Elements of monoidal topology
Lecture 6: separation axioms for generalized spaces

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Abstract

This lecture continues to view $(\mathbb{T}, V)$-categories as generalized spaces and considers the respective generalized versions of low separation axioms (T_0, R_0, T_1, R_1), regulariy, normality, and also extremal disconnectedness.

1. Order separation

Remark 1. Since this lecture considers properties inspired by general topology, given a category $(\mathbb{T}, V)$-**Cat**, its objects (resp. morphisms) will be often referred to as $(\mathbb{T}, V)$-*spaces* (resp. $(\mathbb{T}, V)$-*continuous maps*). ▪

Definition 2.

(1) Recall from Lecture 2 that given a $(\mathbb{T}, V)$-space (X, a), the V-relation $TX \overset{a}{\longrightarrow\!\!\!\!\!\!\!+\;} X$ induces a preorder $\leqslant$ on the set X defined for every $x, y \in X$ by $x \leqslant y$ iff $k \leqslant a(e_X(x), y)$ (where e is the unit of the monad $\mathbb{T}$). This preorder is called the *underlying preorder* induced by a or simply the *induced preorder*.
(2) A $(\mathbb{T}, V)$-space (X, a) is said to be *separated* provided that its underlying preorder is a partial order, i.e., for every $x, y \in X$, if $x \leqslant y$ and $y \leqslant x$, then $x = y$.
(3) The full subcategory of the category $(\mathbb{T}, V)$-**Cat** of separated $(\mathbb{T}$,V)-spaces is denoted $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{sep}}$. ▪

Definition 3. A topological space (X, τ), where τ is a topology on the set X, is called a T_0-*space* provided that for every two distinct points of X, there exists an element of τ containing exactly one of them. ▪

Example 4.

(1) In the category $\mathsf{2}$-**Cat**, which is exactly the category **Prost** of preordered sets and monotone maps, separated $\mathsf{2}$-categories are exactly the partially ordered sets (*posets*, for short).
(2) In the category P_+-**Cat**, which is exactly the category **QPMet** of quasi-pseudo-metric spaces (generalized metric spaces in the sense of F. W. Lawvere) and non-expansive maps, separated P_+-categories are quasi-pseudo-metric spaces (X, ρ) such that for every $x, y \in X$, if $\rho(x, y) = 0$ and $\rho(y, x) = 0$, then $x = y$.
(3) In the category $(\beta, \mathsf{2})$-**Cat**, which is exactly the category **Top** of topological spaces and continuous maps, separated (β, V)-categories are topological spaces (X, τ) such that for every $x, y \in Y$, if the principal ultrafilter $\dot{x}$ converges to y, and the principal ultrafilter $\dot{y}$ converges to x, then $x = y$. Recall from Lecture 3 that an ultrafilter $\mathfrak{x} \in \beta X$ converges to some $x \in X$ provided that $\mathfrak{x}$ contains every $U \in \tau$ such that $x \in U$. In view of Definition 3, separated topological spaces are precisely the T_0-spaces.

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Preprint submitted to the Czech University of Life Sciences Prague (CZU) *January 3, 2022*

(4) In the category $(\mathbb{\beta}, \mathsf{P}_+)$-**Cat**, which is exactly the category **App** of approach spaces and non-expansive maps, an approach space (X, a) is separated iff for every $x, y \in X$, $a(\dot{x}, y) = 0$ and $a(\dot{y}, x) = 0$ imply $x = y$. Equivalently, in terms of the approach distance δ, $\delta(x, \{y\}) = 0$ and $\delta(y, \{x\}) = 0$ imply $x = y$, where the approach distance $X \times PX \xrightarrow{\delta} [0, \infty]$ of a P_+-category (X, a) is defined by the formula $\delta(z, C) = \inf\{a(\mathfrak{y}, z) \mid \mathfrak{y} \in \beta C\}$ for every $z \in X$ and every $C \subseteq X$. ▪

Proposition 5. *For every $(\mathbb{T}, V)$-space (X, a), the following holds.*

(1) *If (X, a) is Hausdorff, then (X, a) is separated.*

(2) *If (X, a) is separated, then every $(\mathbb{T}, V)$-continuous map $(2 = \{0, 1\}, \underline{\top_V}) \xrightarrow{f} (X, a)$ from a two-element indiscrete $(\mathbb{T}, V)$-space (recall from Lecture 2 that $\underline{\top_V}$ stands for the constant map $T2 \times 2 \xrightarrow{\underline{\top_V}} V$ with value $\top_V$) is constant. If the quantale V is strictly two-sided ($k = \top_V$), then the latter property is equivalent to (X, a) being separated provided that for every map $2 \xrightarrow{f} X$, it follows that $a(Tf(\mathfrak{x}), f(i)) = \top_V$ for every $\mathfrak{x} \in T2$ such that $\mathfrak{x} \notin \{e_2(0), e_2(1)\}$ and every $i \in 2$.*

(3) *The full subcategory $(\mathbb{T}, V)$-**Cat**$_{\mathrm{sep}}$ of separated $(\mathbb{T}, V)$-spaces is closed under mono-sources in $(\mathbb{T}, V)$-**Cat**.*

Proof.

