Elements of monoidal topology
Lecture 5: $(\mathbb{T}, V)$-categories as generalized spaces

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Abstract

This lecture takes a look at $(\mathbb{T}, V)$-categories as generalized spaces and considers two well-known topological properties in this new generalized setting, e.g., Hausdorff separation axiom and compactness. In particular, this lecture provides generalized analogues of the Tychonoff Theorem and the Čech–Stone compactification.

1. Hausdorff and compact spaces

Remark 1. Since this lecture considers properties inspired by general topology, given a category $(\mathbb{T}, V)$-**Cat**, its objects (resp. morphisms) will be often referred to as $(\mathbb{T}, V)$-*spaces* (resp. $(\mathbb{T}, V)$-*continuous maps*). ▪

Definition 2. A topological space (X, τ), where τ is a topology on a set X, is *Hausdorff* (or T_2-*space*) provided that for every distinct $x, y \in X$, there exist disjoint elements $U, V \in \tau$ such that $x \in U$, $y \in V$. ▪

Definition 3. A topological space (X, τ) is *compact* provided that for every subset $\{U_i \mid i \in I\} \subseteq \tau$ such that $X \subseteq \bigcup_{i \in I} U_i$ (an *open cover of* X), there exists a finite set $\{i_1, \ldots, i_n\} \subseteq I$ such that $X \subseteq U_{i_1} \bigcup \ldots \bigcup U_{i_n}$ (a *subcover*). Briefly speaking, every open cover of the set X has a finite subcover. ▪

Proposition 4. *Every topological space (X, τ), where τ is a topology on a set X, has the following properties:*

(1) *(X, τ) is Hausdorff iff every ultrafilter on X has at most one convergence point;*
(2) *(X, τ) is compact provided that every ultrafilter on X converges to some point of X.* ▪

Remark 5. Lecture 1 described topological spaces as $(\beta, 2)$-categories for the ultrafilter monad β and the two-element unital quantale $2 = (\{\perp, \top\}, \wedge, \top)$. In particular, $(\beta, 2)$-categories are sets X equipped with a relation $\beta X \overset{a}{\longrightarrow\!\!\!\!\!\!\!+\,\,} X$, which satisfies the two properties of a $(\beta, 2)$-category. This relation a will be a map (written $\mathfrak{x} \longrightarrow x$ instead of $\mathfrak{x}\, a\, x$ and meaning "an ultrafilter $\mathfrak{x}$ converges to a point x") provided that

(1) for every $x_1, x_2 \in X$ and every $\mathfrak{y} \in \beta X$, if $\mathfrak{y} \longrightarrow y_1$ and $\mathfrak{y} \longrightarrow y_2$, then $y_1 = y_2$, which means $a \cdot a^\circ \leqslant 1_X$;
(2) for every $\mathfrak{x} \in \beta X$, there exists $x \in X$ such that $\mathfrak{x} \longrightarrow x$, which means $1_{\beta X} \leqslant a^\circ \cdot a$.

By Proposition 4, the above item (1) (resp. item (2)) makes the space (X, a) Hausdorff (resp. compact). ▪

Definition 6. A $(\mathbb{T}, V)$-space (X, a) is said to be

(1) *Hausdorff* provided that $a \cdot a^\circ \leqslant 1_X$;
(2) *compact* provided that $1_{TX} \leqslant a^\circ \cdot a$.

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The full subcategory of $(\mathbb{T}, V)$-**Cat** of Hausdorff (resp. compact) spaces is denoted $(\mathbb{T}, V)$-$\mathbf{Cat}_{\text{Haus}}$ (resp. $(\mathbb{T}, V)$-$\mathbf{Cat}_{\text{Comp}}$). The intersection of the above two subcategories is denoted $(\mathbb{T}, V)$-$\mathbf{Cat}_{\text{CompHaus}}$. ∎

Proposition 7. *Given a $(\mathbb{T}, V)$-space (X, a), the following holds:*

(1) *(X, a) is Hausdorff iff for every $x_1, x_2 \in X$ and every $\mathfrak{y} \in TX$,*

$$\perp_V < a(\mathfrak{y}, x_1) \otimes a(\mathfrak{y}, x_2) \ \textit{implies} \ x_1 = x_2, \tag{1.1}$$

$$a(\mathfrak{y}, x_1) \otimes a(\mathfrak{y}, x_2) \leqslant k, \tag{1.2}$$

where the latter condition is always satisfied in case the quantale V is strictly two-sided ($k = \top_V$).

(2) *(X, a) is compact iff for every $x \in TX$,*

$$k \leqslant \bigvee_{x \in X} a(\mathfrak{x}, x) \otimes a(\mathfrak{x}, x). \tag{1.3}$$

Proof. To show item (1), notice that given $x_1, x_2 \in X$, it follows that $(a \cdot a^\circ)(x_1, x_2) = \bigvee_{\mathfrak{y} \in TX} a^\circ(x_1, \mathfrak{y}) \otimes a(\mathfrak{y}, x_2) = \bigvee_{\mathfrak{y} \in TX} a(\mathfrak{y}, x_1) \otimes a(\mathfrak{y}, x_2)$, i.e., (X, a) is Hausdorff provided that

$$\bigvee_{\mathfrak{y} \in TX} a(\mathfrak{y}, x_1) \otimes a(\mathfrak{y}, x_2) \leqslant 1_X(x_1, x_2) = \begin{cases} k, & x_1 = x_2 \\ \perp_V, & \text{otherwise} \end{cases}$$

which is equivalent to conditions (1.1), (1.2).

To show item (2), notice that given $\mathfrak{x}_1, \mathfrak{x}_2 \in TX$, it follows that $(a^\circ \cdot a)(\mathfrak{x}_1, \mathfrak{x}_2) = \bigvee_{x \in X} a(\mathfrak{x}_1, x) \otimes a^\circ(x, \mathfrak{x}_2) = \bigvee_{x \in X} a(\mathfrak{x}_1, x) \otimes a(\mathfrak{x}_2, x)$, i.e., (X, a) is compact provided that

$$1_{TX}(\mathfrak{x}_1, \mathfrak{x}_2) = \begin{cases} k, & \mathfrak{x}_1 = \mathfrak{x}_2 \\ \perp_V, & \text{otherwise} \end{cases} \leqslant \bigvee_{x \in X} a(\mathfrak{x}_1, x) \otimes a(\mathfrak{x}_2, x) \ \text{ iff } \ k \leqslant \bigvee_{x \in X} a(\mathfrak{x}, x) \otimes a(\mathfrak{x}, x).$$

□

Definition 8. A unital quantale V is said to be *superior* provided that for every subset $\{u_i \mid i \in I\} \subseteq V$,

$$k \leqslant \bigvee_{i \in I} u_i \otimes u_i \ \text{ iff } \ k \leqslant \bigvee_{i \in I} u_i. \tag{1.4}$$

∎

Example 9.

