,g] = \bigwedge_{x_1,x_2\in X} b(f(x_1),g(x_2)) \multimapinv a(x_1,x_2)$ for every $f,g \in V\text{-}\mathbf{Cat}((X,a),(Y,b))$. ▪

Theorem 29. *Given a commutative unital quantale V, the category V-**Cat** is symmetric monoidal closed.*

Proof. First, define the tensor product functor $V\text{-}\mathbf{Cat}\times V\text{-}\mathbf{Cat} \xrightarrow{\otimes} V\text{-}\mathbf{Cat}$ by $\otimes(((X_1,a_1),(Y_1,b_1)) \xrightarrow{(f,g)} ((X_2,a_2),(Y_2,b_2))) = (X_1\times Y_1, a_1\otimes b_1) \xrightarrow{f\times g} (X_2\times Y_2, a_2\otimes b_2)$. To show that the functor is correct on morphisms, notice that given $x_1,x_1' \in X$ and $y_1,y_1' \in Y_1$, it follows that $(a_1\otimes b_1)((x_1,y_1),(x_1',y_1')) = a_1(x_1,x_1')\otimes b_1(y_1,y_1') \leqslant$ (both f and g are V-functors) $\leqslant a_2(f(x_1),f(x_1'))\otimes b_2(g(y_1),g(y_1')) = (a_2\otimes b_2)((f(x_1),g(y_1)),(f(x_1'),g(y_1'))) = (a_2\otimes b_2)((f\times g)(x_1,y_1),(f\times g)(x_1',y_1'))$.

Second, define the unit $E = (\{*\}, \underline{k})$, where $\underline{k}(*,*) = k$.

Third, given V-categories (X,a), (Y,b), and (Z,c), define the natural isomorphism $(X,a)\otimes((Y,b)\otimes(Z,c)) \xrightarrow{\alpha_{(X,a),(Y,b),(Z,c)}} ((X,a)\otimes(Y,b))\otimes(Z,c)$ by $\alpha_{(X,a),(Y,b),(Z,c)}(x,(y,z)) = ((x,y),z)$. Further, define the natural isomorphism $E\otimes(X,a) \xrightarrow{\lambda_{(X,a)}} (X,a)$ by $\lambda_{(X,a)}(*,x) = x$ (the projection map) and the natural transformation $(X,a)\otimes E \xrightarrow{\rho_{(X,a)}} (X,a)$ by $\rho_{(X,a)}(x,*) = x$ (the projection map again). To show that, e.g., the map $\lambda_{(X,a)}$ is a V-functor, notice that for every $x_1,x_2 \in X$, it follows that $(\underline{k}\otimes a)((*,x_1),(*,x_2)) = \underline{k}(*,*)\otimes a(x_1,x_2) = k\otimes a(x_1,x_2) = a(x_1,x_2) = a(\lambda_{(X,a)}(*,x_1),\lambda_{(X,a)}(*,x_2))$. Moreover, commutativity of the diagrams of Definition 1 is immediate. For example, for the triangle, notice that $(\rho_{(X,a)}\otimes 1_{(Y,b)})\cdot\alpha_{(X,a),E,(Y,b)}(x,(*,y)) = \rho_{(X,a)}\times 1_{(Y,b)}((x,*),y) = (x,y) = 1_{(X,a)}\times\lambda_{(Y,b)}(x,(*,y)) = 1_{(X,a)}\otimes\lambda_{(Y,b)}(x,(*,y))$.

Fourth, given V-categories (X, a), (Y, b), define the natural isomorphism $(X, a)\otimes(Y, b) \xrightarrow{\sigma_{(X,a),(Y,b)}} (Y, b)\otimes (X, a)$ by $\sigma_{(X,a),(Y,b)}(x, y) = (y, x)$. The above structure then makes V-**Cat** a symmetric monoidal category.

Fifth, to show that the category V-**Cat** is closed, by Remark 14 (3), it is enough to show that given V-categories (X, a) and (Y, b), the map $(X, a)\otimes[(X, a),(Y, B)] \xrightarrow{\mathrm{ev}_{(Y,b)}} (Y, b)$, defined by $\mathrm{ev}_{(Y,b)}(x, f) = f(x)$, provides an $(X, a) \otimes (-)$-co-universal arrow for (Y, b).