(1) Recall from Lecture 5 that a $(\mathbb{T}, V)$-space (X, a) is Hausdorff provided that $a \cdot a^\circ \leqslant 1_X$, which implies, in particular, that for every $x_1, x_2 \in X$ and every $\mathfrak{y} \in TX$, if $\bot_V < a(\mathfrak{y}, x_1) \otimes a(\mathfrak{y}, x_2)$, then $x_1 = x_2$. Given now $x, y \in X$ such that $x \leqslant y$ and $y \leqslant x$, it follows that $k \leqslant a(e_X(x), y)$ and $k \leqslant a(e_X(y), x)$, i.e., $\bot_V < k = k \otimes k \leqslant a(e_X(x), y) \otimes a(e_X(y), x)$, which gives $y = x$ by the above Hausdorffness property.

(2) Given a $(\mathbb{T}, V)$-continuous map $(2, \underline{\top_V}) \xrightarrow{f} (X, a)$, it follows that $\underline{\top_V} \leqslant f^\circ \cdot a \cdot Tf$ (recall Lecture 5), which gives $k \leqslant \top_V = \underline{\top_V}(e_2(0), 1) \leqslant (f^\circ \cdot a \cdot Tf)(e_2(0), 1) =$ (recall Lecture 2) $= a(Tf(e_2(0)), f(1)) = a((Tf \cdot e_2)(0), f(1)) =$ (since $1_{\mathbf{Set}} \xrightarrow{e} T$ is a natural transformation, the diagram

$$\begin{array}{ccc} 2 & \xrightarrow{e_2} & T2 \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle Tf} \\ X & \xrightarrow[e_X]{} & TX \end{array} \tag{1.1}$$

commutes, i.e., $Tf \cdot e_2 = e_X \cdot f$) $= a((e_X \cdot f)(0), f(1)) = a(e_X(f(0)), f(1))$, i.e., $f(0) \leqslant f(1)$. In a similar way, one obtains that $f(1) \leqslant f(0)$, which implies $f(0) = f(1)$, since the $(\mathbb{T}, V)$-space (X, a) is separated.
For the second statement, to show that (X, a) is separated, take $x, y \in X$ such that $x \leqslant y$ and $y \leqslant x$. Define a map $2 \xrightarrow{f} X$ by $f(0) = x$ and $f(1) = y$. If f is $(\mathbb{T}, V)$-continuous, then (by the assumption) f is constant, i.e., $x = f(0) = f(1) = y$. Thus, it is enough to prove that f is $(\mathbb{T}, V)$-continuous, i.e., $\underline{\top_V} \leqslant f^\circ \cdot a \cdot Tf$, which is equivalent to $a(Tf(\mathfrak{x}), f(i)) = \top_V$ for every $\mathfrak{x} \in T2$ and every $i \in \{0, 1\}$.
Since $x \leqslant y$ implies $k \leqslant a(e_X(x), y) = a(e_X(f(0)), y) = a((e_X \cdot f)(0), f(1)) =$ (diagram (1.1)) $= a((Tf \cdot e_2)(0), f(1)) = a(Tf(e_2(0)), f(1))$, and, similarly, $y \leqslant x$ implies $k \leqslant a(Tf(e_2(1)), f(0))$, one gets $a(Tf(e_2(0)), f(1)) = \top_V$ and $a(Tf(e_2(1)), f(0)) = \top_V$, since V is strictly two-sided.
Moreover, since (X, a) is a $(\mathbb{T}, V)$-space, $k \leqslant a(e_X(x), x) = a(Tf(e_2(0)), f(0))$ and $k \leqslant a(e_X(y), y) = a(Tf(e_2(1)), f(1))$ imply $a(Tf(e_2(0)), f(0)) = \top_V$ and $a(Tf(e_2(1)), f(1)) = \top_V$, since $k = \top_V$.
Lastly, by the assumption, it follows that $a(Tf(\mathfrak{x}), f(i)) = \top_V$ for every $\mathfrak{x} \in T2$ such that $\mathfrak{x} \notin \{e_2(0), e_2(1)\}$ and every $i \in 2$, which finishes the proof of $(\mathbb{T}, V)$-continuity of f.

(3) Given a mono-source $\mathcal{S} = ((X, a) \xrightarrow{f_i} (Y_i, b_i))_{i \in I}$ in $(\mathbb{T}, V)$-**Cat** with the property that (Y_i, a_i) is a separated $(\mathbb{T}, V)$-space for every $i \in I$, since the forgetful functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{U}$ **Set** has a left adjoint (see Lecture 2), it preserves mono-sources, and, therefore, $U\mathcal{S} = (X \xrightarrow{f_i} Y_i)_{i \in I}$ is a mono-source in **Set**. If $I = \varnothing$, then the set X has at most one element (since $U\mathcal{S}$ is a mono-source), i.e., (X, a) is separated. If $I \neq \varnothing$, then to show that (X, a) is separated, take $x_1, x_2 \in X$ such that $x_1 \leqslant x_2$ and $x_2 \leqslant x_1$, i.e., $k \leqslant a(e_X(x_1), x_2)$ and $k \leqslant a(e_X(x_2), x_1)$. Given $i \in I$, since $(X, a) \xrightarrow{f_i} (Y_i, b_i)$ is a $(\mathbb{T}, V)$-continuous