(1) Every *idempotent* unital quantale V (i.e., $v \otimes v = v$ for every $v \in V$) is superior.
(2) Every *frame* V, i.e., a complete lattice such that $u \wedge (\bigvee S) = \bigvee_{s \in S} u \wedge s$ for every $u \in V$ and every subset $S \subseteq V$ (namely, finite meets distribute over arbitrary joins), is a superior quantale.
(3) In every strictly two-sided quantale V ($k = \top_V$), $k \leqslant \bigvee_{i \in I} u_i \otimes u_i$ implies $k \leqslant \bigvee_{i \in I} u_i$, since given $i \in I$, $u_i \otimes u_i \leqslant u_i \otimes \top_V = u_i$. The converse implication is generally not valid.
(4) An example of a non-superior unital quantale provides the standard construction of the free unital quantale over a monoid. Given a monoid $M = (M, \otimes, k)$, let V be the powerset PM of M with the operation $PM \times PM \xrightarrow{\hat{\otimes}} PM$ defined by $U \hat{\otimes} V = \{u \otimes v \mid u \in U, v \in V\}$ and with the unit $\hat{k} = \{k\}$. Then $V = (PM, \bigcup, \hat{\otimes}, \hat{k})$ is a unital quantale. Let M be now a group with more than one element, and let $U = \{m, m^{-1}\}$ with $m \neq k$. Then, $\hat{k} = \{k\} \nsubseteq U$, but $\hat{k} = \{k\} \subseteq \{m \otimes m, m^{-1} \otimes m^{-1}, k\} = U \hat{\otimes} U$. ∎

Remark 10.

(1) If the unital quantale V is superior, then the compactness condition (1.3) simplifies to

$$k \leqslant \bigvee_{x \in X} a(\mathfrak{x}, x). \tag{1.5}$$

(2) In view of Example 9, if V is a frame (for example, the two-element quantale $2 = (\{\bot, \top\}, \wedge, \top)$ of Lecture 1) or V is the extended real half-line $\mathsf{P}_+ = ([0,\infty]^{op}, +, 0)$ (recall Lecture 1), then the compactness condition (1.3) reduces to condition (1.5). ∎

Proposition 11. *Given a category V-**Cat**, the following holds.*

(1) *A V-space (X, a) is Hausdorff iff $a = 1_X$ (a is the* discrete *V-category structure on X of Lecture 2).*
(2) *Every V-space (X, a) is compact.*

Proof. To show item (1), on the one hand, if (X, a) is Hausdorff, then $1_X \leqslant a$ (since (X, a) is a V-category), and $a^\circ = 1_X \cdot a^\circ \leqslant a \cdot a^\circ \leqslant ((X, a)$ is Hausdorff$) \leqslant 1_X$ implies $a = (a^\circ)^\circ \leqslant (1_X)^\circ = 1_X$; and, on the other hand, if $a = 1_X$, then $1_X \cdot (1_X)^\circ = 1_X \cdot 1_X = 1_X$.

To show item (2), notice that $1_X \leqslant a$ (since (X, a) is a V-category) implies $1_X \leqslant a^\circ = a^\circ \cdot 1_X \leqslant a^\circ \cdot a$. □

Example 12.

(1) In view of Proposition 11, given a category V-**Cat**, V-$\mathbf{Cat}_{\mathrm{Comp}} = V$-**Cat**, and V-$\mathbf{Cat}_{\mathrm{Haus}}$ is the full (coreflective) subcategory comprising discrete V-categories in V-**Cat**.
Since 2-**Cat** is the category **Prost** of preordered sets and monotone maps (see Lecture 1), a preordered set $(X, \leqslant)$ is Hausdorff iff "$\leqslant$" is given by the equality, and, moreover, every preordered set is compact.
Since P_+-**Cat** is the category **QPMet** of quasi-pseudo-metric spaces (generalized metric spaces in the sense of F. W. Lawvere) and non-expansive maps (see Lecture 1), a quasi-pseudo-metric space (X, ρ) is Hausdorff iff the quasi-pseudo-metric ρ is given by

$$\rho(x_1, x_2) = \begin{cases} 0, & x_1 = x_2 \\ \infty, & \text{otherwise,} \end{cases}$$

and, moreover, every quasi-pseudo-metric space is compact.
(2) In the category **Top** of topological spaces and continuous maps, which is exactly the category $(\beta, 2)$-**Cat**, Hausdorffness and compactness for $(\beta, 2)$-spaces of Definition 6 coincide with their classical topological analogues of Definitions 2, 3. In particular, $(\beta, 2)$-$\mathbf{Cat}_{\mathrm{Haus}}$ is the category **Haus** of Hausdorff spaces, and $(\beta, 2)$-$\mathbf{Cat}_{\mathrm{Comp}}$ is the category **Comp** of compact spaces.
(3) In the category **App** of approach spaces and non-expansive maps, which is precisely the category (β, P_+)-**Cat**, a (β, P_+)-space is Hausdorff provided that for every $\mathfrak{x} \in \beta X$ and every $x_1, x_2 \in X$, $a(\mathfrak{x}, x) < \infty$ and $a(\mathfrak{x}, x_2) < \infty$ together imply $x = y$. Moreover, a (β, P_+)-space is compact provided that $\inf_{x \in X} a(\mathfrak{x}, x) = 0$ for every $\mathfrak{x} \in \beta X$. Such a property is called 0-*compactness* in the theory of approach spaces. In particular, in view of the full embedding **Top** $\hookrightarrow$ **App** of Lecture 1, an approach space induced by a topological space is 0-compact iff its underlying topology is compact. ∎

2. An excursus into category theory

Definition 13.