To check that the map $\mathrm{ev}_{(Y,b)}$ provides a V-functor, notice that given $x_1, x_2 \in X$ and $f, g \in V\text{-}\mathbf{Cat}((X, a), (Y, b))$, it follows that $(a \otimes [-,-])((x_1, f),(x_2, g)) = a(x_1, x_2) \otimes [f, g] =$ (Proposition 27) $= a(x_1, x_2) \otimes (\bigwedge_{x_1',x_2'\in X} a(x_1', x_2') \multimapdotinv b(f(x_1'), g(x_2'))) \leqslant a(x_1, x_2) \otimes (a(x_1, x_2) \multimapdotinv b(f(x_1), g(x_2))) \leqslant$ (Example 17 (7)) $\leqslant b(f(x_1), g(x_2)) = b(\mathrm{ev}_{(Y,b)}(x_1, f), \mathrm{ev}_{(Y,b)}(x_2, g))$.

Given a V-functor $(X, a) \otimes (Z, c) \xrightarrow{f} (Y, b)$, define a map $Z \xrightarrow{\hat{f}} V\text{-}\mathbf{Cat}((X, a),(Y, b))$ by $\hat{f}(z) = f(-, z)$.

To show that the map $\hat{f}$ is correct, i.e., $\hat{f}(z)$ is a V-functor for every $z \in Z$, notice that given $x_1, x_2 \in X$, it follows that $b((\hat{f}(z))(x_1),(\hat{f}(z))(x_2)) = b(f(x_1, z), f(x_2, z)) \geqslant$ (f is a V-functor) $\geqslant (a\otimes c)((x_1, z),(x_2, z)) = a(x_1, x_2) \otimes c(z, z) \geqslant$ ((Z, c) is a V-category) $\geqslant a(x_1, x_2) \otimes k = a(x_1, x_2)$.

To show that the map $\hat{f}$ is a V-functor, one could observe that given $z_1, z_2 \in Z$, it follows that $[\hat{f}(z_1), \hat{f}(z_2)] =$ (Definition 23 (1)) $= \bigwedge_{x\in X} b((\hat{f}(z_1))(x),(\hat{f}(z_2))(x)) = \bigwedge_{x\in X} b(f(x, z_1), f(x, z_2)) \geqslant$ (f is a V-functor) $\geqslant \bigwedge_{x\in X}(a \otimes c)((x, x),(z_1, z_2)) \geqslant \bigwedge_{x\in X}(a(x, x) \otimes c(z_1, z_2)) \geqslant$ ((X, a) is a V-category) $\geqslant \bigwedge_{x\in X}(k \otimes c(z_1, z_2)) = \bigwedge_{x\in X} c(z_1, z_2) \geqslant c(z_1, z_2)$.

Lastly, it is easy to see that $\hat{f}$ is the unique V-functor making the following triangle commute

$$\begin{array}{ccc} (X,a)\otimes(Z,c) & & \\ {\scriptstyle 1_{(X,a)}\otimes\hat{f}}\downarrow & \searrow^{f} & \\ (X,a)\otimes[(X,a),(Y,b)] & \xrightarrow[\mathrm{ev}_{(Y,b)}]{} & (Y,b). \end{array}$$

□

Example 30.

(1) If V is the two-element unital quantale $2 = (\{\bot, \top\}, \wedge, \top)$ (recall Lecture 1), then Theorem 29 confirms that the category **Prost** of preordered sets is cartesian closed (cf. Example 17 (2)).