map, it follows that $a \leqslant f_i^\circ \cdot b_i \cdot Tf_i$, which implies $k \leqslant a(e_X(x_1), x_2) \leqslant (f_i^\circ \cdot b_i \cdot Tf_i)(e_X(x_1), x_2) = b_i((Tf_i \cdot e_X)(x_1), f_i(x_2)) =$ (since $1_{\mathbf{Set}} \xrightarrow{e} T$ is a natural transformation, the diagram

$$\begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle Tf_i} \\ Y_i & \xrightarrow[e_{Y_i}]{} & TY_i \end{array} \tag{1.2}$$

commutes, i.e., $Tf_i \cdot e_X = e_{Y_i} \cdot f_i) = b_i((e_{Y_i} \cdot f_i)(x_1), f_i(x_2)) = b_i(e_{Y_i}(f_i(x_1)), f_i(x_2))$ and, similarly, $k \leqslant b_i(e_{Y_i}(f_i(x_2)), f_i(x_1))$. Thus, $f_i(x_1) \leqslant f_i(x_2)$ and $f_i(x_2) \leqslant f_i(x_1)$, which implies $f_i(x_1) = f_i(x_2)$, since (Y_i, b_i) is separated. As a consequence, $f_i(x_1) = f_i(x_2)$ for every $i \in I$, which provides $x_1 = x_2$, since $U\mathcal{S}$ is a mono-source in **Set**, i.e., point-separating. □

Remark 6.

(1) Since the category $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{sep}}$ is closed under mono-sources in the category $(\mathbb{T}, V)$-**Cat**, $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{sep}}$ is a strongly epireflective subcategory of the category $(\mathbb{T}, V)$-**Cat** (see Lecture 5).

(2) In the category **Top**, for a topological space (X, τ), the respective $\mathbf{Top}_{\mathrm{sep}}$-reflection arrow is given by the quotient map $X \xrightarrow{p} X/{\sim}$, where the equivalence relation $\sim$ on X is defined by $x \sim y$ iff $\mathrm{cl}(\{x\}) = \mathrm{cl}(\{y\})$, in which $\mathrm{cl}(S)$ is the closure of a set S. Moreover, the quotient topology of the T_0-space $X/{\sim}$ makes the above map p both U-final and U-initial w.r.t. the forgetful functor $\mathbf{Top} \xrightarrow{U} \mathbf{Set}$ (see Lecture 2). ▪

Proposition 7. *Given a V-relation $TX \overset{a}{\longrightarrow\!\!\!\!\!\!\!+\;\;} X$, the following are equivalent:*

(1) $a \cdot \hat{T}a \leqslant a \cdot m_X$;

(2) $a \cdot \hat{T}a \cdot m_X^\circ \leqslant a$.

Proof.

(1) ⇒ (2): $a \cdot \hat{T}a \leqslant a \cdot m_X$ implies $a \cdot \hat{T}a \cdot m_X^\circ \leqslant a \cdot m_X \cdot m_X^\circ \leqslant a$, since $m_X \cdot m_X^\circ \leqslant 1_{TX}$.

(2) ⇒ (1): $a \cdot \hat{T}a \cdot m_X^\circ \leqslant a$ implies $a \cdot \hat{T}a \leqslant a \cdot \hat{T}a \cdot m_X^\circ \cdot m_X \leqslant a \cdot m_X$, since $1_{TTX} \leqslant m_X^\circ \cdot m_X$. □

Theorem 8. *Given a $(\mathbb{T}, V)$-space (X, a), the quotient map $X \xrightarrow{p} X/{\sim}$, induced by the equivalence relation $\sim$ on the set X defined by $x \sim y$ iff $x \leqslant y$ and $y \leqslant x$, provides a $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{sep}}$-reflection arrow for (X, a), when $X/{\sim}$ is equipped with the $(\mathbb{T}, V)$-space structure $\tilde{a} = p \cdot a \cdot (Tp)^\circ$, i.e., such that the following diagram*

$$\begin{array}{ccc} TX & \xleftarrow{(Tp)^\circ} & T(X/{\sim}) \\ {\scriptstyle a}\downarrow & & \vdots\,{\scriptstyle \tilde{a}} \\ X & \xrightarrow[p]{} & X/{\sim} \end{array}$$

commutes. This structure makes p both U-final and U-initial w.r.t. the forgetful functor $(\mathbb{T}, V)$-$\mathbf{Cat} \xrightarrow{U} \mathbf{Set}$.

Proof. One begins with the proofs of several inequalities used later on.