(1) A source $(C \xrightarrow{f_i} C_i)_{i \in I}$ in a category **C** is said to be a *mono-source* provided that for every **C**-morphisms $A \underset{h}{\overset{g}{\rightrightarrows}} C$, if $f_i \cdot g = f_i \cdot h$ for every $i \in I$, then $g = h$.
(2) Dually, a sink $(C_i \xrightarrow{f_i} C)_{i \in I}$ in a category **C** is said to be an *epi-sink* provided that for every **C**-morphisms $C \underset{h}{\overset{g}{\rightrightarrows}} A$, if $g \cdot f_i = h \cdot f_i$ for every $i \in I$, then $g = h$. ∎

Definition 14. Let $\mathcal{M}$ be a conglomerate of sources in a category $\mathbf{C}$. A subcategory $\mathbf{B}$ of $\mathbf{C}$ is said to be *closed under the formation of $\mathcal{M}$-sources* provided that whenever $(C \xrightarrow{f_i} B_i)_{i\in I}$ is a source in $\mathcal{M}$ such that every B_i belongs to $\mathbf{B}$, then C belongs to $\mathbf{B}$. Dually, one defines the closure under the formation of $\mathcal{C}$-sinks. ▪

Definition 15. An epimorphism e of a category $\mathbf{C}$ is said to be *strong* provided that whenever $g \cdot e = m \cdot f$ with m a $\mathbf{C}$-monomorphism, there exists a $\mathbf{C}$-morphism h such that the diagram

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d, "f"'] & \bullet \arrow[d, "f"] \arrow[dl, dotted, "h"'] \\ \bullet \arrow[r, "m"'] & \bullet \end{tikzcd}$$

commutes. ▪

Definition 16.

(1) A full subcategory $\mathbf{B}$ of a category $\mathbf{C}$ is said to be *reflective* in $\mathbf{C}$ provided that the inclusion functor $\mathbf{B} \overset{E}{\hookrightarrow} \mathbf{C}$ has a left adjoint, i.e., every $\mathbf{C}$-object C has a $\mathbf{B}$-*reflection arrow* $C \xrightarrow{r_C} EB$, which means that for every $\mathbf{C}$-morphism $C \xrightarrow{f} EB'$, there exists a unique $\mathbf{B}$-morphism $B \xrightarrow{f'} B'$ making the triangle

$$\begin{tikzcd} C \arrow[r, "r_C"] \arrow[dr, "f"'] & EB \arrow[d, dotted, "Ef'"] \\ & EB' \end{tikzcd}$$

commute.

(2) Let $\mathbf{C}$ be a category, and let $\mathcal{E}$ be a class of $\mathbf{C}$-morphisms. An isomorphism-closed, full subcategory $\mathbf{B}$ of $\mathbf{C}$ is $\mathcal{E}$*-reflective* in $\mathbf{C}$ provided that every $\mathbf{C}$-object has a $\mathbf{B}$-reflection arrow in $\mathcal{E}$. If $\mathcal{E}$ is the class of all (strong) $\mathbf{C}$-epimorphisms, then one uses the term (*strongly*) *epireflective* instead of $\mathcal{E}$-reflective. ▪

Proposition 17. *Given a category* $\mathbf{C}$ *with a factorization system* $(\mathcal{E}, \mathcal{M})$ *for sources, a full isomorphism-closed subcategory* $\mathbf{B}$ *of* $\mathbf{C}$ *is* $\mathcal{E}$*-reflective iff* $\mathbf{B}$ *is closed under the formation of* $\mathcal{M}$*-sources in* $\mathbf{C}$.

Proposition 18. *Given a topological construct* $\mathbf{C}$*, if* $\mathcal{E}$ *is the class of strong* $\mathbf{C}$*-epimorphisms, and* $\mathcal{M}$ *is the conglomerate of mono-sources in* $\mathbf{C}$*, then* $(\mathcal{E}, \mathcal{M})$ *is a factorization system for sources in* $\mathbf{C}$.

3. Properties of Hausdorff and compact $(\mathbb{T}, V)$-spaces

Proposition 19.

(1) *A source* $\mathcal{S} = (X \xrightarrow{f_i} Y_i)_{i\in I}$ *with* $I \neq \varnothing$ *in* $\mathbf{Set}$ *is a mono-source iff* $\bigwedge_{i\in I} f_i^\circ \cdot f_i = 1_X$.

(2) *A sink* $\mathcal{T} = (X_i \xrightarrow{g_i} Y)_{i\in I}$ *is an epi-sink in* $\mathbf{Set}$ *iff* $\bigvee_{i\in I} g_i \cdot g_i^\circ = 1_Y$.

Proof. For (1), notice that $\mathcal{S}$ is a mono-source in $\mathbf{Set}$ iff given $x, x' \in X$, "$f_i(x) = f_i(x')$ for every $i \in I$" is equivalent to "$x = x'$". Further, $(\bigwedge_{i\in I} f_i^\circ \cdot f_i)(x, x') = \bigwedge_{i\in I}(f_i^\circ \cdot f_i)(x, x') = \bigwedge_{i\in I}(f_i^\circ \cdot 1_{Y_i} \cdot f_i)(x, x') = \bigwedge_{i\in I} 1_{Y_i}(f_i(x), f_i(x')) = k$ iff $f_i(x) = f_i(x')$ for every $i \in I$; and $1_X(x, x') = k$ iff $x = x'$.

For (2), notice that $\mathcal{T}$ is an epi-sink in $\mathbf{Set}$ iff $Y = \bigcup_{i\in I} g_i(X_i)$. Further, given $y, y' \in Y$, it follows that $(\bigvee_{i\in I} g_i \cdot g_i^\circ)(y, y') = \bigvee_{i\in I}(g_i \cdot g_i^\circ)(y, y') = \bigvee_{i\in I} \bigvee_{x\in X_i} g_i^\circ(y, x) \otimes g_i(x, y') = \bigvee_{i\in I} \bigvee_{x\in X_i} g_i(x, y) \otimes g_i(x, y') = \bigvee_{i\in I} \bigvee\{k \mid x \in X_i : g_i(x) = y \text{ and } g_i(x) = y'\} = k$ iff $y = y'$ and $g_i^{-1}(y) \neq \varnothing$; and $1_Y(y, y') = k$ iff $y = y'$. □

Proposition 20. *Given* $(\mathbb{T}, V)$*-spaces* (X, a), (Y, b) *and a map* $X \xrightarrow{f} Y$*, the following are equivalent:*

(1) $(X,a) \xrightarrow{f} (Y,b)$ *is a* $(\mathbb{T},V)$*-continuous map;*
(2) $a \leqslant f^\circ \cdot b \cdot Tf$*;*
(3) $f \cdot a \cdot (Tf)^\circ \leqslant b$.

PROOF. (1) $\Rightarrow$ (2): Since $(X,a) \xrightarrow{f} (Y,b)$ is a $(\mathbb{T},V)$-continuous map,

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "a"', "|"] \arrow[dr, phantom, "\leqslant"] & TY \arrow[d, "b", "|"'] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

implies $f \cdot a \leqslant b \cdot Tf$, which gives $a \leqslant f^\circ \cdot f \cdot a \leqslant f^\circ \cdot b \cdot Tf$, since $1_X \leqslant f^\circ \cdot f$.