(2) If V is the extended real half-line $\mathsf{P}_+ = ([0, \infty]^{op}, +, 0)$ (recall Lecture 1), then Theorem 29 confirms that the category **QPMet** of quasi-pseudo-metric spaces (generalized metric spaces in the sense of F. W. Lawvere) is monoidal closed. Following Remark 25, notice that given P_+-categories (X, a), (Y, b), the monoidal product $(X, a) \otimes (Y, b)$ is *not* the product P_+-category $(X, a) \times (Y, b)$, since V-category structure on $(X, a) \otimes (Y, b)$ is given by the "+-metric" (cf. Example 26), whereas V-category structure on $(X, a) \times (Y, b)$ is given by $c((x_1, y_1),(x_2, y_2)) = \max\{a(x_1, x_2), b(y_1, y_2)\}$. ▪

2.2. Dual V-categories and their induced functor

Definition 31. Given a V-category (X, a), define a V-relation a° on X by $a^\circ(x, y) = a(y, x)$. ▪

Proposition 32. *Given a V-category (X, a) over a commutative unital quantale V, (X, a°) is a V-category.*

Proof. First, given $x \in X$, it follows that $a^\circ(x, x) = a(x, x) \geqslant$ ((X, a) is a V-category) $\geqslant k$. Second, given $x, y, z \in X$, it follows that $a^\circ(x, y) \otimes a^\circ(y, z) = a(y, x) \otimes a(z, y) =$ (V is commutative) $= a(z, y) \otimes a(y, x) \leqslant$ ((X, a) is a V-category) $\leqslant a(z, x) = a^\circ(x, z)$, i.e., $a^\circ(x, y) \otimes a^\circ(y, z) \leqslant a^\circ(x, z)$. □

Remark 33. Given a V-category (X, a) over a commutative unital quantale V, $(X, a)^{op} = (X, a^\circ)$ is called the *dual* V-category of (X, a). ▪

Proposition 34. *Given a commutative unital quantale V, there exists a functor V-**Cat** $\xrightarrow{(-)^{op}}$ V-**Cat** defined by $((X, a) \xrightarrow{f} (Y, b)) = (X, a^\circ) \xrightarrow{f} (Y, b^\circ)$.*

Proof. Given a V-functor $(X, a) \xrightarrow{f} (Y, b)$, it is enough to show that $(X, a^\circ) \xrightarrow{f} (Y, b^\circ)$ is a V-functor. Given $x_1, x_2 \in X$, $a^\circ(x_1, x_2) = a(x_2, x_1) \leqslant$ (f is a V-functor) $\leqslant b(f(x_2), f(x_1)) = b^\circ(f(x_1), f(x_2))$. □

2.3. Unital quantale V as a V-category

Proposition 35. *Given a unital quantale V, the pair $(V, \multimap)$ is a V-category.*

Proof. Recall that Example 17 (7) defined the map $V \xrightarrow{(-)\multimap(-)} V$ by $a \multimap b = \bigvee\{c \in V \mid a \otimes c \leqslant b\}$, which is a V-relation $V \xrightarrow{(-)\multimap(-)} V$. We show the two required properties of a V-category (see Lecture 1).

(1) Given $v \in V$, $v \otimes k = v \leqslant v$ implies $k \leqslant v \multimap v$.
(2) Given $u, v, w \in V$, $u \multimap v \leqslant u \multimap v$ and $v \multimap w \leqslant v \multimap w$ imply $u \otimes (u \multimap v) \leqslant v$ and $v \otimes (v \multimap w) \leqslant w$ imply $u \otimes (u \multimap v) \otimes (v \multimap w) \leqslant v \otimes (v \multimap w) \leqslant w$ implies $(u \multimap v) \otimes (v \multimap w) \leqslant u \multimap w$. □

Example 36.

(1) If V is the two-element unital quantale $2 = (\{\bot, \top\}, \wedge, \top)$ (recall Lecture 1), then $(-) \multimap (-)$ is the partial order of 2, i.e., for every $u, v \in 2$, it follows that $u \multimap v = \top$ iff $u \leqslant v$.
(2) If V is the extended real half-line $\mathsf{P}_+ = ([0, \infty]^{op}, +, 0)$ (recall Lecture 1), then $(-) \multimap (-)$ is the truncated difference, i.e., for every $u, v \in \mathsf{P}_+$,

$$u \multimap v = \inf\{w \in [0, \infty] \mid v \leqslant u + w\} = \begin{cases} v - u, & u \leqslant v < \infty \\ 0, & v \leqslant u \\ \infty, & u < v = \infty. \end{cases}$$