(1) Notice that $p^\circ \cdot p \leqslant a \cdot e_X$, since given $x, y \in X$, it follows that $(p^\circ \cdot p)(x, y) = \bigvee_{[z]_\sim \in X/\sim} p(x, [z]_\sim) \otimes p^\circ([z]_\sim, y) = \bigvee_{[z]_\sim \in X/\sim} p(x, [z]_\sim) \otimes p(y, [z]_\sim) = \begin{cases} k, & p(x) = p(y) \\ \bot_V, & \text{otherwise} \end{cases} = \begin{cases} k, & x \sim y \\ \bot_V, & \text{otherwise} \end{cases} \leqslant (a \cdot e_X)(x, y)$, since $x \sim y$ implies $x \leqslant y$, which gives $k \leqslant a(e_X(x), y) = (a \cdot e_X)(x, y)$. Given $x \in X$, one uses here the notation $[x]_\sim$ for the equivalence class of x w.r.t. the equivalence relation $\sim$, i.e., the set $\{y \in X \mid x \sim y\}$.

(2) Observe that $1_{X/\sim} \leqslant \tilde{a} \cdot e_{X/\sim}$, since $\tilde{a} \cdot e_{X/\sim} = p \cdot a \cdot (Tp)^{\circ} \cdot e_{X/\sim} \geqslant$ (since $1_{\mathbf{Set}} \xrightarrow{e} T$ is a natural transformation, the following diagram

$$\begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Tp} \\ X/\sim & \xrightarrow[e_{X/\sim}]{} & T(X/\sim) \end{array} \tag{1.3}$$

commutes, i.e., $e_{X/\sim} \cdot p = Tp \cdot e_X$, which implies $(Tp)^{\circ} \cdot e_{X/\sim} \cdot p \cdot p^{\circ} = (Tp)^{\circ} \cdot Tp \cdot e_X \cdot p^{\circ}$, which gives (since p is surjective, and, therefore, $p \cdot p^{\circ} = 1_{X/\sim}$) $(Tp)^{\circ} \cdot e_{X/\sim} = (Tp)^{\circ} \cdot Tp \cdot e_X \cdot p^{\circ}$, which provides $(Tp)^{\circ} \cdot e_{X/\sim} \geqslant e_X \cdot p^{\circ}$, since $(Tp)^{\circ} \cdot Tp \geqslant 1_{TX}$) $\geqslant p \cdot a \cdot e_X \cdot p^{\circ} \geqslant$ ($a \cdot e_X \geqslant p^{\circ} \cdot p$ by item (1)) $\geqslant p \cdot p^{\circ} \cdot p \cdot p^{\circ} =$ ($p \cdot p^{\circ} = 1_{X/\sim}$, since p is surjective) $= 1_{X/\sim} \cdot 1_{X/\sim} = 1_{X/\sim}$.

(3) Notice that $p^{\circ} \cdot \tilde{a} \cdot Tp \leqslant a$, since $p^{\circ} \cdot \tilde{a} \cdot Tp = p^{\circ} \cdot p \cdot a \cdot (Tp)^{\circ} \cdot Tp \leqslant$ ($p^{\circ} \cdot p \leqslant a \cdot e_X$) $\leqslant a \cdot e_X \cdot a \cdot (Tp)^{\circ} \cdot Tp \leqslant$ (properties of lax extensions of functors imply $(Tp)^{\circ} \cdot Tp \leqslant \hat{T}(p^{\circ}) \cdot \hat{T}p \leqslant \hat{T}(p^{\circ} \cdot p) \leqslant \hat{T}(a \cdot e_X)$ by item (1)) $\leqslant a \cdot e_X \cdot a \cdot \hat{T}(a \cdot e_X) =$ (Lecture 2) $= a \cdot e_X \cdot a \cdot \hat{T}a \cdot Te_X \leqslant$ (since (X, a) is a $(\mathbb{T}, V)$-space, $a \cdot \hat{T}a \leqslant a \cdot m_X$) $\leqslant a \cdot e_X \cdot a \cdot m_X \cdot Te_X =$ (since $\mathbb{T}$ is a monad, $m_X \cdot Te_X = 1_{TX}$) $= a \cdot e_X \cdot a \leqslant$ ($e_X \cdot a \leqslant \hat{T}a \cdot e_{TX}$ by a property of lax extensions of monads) $\leqslant a \cdot \hat{T}a \cdot e_{TX} \leqslant$ (since (X, a) is a $(\mathbb{T}, V)$-space, $a \cdot \hat{T}a \leqslant a \cdot m_X$) $\leqslant a \cdot m_X \cdot e_{TX} =$ (since $\mathbb{T}$ is a monad, $m_X \cdot e_{TX} = 1_{TX}$) $= a$.