(2) $\Rightarrow$ (3): $a \leqslant f^\circ \cdot b \cdot Tf$ implies $f \cdot a \cdot (Tf)^\circ \leqslant f \cdot f^\circ \cdot b \cdot Tf \cdot (Tf)^\circ \leqslant b$, since $f \cdot f^\circ \leqslant 1_Y$ and $Tf \cdot (Tf)^\circ \leqslant 1_{TY}$.

(3) $\Rightarrow$ (1): $f \cdot a \cdot (Tf)^\circ \leqslant b$ implies $f \cdot a \leqslant f \cdot a \cdot (Tf)^\circ \cdot Tf \leqslant b \cdot Tf$, since $1_{TX} \leqslant (Tf)^\circ \cdot Tf$. $\square$

Proposition 21.

(1) $(\mathbb{T},V)$-$\mathbf{Cat}_{\mathrm{Haus}}$ *is closed under non-empty mono-sources in* $(\mathbb{T},V)$-$\mathbf{Cat}$. $(\mathbb{T},V)$-$\mathbf{Cat}_{\mathrm{Haus}}$ *is closed under all mono-sources (and, therefore, is strongly epireflective in* $(\mathbb{T},V)$-$\mathbf{Cat}$*) if V is strictly two-sided.*

(2) $(\mathbb{T},V)$-$\mathbf{Cat}_{\mathrm{Comp}}$ *is closed under those sinks* $((X_i,a_i) \xrightarrow{g_i} (Y,b))_{i\in I}$ *in* $(\mathbb{T},V)$-$\mathbf{Cat}$ *for which* $(TX_i \xrightarrow{Tg_i} TY)_{i\in I}$ *is an epi-sink in* $\mathbf{Set}$.

PROOF.

(1) Observe that given a mono-source $\mathcal{S} = ((X,a) \xrightarrow{f_i} (Y_i,b_i))_{i\in I}$ in $(\mathbb{T},V)$-$\mathbf{Cat}$ with $I \neq \varnothing$ and such that (Y_i,a_i) is Hausdorff for every $i \in I$, since the forgetful functor $(\mathbb{T},V)$-$\mathbf{Cat} \xrightarrow{U} \mathbf{Set}$ has a left adjoint (see Lecture 2), it preserves mono-sources, i.e., $U\mathcal{S} = (X \xrightarrow{f_i} Y_i)_{i\in I}$ is a mono-source in $\mathbf{Set}$ with $I \neq \varnothing$. By Proposition 19 (1), it follows that $\bigwedge_{i\in I} f_i^\circ \cdot f_i = 1_X$. Given $i \in I$, since (Y_i,a_i) is Hausdorff, $a \cdot a^\circ \leqslant$ (Proposition 20 (2)) $\leqslant (f_i^\circ \cdot b_i \cdot Tf_i) \cdot (f_i^\circ \cdot b_i \cdot Tf_i)^\circ = f_i^\circ \cdot b_i \cdot Tf_i \cdot (Tf_i)^\circ \cdot b_i^\circ \cdot f_i \leqslant (Tf_i \cdot (Tf_i)^\circ \leqslant 1_{TY_i})$ $\leqslant f_i^\circ \cdot b_i \cdot b_i^\circ \cdot f_i \leqslant ((Y_i,b_i)$ is Hausdorff$) \leqslant f_i^\circ \cdot f_i$. Thus, $a \cdot a^\circ \leqslant \bigwedge_{i\in I} f_i^\circ \cdot f_i = 1_X$, i.e., (X,a) is Hausdorff. If $I = \varnothing$, then $|X| \leqslant 1$, since $\mathcal{S}$ is a mono-source. Therefore, Hausdorff condition (1.1) is satisfied. Moreover, Hausdorff condition (1.2) is also satisfied provided that $k = \top_V$ (V is strictly two-sided). The claim on strong epireflectivity follows immediately from Propositions 17, 18.

(2) Notice that given a sink $((X_i,a_i) \xrightarrow{g_i} (Y,b))_{i\in I}$ in $(\mathbb{T},V)$-$\mathbf{Cat}$ with (X_i,a_i) compact for every $i \in I$, and such that $(TX_i \xrightarrow{Tg_i} TY)_{i\in I}$ is an epi-sink in $\mathbf{Set}$, by Proposition 19 (2), one gets $\bigvee_{i\in I} Tg_i \cdot (Tg_i)^\circ = 1_Y$. Given $i \in I$, since (X_i,a_i) is compact, $b^\circ \cdot b \geqslant$ (Proposition 20 (3)) $\geqslant (g_i \cdot a_i \cdot (Tg_i)^\circ)^\circ \cdot g_i \cdot a_i \cdot (Tg_i)^\circ =$ $Tg_i \cdot a_i{}^\circ \cdot g_i{}^\circ \cdot g_i \cdot a_i \cdot (Tg_i)^\circ \geqslant (g_i{}^\circ \cdot g_i \geqslant 1_{X_i}) \geqslant Tg_i \cdot a_i{}^\circ \cdot a_i \cdot (Tg_i)^\circ \geqslant ((X_i,a_i)$ is compact$) \geqslant Tg_i \cdot (Tg_i)^\circ$. As a result, it follows that $b^\circ \cdot b \geqslant \bigvee_{i\in I} Tg_i \cdot (Tg_i)^\circ = 1_{TY}$, i.e., the $(\mathbb{T},V)$-space (Y,b) is compact. $\square$

Corollary 22.

(1) *Given a surjective* $(\mathbb{T},V)$*-continuous map* $(X,a) \xrightarrow{g} (Y,B)$*, if* (X,a) *is compact, then* (Y,b) *is compact.*

(2) *If the functor* $\mathbf{Set} \xrightarrow{T} \mathbf{Set}$ *preserves small coproducts, then* $(\mathbb{T},V)$-$\mathbf{Cat}_{\mathrm{Comp}}$ *is closed under small epi-sinks in* $(\mathbb{T},V)$-$\mathbf{Cat}$*. The same statement holds for finite coproducts with closure under finite epi-sinks.*

PROOF.

(1) Since $X \xrightarrow{g} Y$ is surjective, it is a *retraction* in $\mathbf{Set}$, i.e., there exists a map $Y \xrightarrow{f} X$ such that $g \cdot f = 1_Y$. Every functor preserves retractions, and, therefore, T also does, i.e., Tg is a retraction in $\mathbf{Set}$. Moreover, every retraction is an epimorphism. Thus, by Proposition 21 (2), (Y,b) must be compact.