■

*2.4. The category V-**Mod***

2.4.1. Ordered categories and quantaloids

Definition 37. A category $\mathbf{C}$ is called *preordered* provided that every its hom-set $\mathbf{C}(A, B)$ is a preordered set, and the composition of morphisms is monotone in both variables, i.e., given $\mathbf{C}$-morphisms $A \xrightarrow{f} B \overset{g_1}{\underset{g_2}{\rightrightarrows}} C \xrightarrow{h} D$, if $g_1 \leqslant g_2$, then $h \cdot g_1 \leqslant h \cdot g_2$ and $g_1 \cdot f \leqslant g_2 \cdot f$. Moreover, if the preorder on the hom-sets of $\mathbf{C}$ is a partial order, then the category $\mathbf{C}$ is called *partially ordered.* ■

Remark 38. The condition on composition of morphism in a preordered category $\mathbf{C}$ of Definition 37 is equivalent to the following: given $\mathbf{C}$-morphisms $A \xrightarrow{f} B \overset{g_1}{\underset{g_2}{\rightrightarrows}} C \xrightarrow{h} D$, if $g_1 \leqslant g_2$, then $h \cdot g_1 \cdot f \leqslant h \cdot g_2 \cdot f$. ■

Example 39.

(1) **Prost** is a preordered category, where $\mathbf{Prost}((X, \leqslant_X), (Y, \leqslant_Y))$ is equipped with a pointwise order.
(2) **Sup** is a partially ordered category.
(3) The category V-**Rel** of sets (as objects) and V-relations (as morphisms) is partially ordered by pointwise evaluation of V-relations (for V-relations $X \overset{r}{\underset{s}{\rightrightarrows}} Y$, $r \leqslant s$ iff $r(x, y) \leqslant s(x, y)$ for every $x \in X$, $y \in Y$).
(4) V-**Cat** is a partially ordered category with the partial order inherited from the category V-**Rel**.
(5) Every category is partially ordered if equipped with the partial order given by equality. ■

Remark 40.

(1) Given a preordered category $\mathbf{C}$, its dual category $\mathbf{C}^{op}$ is also preordered.
(2) For every preordered category $\mathbf{C}$, there exists the *conjugate* preordered category $\mathbf{C}^{co}$, which has the same morphisms but employs the dual preorder on hom-sets, i.e., $\mathbf{C}^{co}(A, B) = (\mathbf{C}(A, B), \geqslant) = (\mathbf{C}(A, B))^{op}$. Moreover, it is easy to verify that $\mathbf{C}^{op\,co} = \mathbf{C}^{co\,op}$. ■

Definition 41. A morphism $A \xrightarrow{f} B$ of a preordered category $\mathbf{C}$ is said to be a *map* provided that there exists a $\mathbf{C}$-morphism $B \xrightarrow{g} A$ such that $1_A \leqslant g \cdot f$ and $f \cdot g \leqslant 1_B$. One uses the notation $f \dashv g$, where f is the *left adjoint* and g is the *right adjoint* of the *adjunction*. ▪

Remark 42.

(1) The terminology of Definition 41 is motivated by the category **Rel**, where a relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$ is a map in **Rel** exactly when it is the graph of a morphism $X \xrightarrow{r} Y$ in **Set**. The existence of a relation $Y \overset{s}{\longrightarrow\!\!\!\!\!\!\!+\;\;} X$ such that $1_X \leqslant s \cdot r$ means that for every $x \in X$, there exists $y \in Y$ such that $x\,r\,y$ and $y\,s\,x$; and $r \cdot s \leqslant 1_Y$ means that for every $x \in X$ and every $y_1, y_2 \in Y$, if $y_1\,s\,x$ and $x\,r\,y_2$, then $y_1 = y_2$. One can thus define a unique map $X \xrightarrow{r} Y$ in **Set** by $r(x) = y$ iff $x\,r\,y$. One can also check that $s = r^{\circ}$.
(2) In every preordered category $\mathbf{C}$, a right adjoint g of a map f as in Definition 41 is uniquely determined up to "$\cong$", i.e., if both g_1 and g_2 are right adjoints of f, then $g_1 \leqslant g_2$ and $g_2 \leqslant g_1$. Moreover, in a partially ordered category $\mathbf{C}$, a right adjoint g to a map f is determined uniquely. ▪