(4) Observe that $\tilde{a} \cdot \hat{T}\tilde{a} \leqslant \tilde{a} \cdot m_{X/\sim}$, since $\tilde{a} \cdot \hat{T}\tilde{a} \cdot m^{\circ}_{X/\sim} =$ (since p is surjective, $p \cdot p^{\circ} = 1_{X/\sim}$, and, moreover, the same holds for Tp and TTp, since **Set**-functors preserve surjective maps) $= p \cdot p^{\circ} \cdot \tilde{a} \cdot Tp \cdot (Tp)^{\circ} \cdot \hat{T}\tilde{a} \cdot TTp \cdot (TTp)^{\circ} \cdot m^{\circ}_{X/\sim} \leqslant$ ($p^{\circ} \cdot \tilde{a} \cdot Tp \leqslant a$ by item (3)) $\leqslant p \cdot a \cdot (Tp)^{\circ} \cdot \hat{T}\tilde{a} \cdot TTp \cdot (TTp)^{\circ} \cdot m^{\circ}_{X/\sim} \leqslant$ ($(Tp)^{\circ} \cdot \hat{T}\tilde{a} \cdot TTp \leqslant \hat{T}(p^{\circ}) \cdot \hat{T}\tilde{a} \cdot \hat{T}Tp \leqslant \hat{T}(p^{\circ} \cdot \tilde{a} \cdot Tp)$ by the properties of lax extensions of functors of Lecture 2) $\leqslant p \cdot a \cdot \hat{T}(p^{\circ} \cdot \tilde{a} \cdot Tp) \cdot (TTp)^{\circ} \cdot m^{\circ}_{X/\sim} = p \cdot a \cdot \hat{T}(p^{\circ} \cdot \tilde{a} \cdot Tp) \cdot (m_{X/\sim} \cdot TTp)^{\circ} =$ (since $TT \xrightarrow{m} T$ is a natural transformation, the following diagram

$$\begin{array}{ccc} TTX & \xrightarrow{m_X} & TX \\ {\scriptstyle TTp}\downarrow & & \downarrow{\scriptstyle Tp} \\ TTX/\sim & \xrightarrow[m_{X/\sim}]{} & T(X/\sim) \end{array}$$

commutes, i.e., $m_{X/\sim} \cdot TTp = Tp \cdot m_X$) $= p \cdot a \cdot \hat{T}(p^{\circ} \cdot \tilde{a} \cdot Tp) \cdot (Tp \cdot m_X)^{\circ} = p \cdot a \cdot \hat{T}(p^{\circ} \cdot \tilde{a} \cdot Tp) \cdot m^{\circ}_X \cdot (Tp)^{\circ} \leqslant$ ($p^{\circ} \cdot \tilde{a} \cdot Tp \leqslant a$ by item (3)) $\leqslant p \cdot a \cdot \hat{T}a \cdot m^{\circ}_X \cdot (Tp)^{\circ} \leqslant$ ((X, a) is a $(\mathbb{T}, V)$-space backed by Proposition 7) $\leqslant p \cdot a \cdot (Tp)^{\circ} = \tilde{a}$, i.e., $\tilde{a} \cdot \hat{T}\tilde{a} \cdot m^{\circ}_{X/\sim} \leqslant \tilde{a}$, which implies $\tilde{a} \cdot \hat{T}\tilde{a} \leqslant \tilde{a} \cdot m_{X/\sim}$ by Proposition 7.

By the above items (2) and (4), $(X, \tilde{a})$ is a $(\mathbb{T}, V)$-space. Moreover, $\tilde{a}$ provides an U-final $(\mathbb{T}, V)$-space structure on the set $X/\sim$ w.r.t. the map $U(X, a) \xrightarrow{p} X/\sim$ (see Lecture 2). To show that a provides an U-initial $(\mathbb{T}, V)$-space structure on the set X w.r.t. the map $X \xrightarrow{p} U(X/\sim, \tilde{a})$, it is enough to check that $a = p^{\circ} \cdot \tilde{a} \cdot Tp$ (see Lecture 2). Notice that $p^{\circ} \cdot \tilde{a} \cdot Tp \leqslant a$ by the above item (3). Moreover, $\tilde{a} = p \cdot a \cdot (Tp)^{\circ}$ implies $p^{\circ} \cdot \tilde{a} \cdot Tp = p^{\circ} \cdot p \cdot a \cdot (Tp)^{\circ} \cdot Tp \geqslant a$, since $p^{\circ} \cdot p \geqslant 1_X$ as well as $(Tp)^{\circ} \cdot Tp \geqslant 1_{TX}$.

To show that $(X/\sim, \tilde{a})$ is separated, notice that $[x]_\sim \leqslant_{\tilde{a}} [y]_\sim$ implies $p(x) \leqslant_{\tilde{a}} p(y)$, which gives $k \leqslant \tilde{a}(e_{X/\sim}(p(x)), p(y)) = (p^{\circ} \cdot \tilde{a} \cdot e_{X/\sim} \cdot p)(x, y) =$ (diagram (1.3)) $= (p^{\circ} \cdot \tilde{a} \cdot Tp \cdot e_X)(x, y) = (a \cdot e_X)(x, y) = a(e_X(x), y)$, i.e., $x \leqslant_a y$. If also $[y]_\sim \leqslant_{\tilde{a}} [x]_\sim$, then, similarly, $y \leqslant_a x$, and, therefore, $x \sim y$, i.e., $[x]_\sim = [y]_\sim$.