(2) Given a small epi-sink $\mathcal{T} = ((X_i, a_i) \xrightarrow{g_i} (Y, b))_{i \in I}$ in $(\mathbb{T}, V)$-**Cat**, where small means that I is a set, since the forgetful functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{U}$ **Set** has a right adjoint (see Lecture 2), it preserves epi-sinks, and, therefore, $U\mathcal{T} = (X_i \xrightarrow{g_i} Y)_{i \in I}$ is a small epi-sink in **Set**. Thus, forming a coproduct of $(X_i)_{i \in I}$ in **Set**, the unique morphism $\coprod_{i \in I} X_i \xrightarrow{g} Y$, making the triangle

$$\begin{tikzcd} X_i \arrow[r, "\mu_i"] \arrow[dr, "g_i"'] & \coprod_{i \in I} X_i \arrow[d, dotted, "g"] \\ & Y \end{tikzcd}$$

commute for every $i \in I$ (notice that μ_i are the coproduct injections), is an epimorphism in **Set**, i.e., g is surjective. Applying the functor T to the above triangle, one gets

$$\begin{tikzcd} TX_i \arrow[r, "T\mu_i"] \arrow[dr, "Tg_i"'] & T(\coprod_{i \in I} X_i) \arrow[d, dotted, "Tg"] \\ & TY, \end{tikzcd}$$

where $(T\mu_i)_{i \in I}$ is an epi-sink, since T preserves small coproducts, and Tg is an epimorphism in **Set** by the argument used in item (1) above. Since composition of epi-sinks is an epi-sink, it follows that $(Tg_i)_{i \in I}$ is an epi-sink. To establish the claim of item (2), it remains to apply Proposition 21 (2). $\square$

Example 23.

(1) Compact spaces are closed under finite epi-sinks in **Top** $\cong (\beta, V)$-**Cat**, since the functor **Set** $\xrightarrow{\beta}$ **Set** preserves finite coproducts. In particular, one obtains the classical result of general topology that given an onto continuous map $(X, \tau) \xrightarrow{f} (Y, \sigma)$ between topological spaces, if (X, τ) is compact, then (Y, σ) is compact. However, compact spaces are not closed under countable epi-sinks. As a counterexample, consider, e.g., the epi-sink $(1 \xrightarrow{\underline{n}} \mathbb{N})_{n \in \mathbb{N}}$ in **Top**, where $1 = \{*\}$, $\underline{n}(*) = n$, and $\mathbb{N}$ (the set of natural numbers) is given the discrete topology. It is easy to see that the space 1 is compact, but $\mathbb{N}$ is not.

(2) In a similar way, 0-compact spaces are closed under finite epi-sinks in **App** $\cong (\beta, \mathcal{P}_+)$-**Cat**, but not under countable epi-sinks (recall Example 12 (3) for the notion of 0-compactness). ▪

Proposition 24. *Given a morphism of lax extensions of monads $\hat{\mathbb{S}} \xrightarrow{\alpha} \hat{\mathbb{T}}$, the respective algebraic functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{A_\alpha}$ $(\mathbb{S}, V)$-**Cat**, which is defined by $A_\alpha((X, a) \xrightarrow{f} (Y, b)) = (X, a \cdot \alpha_X) \xrightarrow{f} (Y, b \cdot \alpha_Y)$, preserves both Hausdorffness and compactness property.*

Proof. Given a Hausdorff $(\mathbb{T}, V)$-space (X, a), to show that the $(\mathbb{T}, V)$-space $(X, a \cdot \alpha_X)$ is Hausdorff, notice that $(a \cdot \alpha_X) \cdot (a \cdot \alpha_X)^\circ = a \cdot \alpha_X \cdot \alpha_X^\circ \cdot a^\circ \leqslant (\alpha_X \cdot \alpha_X^\circ \leqslant 1_{TX}) \leqslant a \cdot a^\circ \leqslant 1_X$.

Given a compact $(\mathbb{T}, V)$-space (X, a), to show that the $(\mathbb{T}, V)$-space $(X, a \cdot \alpha_X)$ is compact, notice that $(a \cdot \alpha_X)^\circ \cdot (a \cdot \alpha_X) = \alpha_X^\circ \cdot a^\circ \cdot a \cdot \alpha_X \geqslant (a^\circ \cdot a \geqslant 1_{TX}$, since (X, a) is compact$) \geqslant \alpha_X^\circ \cdot \alpha_X \geqslant 1_{SX}$. $\square$

4. Tychonoff Theorem, Čech–Stone compactification

Definition 25. A quantale V is said to be *lean* provided that for every $u, v \in V$, if $u \vee v = \top_V$ and $u \otimes v = \bot_V$, then $u = \top_V$ or $v = \top_V$. ▪

Remark 26. Given a strictly two-sided and lean quantale V, $\top_V$ and $\bot_V$ are the only its *complemented* elements, i.e., elements $u \in V$ such that there exists $v \in V$ with $u \vee v = \top_V$, $u \wedge v = \bot_V$. Indeed, since V is strictly two-sided, for every $u, v \in V$, $u \otimes v \leqslant u \otimes \top_V = u$ and $u \otimes v \leqslant \top_V \otimes v = v$ imply $u \otimes v \leqslant u \wedge v$. ▪

Example 27.

(1) The quantales 2 and P_+ are strictly two-sided and lean.

(2) The quantale $3 = (\{\bot, k, \top\}, \otimes, k)$, where $\bot < k < \top$, and the multiplication $\otimes$ is given by the table

$\otimes$	$\bot$	k	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
k	$\bot$	k	$\top$
$\top$	$\bot$	$\top$	$\top$

is lean but not strictly two-sided.

(3) The quantale 2×2 is strictly two-sided but fails to be lean, since for $u = (\top, \bot) \neq (\top, \top)$ and $v = (\bot, \top) \neq (\top, \top)$, it follows that $u \vee v = (\top, \top)$ and $u \otimes v = u \wedge v = (\bot, \bot)$. ■

Proposition 28. *Let V be a strictly two-sided quantale.*

(1) *If V is lean, then all maps in V-**Rel** are **Set**-maps*

(2) *If V is commutative and all maps in V-**Rel** are **Set**-maps, then V is lean.*

Proof. Recall from Lecture 4 that since V-**Rel** is an ordered category, a V-relation $X \xrightarrow{r} Y$ is called a *map* provided that there exists a V-relation $Y \xrightarrow{s} X$ such that $r \dashv s$, i.e., $1_X \leqslant s \cdot r$ and $r \cdot s \leqslant 1_Y$.