Definition 43. A category $\mathbf{C}$ is a said to be a *quantaloid* provided that every its hom-set $\mathbf{C}(A, B)$ is a $\bigvee$-semilattice, and the composition of morphisms is $\bigvee$-preserving in both variables, i.e., given $\mathbf{C}$-morphisms $A \xrightarrow{f} B \xrightarrow{g_i} C \xrightarrow{h} D$ with $i \in I$, it follows that $h \cdot (\bigvee_{i \in I} g_i) = \bigvee_{i \in I}(h \cdot g_i)$ and $(\bigvee_{i \in I} g_i) \cdot f = \bigvee_{i \in I}(g_i \cdot f)$. ▪

Remark 44. Every quantaloid is a partially ordered category. ▪

Example 45.

(1) The categories **Sup** and V-**Rel** are quantaloids.
(2) A category $\mathbf{C}$ partially ordered by equality is a quantaloid iff its hom-sets have at most one element.
(3) Unital quantales are precisely the quantaloids with one object.
(4) Given a quantaloid $\mathbf{C}$, the dual category $\mathbf{C}^{op}$ is a quantaloid, but the conjugate category $\mathbf{C}^{co}$ is generally not a quantaloid, since composition of morphisms will generally preserve $\bigwedge$ and not $\bigvee$. ▪

Definition 46. A *homomorphism of quantaloids* $\mathbf{C} \xrightarrow{F} \mathbf{D}$ is a functor which preserves $\bigvee$ on hom-sets, i.e., given $\mathbf{C}$-morphisms $A \xrightarrow{f_i} B$ for $i \in I$, it follows that $F(\bigvee_{i \in I} f_i) = \bigvee_{i \in I} F f_i$. ▪

Example 47. Every homomorphism of unital quantales provides a homomorphism of quantaloids. ▪

Definition 48. A quantaloid $\mathbf{C}$ is said to be *involutive* provided that it comes equipped with a quantaloid homomorphism $\mathbf{C}^{op} \xrightarrow{(-)^{\circ}} \mathbf{C}$ (called *involution*) such that $C^{\circ} = C$ for every $\mathbf{C}$-object C and $(f^{\circ})^{\circ} = f$ for every $\mathbf{C}$-morphism $A \xrightarrow{f} B$. In particular, given $\mathbf{C}$-morphisms $A \xrightarrow{f_i} B$ for $i \in I$, $(\bigvee_{i \in I} f_i)^{\circ} = \bigvee_{i \in I} f_i^{\circ}$. ▪

2.4.2. V-modules

Definition 49. Given V-categories (X, a), (Y, b) over a unital quantale V, a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$ is called *V-module* (also *V-bimodule*, *V-profunctor*, or *V-distributor*) provided that $r \cdot a \leqslant r$ and $b \cdot r \leqslant r$. One denotes a V-module between V-categories (X, a) and (Y, b) by $(X, a) \overset{r}{\longrightarrow\!\!\!\!\!\!\!\circ\;\;} (Y, b)$. ▪

Proposition 50. *For a V-relation* $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$ *between V-categories* (X, a) *and* (Y, b) *equivalent are:*

(1) r *is a V-module;*
(2) $r \cdot a = r$ *and* $b \cdot r = r$*;*
(3) $b \cdot r \cdot a = r$.

Proof. For (1) $\Rightarrow$ (2), it will be enough to verify that given a V-module $(X,a) \overset{r}{\longrightarrow\!\!\!\!\circ\,} (Y,b)$, it follows that $r \leqslant r \cdot a$ and $r \leqslant b \cdot r$. For the former, notice that given $x \in X$ and $y \in Y$, one gets that $(r \cdot a)(x,y) = \bigvee_{x' \in X} a(x,x') \otimes r(x',y) \geqslant a(x,x) \otimes r(x,y) \geqslant ((X,a)$ is a V-category$) \geqslant k \otimes r(x,y) = r(x,y)$; and for the latter, observe that given $x \in X$ and $y \in Y$, one obtains that $(b \cdot r)(x,y) = \bigvee_{y' \in Y} r(x,y') \otimes b(y',y) \geqslant r(x,y) \otimes b(y,y) \geqslant ((Y,b)$ is a V-category$) \geqslant r(x,y) \otimes k = r(x,y)$.