To show that $(X, a) \xrightarrow{p} (X/\sim, \tilde{a})$ provides a $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{sep}}$-reflection arrow for (X, a), one has to check that given a $(\mathbb{T}, V)$-continuous map $(X, a) \xrightarrow{f} (Y, b)$ with (Y, b) separated, there exists a unique $(\mathbb{T}, V)$-continuous map $(X/\sim, \tilde{a}) \xrightarrow{\tilde{f}} (Y, b)$, which makes the following triangle commute

$$\begin{array}{ccc} (X, a) & \xrightarrow{p} & (X/\sim, \tilde{a}) \\ & {\scriptstyle f}\searrow & \vdots\,{\scriptstyle \tilde{f}} \\ & & (Y, b). \end{array} \tag{1.4}$$

Define the required map $X/{\sim} \xrightarrow{\tilde{f}} Y$ by $\tilde{f}([x]_\sim) = f(x)$. To show that the definition of the map $\tilde{f}$ is correct, one has to check that $[x_1]_\sim = [x_2]_\sim$ implies $f(x_1) = f(x_2)$. Indeed, $[x_1]_\sim = [x_2]_\sim$ implies $p(x_1) = p(x_2)$, which gives $x_1 \leqslant_a x_2$ and $x_2 \leqslant_a x_1$ by the previous paragraph. Thus, $k \leqslant a(e_X(x_1), x_2) \leqslant$ ($(X,a) \xrightarrow{f} (Y,b)$ is a $(\mathbb{T},V)$-continuous map) $\leqslant b(Tf(e_X(x_1)), f(x_2)) = b((Tf \cdot e_X)(x_1), f(x_2)) =$ ($1_{\mathbf{Set}} \xrightarrow{e} T$ is a natural transformation) $= b((e_Y \cdot f)(x_1), f(x_2)) = b(e_Y(f(x_1)), f(x_2))$, i.e., $f(x_1) \leqslant_b f(x_2)$, and, similarly, $f(x_2) \leqslant_b f(x_1)$. Since (Y,b) is a separated $(\mathbb{T},V)$-space, it follows that $f(x_1) = f(x_2)$.

Commutativity of diagram (1.4) follows from the definition of the map $\tilde{f}$. Moreover, since p is surjective, the map $\tilde{f}$, making diagram (1.4) commute, is unique. Lastly, since the map $(X,a) \xrightarrow{p} (X/{\sim}, \tilde{a})$ is U-final, commutativity of diagram (1.4) implies that $(X/{\sim}, \tilde{a}) \xrightarrow{\tilde{f}} (Y,b)$ is $(\mathbb{T},V)$-continuous (Lecture 2). □

Remark 9. Recall from Lecture 2 that there exists a concrete functor $(\mathbb{T},V)$-$\mathbf{Cat} \xrightarrow{Spec} \mathbf{Prost}$, which is defined by $Spec\,((X,a) \xrightarrow{f} (Y,b)) = (X, \leqslant_a) \xrightarrow{f} (Y, \leqslant_b)$. The functor *Spec* restricts to the subcategories $(\mathbb{T},V)$-$\mathbf{Cat}_{\mathrm{sep}}$ of separated $(\mathbb{T},V)$-spaces and $\mathbf{Prost}_{\mathrm{sep}} = \mathbf{Pos}$ of posets. ▪

Corollary 10. *The diagram*

$$\begin{array}{ccc} (\mathbb{T},V)\text{-}\mathbf{Cat}_{\mathrm{sep}} & \xrightarrow{Spec} & \mathbf{Pos} \\ \uparrow \dashv \downarrow & & \uparrow \dashv \downarrow \\ (\mathbb{T},V)\text{-}\mathbf{Cat} & \xrightarrow[Spec]{} & \mathbf{Prost} \end{array} \tag{1.5}$$

commutes w.r.t. both the solid and the dotted arrows.

Proof. Follows from the construction of $(\mathbb{T},V)$-$\mathbf{Cat}_{\mathrm{sep}}$-reflection arrows in Theorem 8. □

2. Between order separation and Hausdorff separation

Definition 11. A topological space (X,τ) is called

(1) T_1-*space* provided that for every distinct $x, y \in X$, there exists $U \in \tau$ such that $x \in U$ and $y \notin U$;
(2) R_0-*space* or *symmetric space* provided that for every $x, y \in X$, if $x \in \mathrm{cl}(\{y\})$, then $y \in \mathrm{cl}(\{x\})$;
(3) R_1-*space* provided that for every distinct $x, y \in X$, if $\mathrm{cl}(\{x\}) \neq \mathrm{cl}(\{y\})$, then there exists $U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \bigcap V = \varnothing$. ▪

Definition 12. Given a $(\mathbb{T},V)$-space (X,a), one can introduce the following separation axioms:

$$\begin{array}{ll} (T_0)\ (a \cdot e_X) \wedge (a \cdot e_X)^\circ \leqslant 1_X; & (R_0)\ (a \cdot e_X)^\circ \leqslant a \cdot e_X; \\ (T_1)\ a \cdot e_X \leqslant 1_X; & (R_1)\ a \cdot a^\circ \leqslant a \cdot e_X. \end{array}$$

▪

Remark 13.