Suppose that the quantale V is lean, and let $r \dashv s$ be valid in V-**Rel**. If $X = \varnothing$, then r is the inclusion map $\varnothing \hookrightarrow Y$. Thus, one can assume the existence of an element $x \in X$. Then, $\bot_V < \top_V = 1_X(x,x) \leqslant (s \cdot r)(x,x) = \bigvee_{y' \in Y} r(x,y') \otimes s(y',x)$ implies the existence of some $y \in Y$ such that $r(x,y) \otimes s(y,x) = u > \bot_V$. As a result, $\top_V = u \vee v$, where $v = \bigvee_{y' \in Y \setminus \{y\}} r(x,y') \otimes s(y',x)$. Further, for every $y' \in Y$ such that $y \neq y'$, $(r \cdot s)(y,y') \leqslant 1_Y(y,y') = \bot_V$ implies $(r \cdot s)(y,y') = \bot_V$ implies $s(y,x) \otimes r(x,y') = \bot_V$, and, therefore, $u \otimes v = (r(x,y) \otimes s(y,x)) \otimes (\bigvee_{y' \in Y \setminus \{y\}} r(x,y') \otimes s(y',x)) = \bigvee_{y' \in Y \setminus \{y\}} r(x,y) \otimes (s(y,x) \otimes r(x,y')) \otimes s(y',x) = \bigvee_{y' \in Y \setminus \{y\}} r(x,y) \otimes \bot_V \otimes s(y',x) = \bot_V$. Thus, we have found $u, v \in V$ such that $u \vee v = \top_V$ and $u \otimes v = \bot_V$. Since the quantale V is lean, it follows that either $u = \top_V$ or $v = \top_V$. If $v = \top_V$, then $u = u \otimes \top_V = u \otimes v = \bot_V$, which contradicts the above result $u > \bot_V$. Therefore, one obtains that $u = \top_V$ must hold, which implies $v = \top_V \otimes v = u \otimes v = \bot_V$, namely, $v = \bot_V$. It then follows that for every $x \in X$, there exists exactly one $y \in Y$ such that $r(x,y) \otimes s(y,x) > \bot_V$, and, moreover, $r(x,y) \otimes s(y,x) = \top_V$. Lastly, since V is strictly two-sided, it follows that $\top_V = r(x,y) \otimes s(y,x) \leqslant r(x,y) \wedge s(y,x)$ (see Remark 26), i.e., $r(x,y) = \top_V = s(y,x)$. Defining $y = f(x)$, one gets a map $X \xrightarrow{f} Y$ such that $f \leqslant r$ and $f^\circ \leqslant s$. It remains to show that $r \leqslant f$, which can be done as follows: $r = r \cdot 1_X \leqslant (1_X \leqslant f^\circ \cdot f) \leqslant r \cdot f^\circ \cdot f \leqslant r \cdot s \cdot f \leqslant 1_Y \cdot f = f$.

Suppose now that every map in the ordered category V-**Rel** is a **Set**-map. To show that the quantale V is necessarily lean, notice that given $u, v \in V$ such that $u \vee v = \top_V$ and $u \otimes v = \bot_V$, one can set $X = \{u, v\}$ and define a V-relation $\{*\} \xrightarrow{r} X$ by $r(*,x) = x$. It appears that $r \dashv r^\circ$, since, first, $(r \cdot r^\circ)(u,v) = r^\circ(u,*) \otimes r(*,v) = r(*,u) \otimes r(*,v) = u \otimes v = \bot_V$ and $(r \cdot r^\circ)(v,u) = r^\circ(v,*) \otimes r(*,u) = r(*,v) \otimes r(*,u) = v \otimes u = (V$ is commutative$) = u \otimes v = \bot_V$ imply $r \cdot r^\circ \leqslant 1_X$, and, second, $u = u \otimes \top_V = u \otimes (u \vee v) = (u \otimes u) \vee (u \otimes v) = (u \otimes u) \vee \bot_V = u \otimes u$ and $v = \top_V \otimes v = (u \vee v) \otimes v = (u \otimes v) \vee (v \otimes v) = \bot_V \vee (v \otimes v) = v \otimes v$ imply $(r^\circ \cdot r)(*,*) = \bigvee_{x \in X} r(*,x) \otimes r^\circ(x,*) = \bigvee_{x \in X} r(*,x) \otimes r(*,x) = (r(*,u) \otimes r(*,u)) \vee (r(*,v) \otimes r(*,v)) = (u \otimes u) \vee (v \otimes v) = u \vee v = \top_V$ implies $1_{\{*\}} \leqslant r^\circ \cdot r$. Since $r \dashv r^\circ$, r should be a map $X \xrightarrow{f} Y$ in **Set**, i.e., $f(*) = u$ or $f(*) = v$, which then gives the desired $\top_V = r(*,u) = u$ or $\top_V = r(*,v) = v$. □

Proposition 29. *If V is a strictly two-sided and lean quantale, then $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{CompHaus}}$ is the full subcategory of $\mathbf{Set}^{\mathbb{T}}$ of $\mathbb{T}$-algebras (X, a) such that $a \cdot \hat{T}a = a \cdot m_X$. If $\hat{T}$ is flat, then $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{CompHaus}} = \mathbf{Set}^{\mathbb{T}}$.*

Proof. Given a compact Hausdorff $(\mathbb{T}$,V)-space (X, a), it follows that $1_{TX} \leqslant a^\circ \cdot a$ and $a \cdot a^\circ \leqslant 1_X$, and, therefore, $(\mathbb{T}, V)$-relation $TX \xrightarrow{a} X$ is a map in the ordered category V-**Rel**. Since the quantale V is strictly two-sided and lean, $(\mathbb{T}, V)$-relation a is a **Set**-map $TX \xrightarrow{a} X$ by Proposition 28 (1). Thus, the defining two conditions of the $(\mathbb{T}, V)$-space (X, a), i.e., V-relational inequalities $a \cdot Ta \leqslant a \cdot \hat{T}a \leqslant a \cdot m_X$ and $1_X \leqslant a \cdot e_X$ between **Set**-maps must be equalities. Similarly, the defining condition $f \cdot a \leqslant b \cdot Tf$ of a $(\mathbb{T}, V)$-continuous map $(X, a) \xrightarrow{f} (Y, b)$ must be an equality provided that both (X, a) and (Y, b) are compact Hausdorff.