For (2) $\Rightarrow$ (3), notice that $b \cdot r \cdot a = b \cdot (r \cdot a) = b \cdot r = r$.

For (3) $\Rightarrow$ (1), observe that, first, $r \cdot a = 1_Y \cdot r \cdot a \leqslant ((Y,b)$ is a V-category$) \leqslant b \cdot r \cdot a = r$ and, second, $b \cdot r = b \cdot r \cdot 1_X \leqslant ((X,a)$ is a V-category$) \leqslant b \cdot r \cdot a = r$. □

Example 51. If $V = 2$, then 2-modules are precisely the classical *modules* between preordered sets, i.e., relations $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,} Y$ (where $(X, \leqslant_X)$, $(Y, \leqslant_Y)$ are preordered sets) such that $(\leqslant_Y) \cdot r \cdot (\leqslant_X) \leqslant r$. The latter condition means that for every $x_1, x_2 \in X$, $y_1, y_2 \in Y$, $x_2 \leqslant_X x_1$ and $x_1\, r\, y_1$ and $y_1 \leqslant_Y y_2$ together imply $x_2\, r\, y_2$, i.e., the map $X^{op} \times Y \xrightarrow{r} 2$ is monotone, where $X^{op} \times Y$ is given the component-wise preorder. ▪

Proposition 52.

(1) *If* $(X,a) \overset{r}{\longrightarrow\!\!\!\!\circ\,} (Y,b)$, $(Y,b) \overset{s}{\longrightarrow\!\!\!\!\circ\,} (Z,c)$ *are V-modules, then* $X \overset{s \cdot r}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,} Z$ *is a V-module.*

(2) *Given a V-category (X,a),* $X \overset{a}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,} X$ *is a V-module.*

Proof.

(1) In view of Proposition 50, $c \cdot s \cdot r \cdot b = (c \cdot s) \cdot (r \cdot b) =$ (both r and s are V-modules) $= s \cdot r$.
(2) First, $a \cdot a \leqslant a$ ((X,a) is a V-category), and, second, $a = a \cdot 1_X \leqslant ((X,a)$ is a V-category$) \leqslant a \cdot a$. As a consequence, it follows that $a \cdot a = a$. □

Remark 53.

(1) In view of Proposition 50 (2) and Proposition 52, there exists the category V-**Mod** of V-categories (as objects) and V-modules (as morphisms), where given a V-category (X,a), it follows that V-relation $X \overset{a}{\longrightarrow\!\!\!\!\!\!\!+\,\,\,} X$ provides the identity morphism on (X,a).
(2) The category V-**Mod** is a partially ordered category with the partial order on hom-sets inherited from the partially ordered category V-**Rel**. Moreover, the category V-**Mod** is a quantaloid with $\bigvee$ in hom-sets formed by pointwise evaluation precisely as in the category V-**Rel**. ▪

Remark 54. Recall from Lecture 1 that for a unital quantale V, there is a functor **Set** $\xrightarrow{(-)_\circ}$ V-**Rel**, which is a non-full embedding if V has at least two elements. There also exists a functor $\mathbf{Set}^{op} \xrightarrow{(-)^\circ} V$-**Rel**. ▪

Lemma 55. *Given a V-functor $(X,a) \xrightarrow{f} (Y,b)$ over a unital quantale V, it follows that $a \cdot f^\circ \leqslant f^\circ \cdot b$.*

Proof. Since f is a V-functor, it follows that $f \cdot a \leqslant b \cdot f$. Recall from Lecture 1 that $1_X \leqslant f^\circ \cdot f$ and $f \cdot f^\circ \leqslant 1_Y$ for every map $X \xrightarrow{f} Y$. Thus, $a \leqslant f^\circ \cdot f \cdot a \leqslant f^\circ \cdot b \cdot f$ implies $a \cdot f^\circ \leqslant f^\circ \cdot b \cdot f \cdot f^\circ \leqslant f^\circ \cdot b$. □

Proposition 56.