(1) Given a V-space (X,a), the axioms of Definition 12 simplify to the following:

$$\begin{array}{ll} (T_0)\ a \wedge a^\circ \leqslant 1_X; & (R_0)\ a^\circ \leqslant a; \\ (T_1)\ a \leqslant 1_X; & (R_1)\ a \cdot a^\circ \leqslant a. \end{array}$$

(2) The axioms of Definition 12 are inspired by the separation properties of topological spaces in the category $\mathbf{Top} \cong (\beta, 2)$-$\mathbf{Cat}$ mentioned in Definitions 3, 11. ▪

Lemma 14. *For every $(\mathbb{T},V)$-space (X,a), $e_X \leqslant a^\circ$.*

Proof. Since (X,a) is a $(\mathbb{T},V)$-space, it follows that $1_X \leqslant a \cdot e_X$, which implies $1_X = (1_X)^\circ \leqslant (a \cdot e_X)^\circ = e_X^\circ \cdot a^\circ$, which provides $e_X \leqslant e_X \cdot e_X^\circ \cdot a^\circ \leqslant a^\circ$, since $e_X \cdot e_X^\circ \leqslant 1_{TX}$. □

Proposition 15. *For every $(\mathbb{T}, V)$-space (X, a), the following implications hold.*

$$\begin{array}{ccccc} \text{Hausdorff} & \Leftrightarrow & (T_1) & \& & (R_1) \\ \Downarrow & & \Downarrow & & \Downarrow \\ (T_1) & \Leftrightarrow & (T_0) & \& & (R_0) \\ & & \Downarrow & & \\ & & \text{separated} & & \end{array}$$

Proof.

"Hausdorff $\Rightarrow (T_1) \,\&\, (R_1)$": Lemma 14 provides $e_X \leqslant a^\circ$, which implies $a \cdot e_X \leqslant a \cdot a^\circ \leqslant$ ((X,a) is Hausdorff) $\leqslant 1_X \leqslant$ ((X,a) is a $(\mathbb{T}, V)$-space) $\leqslant a \cdot e_X$. It follows that $a \cdot e_X \leqslant 1_X$, which implies (T_1). Additionally, $a \cdot a^\circ \leqslant a \cdot e_X$, which implies (R_1).

"$(T_1) \,\&\, (R_1) \Rightarrow$ Hausdorff": $a \cdot a^\circ \overset{(R_1)}{\leqslant} a \cdot e_X \overset{(T_1)}{\leqslant} 1_X$, i.e., $a \cdot a^\circ \leqslant 1_X$, i.e., (X,a) is Hausdorff.

"$(T_1) \Rightarrow (T_0) \,\&\, (R_0)$": $(a \cdot e_X) \wedge (a \cdot e_X)^\circ \leqslant a \cdot e_X \overset{(T_1)}{\leqslant} 1_X$ gives (X,a) is (T_0); and $a \cdot e_X \overset{(T_1)}{\leqslant} 1_X$ gives $(a \cdot e_X)^\circ \leqslant (1_X)^\circ = 1_X \leqslant$ ((X,a) is a $(\mathbb{T}, V)$-category) $\leqslant a \cdot e_X$, i.e., $(a \cdot e_X)^\circ \leqslant a \cdot e_X$, i.e., (X,a) is (R_0).

"$(T_0) \,\&\, (R_0) \Rightarrow (T_1)$": $a \cdot e_X = (a \cdot e_X)^{\circ\circ} \overset{(R_0)}{\leqslant} (a \cdot e_X)^\circ$ implies $a \cdot e_X = (a \cdot e_X) \wedge (a \cdot e_X)^\circ \overset{(T_0)}{\leqslant} 1_X$, i.e., $a \cdot e_X \leqslant 1_X$, which implies (X,a) is (T_1).

"$(R_1) \Rightarrow (R_0)$": $(a \cdot e_X)^\circ = e_X^\circ \cdot a^\circ \leqslant$ (Lemma 14 gives $e_X \leqslant a^\circ$, which implies $e_X^\circ \leqslant a$) $\leqslant a \cdot a^\circ \overset{(R_1)}{\leqslant} a \cdot e_X$, i.e., $(a \cdot e_X)^\circ \leqslant a \cdot e_X$, which implies that (X,a) is (R_0).

"$(T_0) \Rightarrow$ separated": Given $x, y \in X$ such that $x \leqslant y$ and $y \leqslant x$, it follows that $k \leqslant a(e_X(x), y)$ and $k \leqslant a(e_X(y), x)$, which implies $k \leqslant a(e_X(x), y) \wedge a(e_X(y), x) = (a \cdot e_X)(x,y) \wedge (a \cdot e_X)(y,x) = (a \cdot e_X)(x,y) \wedge (a \cdot e_X)^\circ(x,y) = ((a \cdot e_X) \wedge (a \cdot e_X)^\circ)(x,y) \overset{(T_0)}{\leqslant} 1_X(x,y)$, which gives $x = y$. Thus, (X,a) is separated. □

Corollary 16. *For every V-space (X, a), the following implications hold.*

$$\begin{array}{ccccc} \text{Hausdorff} & \Leftrightarrow & (T_1) & \& & (R_1) \\ \Updownarrow & & \Downarrow & & \Updownarrow \\ (T_1) & \Leftrightarrow & (T_0) & \& & (R_0) \\ & & \Downarrow & & \\ & & \text{separated} & & \end{array}$$

Moreover,

(1) (R_0) *is equivalent to* $a = a^\circ$*;*
(2) (T_1) *is equivalent to* $a = 1_X$*;*
(3) *if* $V = 2$*, then "separated" implies* (T_0)*.*

Proof. In view of Proposition 15, one shows just the additional implications and statements.