Moreover, recall from Lecture 2 that every flat lax extension $\hat{\mathbb{T}}$ of a monad $\mathbb{T}$ on **Set** has a full embedding $\mathbf{Set}^{\mathbb{T}} \overset{E}{\hookrightarrow} (\mathbb{T}, V)\text{-}\mathbf{Cat}$, which is given by $E((X, a) \xrightarrow{f} (Y, b)) = (X, a) \xrightarrow{f} (Y, b)$. As a consequence, every $\mathbb{T}$-algebra (X, a) has the property $a \cdot \hat{T}a = a \cdot m_X$, i.e., $(\mathbb{T}, V)\text{-}\mathbf{Cat}_{\mathrm{CompHaus}} = \mathbf{Set}^{\mathbb{T}}$. $\square$

Example 30. Since the lax extension of the ultrafilter monad β of Lecture 1 is flat, Proposition 29 implies, in particular, the classical result $(\beta, 2)\text{-}\mathbf{Cat}_{\mathrm{CompHaus}} = \mathbf{Set}^{\beta}$, i.e., the category of compact Hausdorff spaces is exactly the category of Eilenberg-Moore algebras for the ultrafilter monad on **Set**. ■

Theorem 31 (Tychonoff Theorem). *Let V be a strictly two-sided and lean quantale, and let the extension of the monad $\mathbb{T}$ to the category V-**Rel** be flat. Given a set-indexed family of compact Hausdorff $(\mathbb{T}, V)$-spaces $((X_i, a_i))_{i \in I}$, the product $\prod_{i \in I}(X_i, a_i)$ in $(\mathbb{T}, V)$-**Cat** is compact Hausdorff.*

Proof. Since (X_i, a_i) is compact Hausdorff for every $i \in I$, by Proposition 29, one has a set-indexed family of $\mathbf{Set}^{\mathbb{T}}$-objects $((X_i, a_i))_{i \in I}$. The Eilenberg-Moore algebra structure on the product $\prod_{i \in I}(X_i, a_i) = (\prod_{i \in I} X_i, a)$ in $\mathbf{Set}^{\mathbb{T}}$ is given by the unique map $TX \xrightarrow{a} X$ making the diagram

$$\begin{tikzcd} T(\prod_{i \in I} X_i) \arrow[r, "T\pi_i"] \arrow[d, dotted, "a"'] & TX_i \arrow[d, "a_i"] \\ \prod_{i \in I} X_i \arrow[r, "\pi_i"'] & X_i \end{tikzcd}$$

commute for every $i \in I$ (notice that π_i are the product projections). Since every product is a mono-source, Proposition 19 (1) implies $\bigwedge_{i \in I} \pi_i^{\circ} \cdot \pi_i = 1_{\prod_{i \in I} X_i}$. Thus, $a = 1_{\prod_{i \in I} X_i} \cdot a = (\bigwedge_{i \in I} \pi_i^{\circ} \cdot \pi_i) \cdot a =$ (a and π_i are **Set**-maps and, therefore, act on the two-element quantale $\{\perp_V, k\}$) $= \bigwedge_{i \in I}(\pi_i^{\circ} \cdot \pi_i \cdot a) = \bigwedge_{i \in I}(\pi_i^{\circ} \cdot a_i \cdot T\pi_i)$, which is exactly the product structure on $\prod_{i \in I} X_i$ formed in the category $(\mathbb{T}, V)$-**Cat** (recall from Lecture 2 that the category $(\mathbb{T}, V)$-**Cat** is a topological construct). Thus, the product $\prod_{i \in I}(X_i, a_i)$ in $(\mathbb{T}, V)$-**Cat** belongs to $\mathbf{Set}^{\mathbb{T}}$, and, therefore, by Proposition 29, $\prod_{i \in I}(X_i, a_i)$ is compact Hausdorff. $\square$

Definition 32. Given a functor $\mathbf{A} \xrightarrow{G} \mathbf{B}$, a *$G$-solution set* for a **B**-object B is a set $\mathcal{L}$ of **A**-objects such that for every **B**-morphism $B \xrightarrow{f} GA$, there exists $L \in \mathcal{L}$, a **B**-morphism $B \xrightarrow{h} GL$, and an **A**-morphism $L \xrightarrow{g} A$ such that the triangle

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[dr, "f"'] & GL \arrow[d, "Gg"] \\ & GA \end{tikzcd}$$

commutes. ■

Theorem 33 (Adjoint Functor Theorem). *Given a functor $\mathbf{A} \xrightarrow{G} \mathbf{B}$, where $\mathbf{A}$ is a complete category, G has a left adjoint iff it satisfies the following two conditions:*

(1) *G preserves small limits;*
(2) *every **B**-object B has a G-solution set.*

Proposition 34. *Given a complete category $\mathbf{A}$, a functor $\mathbf{A} \xrightarrow{G} \mathbf{B}$ preserves small limits (namely, limits of diagrams $\mathbf{I} \xrightarrow{D} \mathbf{A}$, where $\mathbf{I}$ is a small category) iff it preserves small products and equalizers.*

Remark 35. Let $\mathbb{T} = (T, m, e)$ be a monad on the category **Set**.

(1) A $\mathbf{Set}^{\mathbb{T}}$-object (X, a) is *non-trivial* provided that the set X has more than one element.

(2) If there exists at least one non-trivial $\mathbb{T}$-algebra, then the unit $X \xrightarrow{e_X} TX$ is injective for every set X.

(3) There exist exactly two trivial monads on **Set** (admitting only trivial $\mathbb{T}$-algebras), i.e., the monad sending every set to a singleton 1, and the monad sending the empty set to itself and all the other sets to 1. ▪

Theorem 36 (Čech–Stone compactification). *Let V be a strictly two-sided and lean quantale, and let the extension of the monad $\mathbb{T} = (T, m, e)$ to the category V-**Rel** be flat. Then $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{CompHaus}}$ is reflective in $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{Haus}}$, and $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{Haus}}$ is strongly epireflective in $(\mathbb{T}, V)$-**Cat**.*

Proof. By Definition 16, the category $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{CompHaus}}$ is reflective in the category $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{Haus}}$ provided that the inclusion functor $(\mathbb{T}, V)\text{-}\mathbf{Cat}_{\mathrm{CompHaus}} \overset{E}{\hookrightarrow} (\mathbb{T}, V)\text{-}\mathbf{Cat}_{\mathrm{Haus}}$ has a left adjoint. Since $(\mathbb{T}, V)\text{-}\mathbf{Cat}_{\mathrm{CompHaus}} = \mathbf{Set}^{\mathbb{T}}$ by Proposition 29, one considers the inclusion $\mathbf{Set}^{\mathbb{T}} \overset{E}{\hookrightarrow} (\mathbb{T}, V)\text{-}\mathbf{Cat}_{\mathrm{Haus}}$. In view of Theorem 33 and Proposition 34, it will be enough to show, first, that E preserves products and equalizers, and, second, that every $\mathbb{T}$-algebra (X, a) has an E-solution set.