(1) *There exists a functor* V-**Cat** $\xrightarrow{(-)_*}$ V-**Mod** *defined by* $((X,a) \xrightarrow{f} (Y,b))_* = (X,a) \overset{f_*}{\longrightarrow\!\!\!\!\circ\,} (Y,b)$, *where* $f_* = b \cdot f$, *i.e.,* $f_*(x,y) = b(f(x),y)$ *for every $x \in X$ and every $y \in Y$.*

(2) *There exists a functor* $(V\text{-}\mathbf{Cat})^{op} \xrightarrow{(-)^*}$ V-**Mod** *defined by* $((X,a) \xrightarrow{f} (Y,b))^* = (Y,b) \overset{f^*}{\longrightarrow\!\!\!\!\circ\,} (X,a)$, *where* $f^* = f^\circ \cdot b$, *i.e.,* $f^*(y,x) = b(y,f(x))$ *for every $x \in X$ and every $y \in Y$.*

Proof.

(1) To show that f_* is a V-module, consider Definition 49: $f_* \cdot a = b \cdot f \cdot a \leqslant$ (f is a V-functor) $\leqslant b \cdot b \cdot f \leqslant$ ((Y,b) is a V-category) $\leqslant b \cdot f = f_*$ and $b \cdot f_* = b \cdot b \cdot f \leqslant$ ((Y,b) is a V-category) $\leqslant b \cdot f = f_*$.

To show that $(-)_*$ preserves composition of morphisms, notice that given V-functors $(X,a) \xrightarrow{f} (Y,b)$, $(Y,b) \xrightarrow{g} (Z,c)$, it follows that $(g \cdot f)_* = c \cdot g \cdot f = c \cdot g \cdot 1_Y \cdot f \leqslant$ ((Y,b) is a V-category) $\leqslant c \cdot g \cdot b \cdot f = g_* \cdot f_*$. Moreover, $g_* \cdot f_* = c \cdot g \cdot b \cdot f \leqslant$ (g is V-functor) $\leqslant c \cdot c \cdot g \cdot f \leqslant$ ((Z,c) is a V-category) $\leqslant c \cdot g \cdot f = (g \cdot f)_*$. To show that $(-)_*$ preserves identities, notice that given a V-category (X,a), $(1_X)_* = a \cdot 1_X = a$.

(2) To show that f^* is a V-module, consider Definition 49: $f^* \cdot b = f^\circ \cdot b \cdot b \leqslant$ ((Y,b) is a V-category) $\leqslant f^\circ \cdot b = f^*$ and $a \cdot f^* = a \cdot f^\circ \cdot b \leqslant$ (Lemma 55 for f) $\leqslant f^\circ \cdot b \cdot b \leqslant$ ((Y,b) is a V-category) $\leqslant f^\circ \cdot b = f^*$.

To show that $(-)^*$ preserves composition of morphisms, notice that given V-functors $(X,a) \xrightarrow{f} (Y,b)$, $(Y,b) \xrightarrow{g} (Z,c)$, it follows that $(g \cdot f)^* = (g \cdot f)^\circ \cdot c = f^\circ \cdot g^\circ \cdot c = f^\circ \cdot 1_Y \cdot g^\circ \cdot c \leqslant$ ((Y,b) is a V-category) $\leqslant f^\circ \cdot b \cdot g^\circ \cdot c = f^* \cdot g^*$. Moreover, $f^* \cdot g^* = f^\circ \cdot b \cdot g^\circ \cdot c \leqslant$ (Lemma 55 for g) $\leqslant f^\circ \cdot g^\circ \cdot c \cdot c \leqslant$ ((Z,c) is a V-category) $\leqslant f^\circ \cdot g^\circ \cdot c = (g \cdot f)^\circ \cdot c = (g \cdot f)^*$.