Start by considering the inclusion $\mathbf{Set}^{\mathbb{T}} \overset{E'}{\hookrightarrow} (\mathbb{T}, V)\text{-}\mathbf{Cat}$. In view of Theorem 31, it preserves small products. To show that E' preserves equalizers, notice that given $\mathbb{T}$-homomorphisms $(X, a) \underset{g}{\overset{f}{\rightrightarrows}} (Y, b)$, an equalizer of f, g in $\mathbf{Set}^{\mathbb{T}}$ is given by an equalizer $Z \overset{i}{\hookrightarrow} X$ of f, g in **Set**, where $Z = \{x \in X \mid f(x) = g(x)\}$ and i is the inclusion, equipped with a $\mathbb{T}$-algebra structure c on Z, i.e., a map $TZ \xrightarrow{c} Z$ making the diagram

$$\begin{tikzcd} TZ \arrow[r, "Ti"] \arrow[d, dotted, "c"'] & TX \arrow[d, "a"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

commute, namely, $i \cdot c = a \cdot Ti$. Since the map i is injective, Proposition 19 (1) gives $i^\circ \cdot i = 1_Z$, and then $c = 1_Z \cdot c = i^\circ \cdot i \cdot c = i^\circ \cdot a \cdot Ti$, which is exactly the equalizer structure on Z formed in the category $(\mathbb{T}, V)$-**Cat** (recall from Lecture 2 that the category $(\mathbb{T}, V)$-**Cat** is a topological construct). Thus, the inclusion E' preserves equalizers. As a result, the above inclusion E preserves both small products and equalizers, since $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{Haus}}$ is closed in $(\mathbb{T}, V)$-**Cat** under small mono-sources by Proposition 21 (1).

Given a Hausdorff $(\mathbb{T}, V)$-space (X, a), in order to construct an E-solution set for (X, a), consider a $(\mathbb{T}, V)$-continuous map $(X, a) \xrightarrow{f} E(Y, b)$.

Take the least $\mathbb{T}$-subalgebra of (Y, b) containing $M = f(X)$, which can be obtained as follows. Let $M \overset{i}{\hookrightarrow} Y$ be the inclusion map, and consider the next commutative diagram

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "e_M"'] & Y \arrow[d, "e_Y"'] \arrow[dr, "1_Y"] & \\ TM \arrow[r, "Ti"'] & TY \arrow[r, "b"'] & Y, \end{tikzcd}$$

where the left-hand (resp. right-hand) side commutes, since e is the unit of the monad $\mathbb{T}$ (resp. (Y, b) is a $\mathbb{T}$-algebra). Denoting $h = b \cdot Ti$, one gets a factorization $M \overset{i}{\hookrightarrow} Y = M \xrightarrow{e_M} TM \xrightarrow{h} Y$, where e_M is injective for non-trivial monads by Remark 35. In case of any of the two trivial monads of Remark 35 (3), the set M

has at most one element, i.e, the map e_M is injective as well. Consider the following commutative diagram

$$\begin{tikzcd} TTM \arrow[r, "TTi"'] \arrow[rr, bend left, "Th"] \arrow[d, "m_M"'] & TTY \arrow[r, "Tb"'] \arrow[d, "m_Y"'] & TY \arrow[d, "b"] \\ TM \arrow[r, "Ti"] \arrow[rr, bend right, "h"'] & TY \arrow[r, "b"] & Y, \end{tikzcd}$$

where the left-hand (resp. right-hand) rectangle commutes, since m is the multiplication of the monad $\mathbb{T}$ (resp. (Y, b) is a $\mathbb{T}$-algebra). One obtains a $\mathbb{T}$-homomorphism $(TM, m_M) \xrightarrow{h} (Y, b)$ (straightforward computations show that given a set X, (TX, m_X) is a $\mathbb{T}$-algebra). Consider a factorization $TM \xrightarrow{h} Y = TM \xrightarrow{\overline{h}} h(TM) \xhookrightarrow{j} Y$ in **Set**, where $\overline{h}$ is the restriction of the map h to $h(TM)$, and j is the inclusion map. Since $\mathbf{Set}^{\mathbb{T}}$ is monadic over **Set**, this factorization can be lifted to $\mathbf{Set}^{\mathbb{T}}$ as $(TM, m_M) \xrightarrow{h} (Y, b) = (TM, m_M) \xrightarrow{\overline{h}} (h(TM), c) \xhookrightarrow{j} (Y, b)$. The desired least $\mathbb{T}$-subalgebra of (Y, b) containing M is then $(h(TM), c)$.

Since $h(TM)$ contains M, there exists a factorization $X \xrightarrow{f} Y = X \xrightarrow{\overline{f}} h(TM) \xhookrightarrow{j} Y$ in **Set**, where $\overline{f}$ is the restriction of the map f to $h(TM)$. Consider the following diagram

$$\begin{tikzcd} TX \arrow[rr, "Tf"] \arrow[dr, "T\overline{f}"] \arrow[ddd, "a"'] & & TY \arrow[ddd, "b"] \\ & T(h(TM)) \arrow[ur, "Tj"'] \arrow[d, "c"] & \\ & h(TM) \arrow[dr, "j"] & \\ X \arrow[ur, "\overline{f}"] \arrow[rr, "f"'] & & Y, \end{tikzcd}$$

where the two triangles and the right-hand rectangle commute, and the outer rectangle has the property $f \cdot a \leqslant b \cdot Tf$ (since f is a $(\mathbb{T}, V)$-continuous map). Thus, $j \cdot \overline{f} \cdot a \leqslant j \cdot c \cdot T\overline{f}$, which implies $\overline{f} \cdot a \leqslant c \cdot T\overline{f}$, since j is injective, and, therefore, $j^{\circ} \cdot j = 1_{h(TM)}$. As a consequence, one gets a commutative triangle

$$\begin{tikzcd} (X, a) \arrow[r, "\overline{f}"] \arrow[dr, "f"'] & E(h(TM), c) \arrow[d, "j"] \\ & E(Y, b) \end{tikzcd}$$

in the category $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{Haus}}$. Moreover, since the restriction of the map $X \xrightarrow{f} Y$ to M provides a surjective map $X \xrightarrow{\hat{f}} M$, which is a retraction in **Set**, $TX \xrightarrow{T\hat{f}} TM$ should be also a retraction in **Set**, namely, a surjective map. As a consequence, one obtains a surjective map $TX \xrightarrow{T\hat{f}} TM \xrightarrow{\overline{h}} h(TM)$, which implies that the cardinality of the set $h(TM)$ does not exceed the cardinality of the set TX.

As a consequence of the above, a solution set for (X, a) can be given by a representative system of non-isomorphic $\mathbb{T}$-algebras (Z, c), the cardinalities of which do not exceed that of TX. $\square$

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