To show that $(-)^*$ preserves identities, one should observe that given a V-category (X,a), it follows that $(1_X)^* = (1_X)^\circ \cdot a = 1_X \cdot a = a$. □

Remark 57.

(1) The functors of Proposition 56 provide a structured version of the functors of Remark 54, i.e., $\mathbf{Set} \xrightarrow{(-)_\circ} V\text{-}\mathbf{Rel} \xleftarrow{(-)^\circ} \mathbf{Set}^{op}$ is replaced with $V\text{-}\mathbf{Cat} \xrightarrow{(-)_*} V\text{-}\mathbf{Mod} \xleftarrow{(-)^*} (V\text{-}\mathbf{Cat})^{op}$.

(2) In case the quantale V has at least two elements, unlike the functor $\mathbf{Set} \xrightarrow{(-)_\circ} V\text{-}\mathbf{Rel}$, the functor $V\text{-}\mathbf{Cat} \xrightarrow{(-)_*} V\text{-}\mathbf{Mod}$ is not faithful. Consider, e.g., a V-category $(\mathbb{R}, \underline{k})$, where $\underline{k}(x,y) = k$ for every real numbers x, y. Then every map $\mathbb{R} \xrightarrow{f} \mathbb{R}$ provides a V-functor $(\mathbb{R}, \underline{k}) \xrightarrow{f} (\mathbb{R}, \underline{k})$. However, $f_* = \underline{k} \cdot f$ implies $f_*(x,y) = \underline{k}(f(x), y) = k$ for every $x, y \in \mathbb{R}$, which then gives $f_* = \underline{k} = 1_{(\mathbb{R},\underline{k})}$. ▪

Proposition 58. *Given a V-functor $(X,a) \xrightarrow{f} (Y,b)$, it follows that $f_* \cdot f^* \leqslant (1_Y)^*$ and $(1_X)^* \leqslant f^* \cdot f_*$.*

Proof. First, $f_* \cdot f^* = b \cdot f \cdot f^\circ \cdot b \leqslant$ (since $f \cdot f^\circ \leqslant 1_Y$) $\leqslant b \cdot b \leqslant$ ((Y,b) is a V-category) $\leqslant b = 1_Y \cdot b = (1_Y)^\circ \cdot b = (1_Y)^*$. Second, $f^* \cdot f_* = f^\circ \cdot b \cdot b \cdot f \geqslant$ (f is a V-functor) $\geqslant a \cdot f^\circ \cdot f \cdot a \geqslant$ (since $f^\circ \cdot f \geqslant 1_X$) $\geqslant a \cdot a =$ (Proposition 52 (2)) $= a = 1_X \cdot a = (1_X)^\circ \cdot a = (1_X)^*$. □

Remark 59.

(1) Proposition 58 provides a structured version of the adjunction $f_\circ \dashv f^\circ$ valid in V-**Rel** for every map $X \xrightarrow{f} Y$, in the form of $f_* \dashv f^*$ valid in V-**Mod** for every V-functor $(X,a) \xrightarrow{f} (Y,b)$.

(2) $(1_Y)^*$ and $(1_X)^*$ in Proposition 58 can be replaced with $(1_Y)_*$ and $(1_X)_*$. ▪

Proposition 60. *Given the dual V-category functor $V\text{-}\mathbf{Cat} \xrightarrow{(-)^{op}} V\text{-}\mathbf{Cat}$, it follows that $(f^{op})_* = (f^*)^\circ$ and $(f^{op})^* = (f_*)^\circ$ for every V-functor $(X,a) \xrightarrow{f} (Y,b)$.*

Proof. Given a V-functor $(X,a) \xrightarrow{f} (Y,b)$, since $((X,a) \xrightarrow{f} (Y,b))^{op} = (X,a^\circ) \xrightarrow{f} (Y,b^\circ)$ by Proposition 34, it follows that $(f^{op})_* = b^\circ \cdot f = (f^\circ \cdot b)^\circ = (f^*)^\circ$ and $(f^{op})^* = f^\circ \cdot b^\circ = (b \cdot f)^\circ = (f_*)^\circ$. □

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