"$(T_1) \Rightarrow$ Hausdorff": $a \overset{(T_1)}{\leqslant} 1_X$ implies $a^\circ \leqslant (1_X)^\circ = 1_X$ implies $a \cdot a^\circ \leqslant a \overset{(T_1)}{\leqslant} 1_X$, i.e., $a \cdot a^\circ \leqslant 1_X$, which implies that (X,a) is Hausdorff.

"$(R_0) \Rightarrow (R_1)$": $a^\circ \overset{(R_0)}{\leqslant} a$ implies $a \cdot a^\circ \leqslant a \cdot a \leqslant$ ((X,a) is a V-category) $\leqslant a$, i.e., $a \cdot a^\circ \leqslant a$, which implies that (X,a) is (R_1).

"$(R_0) \Leftrightarrow a = a^\circ$": The sufficiency is clear. For the necessity, $a^\circ \overset{(R_0)}{\leqslant} a$ implies $a = a^{\circ\circ} \leqslant a^\circ$, i.e., $a = a^\circ$.

"$(T_1) \Leftrightarrow a = 1_X$": $a \overset{(T_1)}{\leqslant} 1_X$ and $1_X \leqslant a$ ((X,a) is a V-space) imply $a = 1_X$.

"separated $\Rightarrow (T_0)$": By the assumption, $V = 2 = (\{\bot, \top\}, \wedge, \top)$. Given $x, y \in X$, it follows that $(a \wedge a^\circ)(x,y) = a(x,y) \wedge a^\circ(x,y) = a(x,y) \wedge a(y,x) = \top$ iff $a(x,y) = \top$ and $\top = a(y,x)$ iff $x \leqslant y$ and $y \leqslant x$ iff $x \overset{(X,a) \text{ is separated}}{=} y$ iff $1_X(x,y) = \top$. As a consequence, one gets $a \wedge a^\circ \leqslant 1_X$. □

Example 17.

(1) In the category 2-**Cat** $\cong$ **Prost**, (T_0) coincides with the separation axiom of Definition 2 (2) by Corollary 16, i.e., both make posets from preordered sets. Moreover, (R_0) coincides with (R_1) by Corollary 16, i.e., both make a preordered set $(X, \leqslant)$ *symmetric* (i.e., for every $x, y \in X$, $x \leqslant y$ implies $y \leqslant x$), which implies that the preorder $\leqslant$ is an equivalence relation on X. Lastly, Hausdorffness and (T_1) coincide and make an equality relation "=" from a preorder "$\leqslant$" (see Lecture 5).

(2) In the category P_+-**Cat** $\cong$ **QPMet**, (R_0) coincides with (R_1) by Corollary 16 and makes a quasi-pseudo-metric space (X, ρ) *symmetric*, i.e., $\rho(x, y) = \rho(y, x)$ for every $x, y \in X$. If (X, ρ) is symmetric, then even (T_0) makes $\rho = 1_X$ (see Corollary 16), i.e.,

$$\rho(x_1, x_2) = \begin{cases} 0, & x_1 = x_2 \\ \infty, & \text{otherwise,} \end{cases}$$

and is, thus, considerably stronger than being order separated. However, a two-element quasi-pseudo-metric space $(X = \{0, 1\}, \rho)$ such that

$$\rho(x_1, x_2) = \begin{cases} \infty, & x_1 = 0 \text{ and } x_2 = 1 \\ 0, & \text{otherwise,} \end{cases}$$

is (T_0) but not (T_1), since $\rho(1, 0) = 0 \neq \infty$.

(3) In the category **Top** $\cong (\beta, 2)$-**Cat**, the axioms of Definition 12 are equivalent to their classical analogues of general topology, which are mentioned in Definitions 3, 11.

(4) In the category **App** $\cong (\beta, \mathsf{P}_+)$-**Cat**, one has the following straightforward characterizations:
   - (X, a) is (T_0) provided that for every $x, y \in X$, if $a(\dot{x}, y) < \infty$ and $a(\dot{y}, x) < \infty$, then $x = y$;
   - (X, a) is (T_1) provided that for every $x, y \in X$, if $a(\dot{x}, y) < \infty$, then $x = y$;
   - (X, a) is (R_0) provided that for every $x, y \in X$, $a(\dot{x}, y) = a(\dot{y}, x)$;
   - (X, a) is (R_1) provided that for every $x, y \in X$ and every $\mathfrak{z} \in \beta X$, $a(\dot{x}, y) \leqslant a(\mathfrak{z}, x) + a(\mathfrak{z}, y)$, which is equivalent to $\delta(y, \{x\}) \leqslant a(\mathfrak{z}, x) + a(\mathfrak{z}, y)$, where $X \times PX \xrightarrow{\delta} [0, \infty]$ is the approach distance of the P_+-category (X, a), defined by $\delta(z, C) = \inf\{a(\mathfrak{y}, z) \mid \mathfrak{y} \in \beta C\}$ for every $z \in X$ and every $C \subseteq X$. ▪

Proposition 18. *Given a topological construct* **C**, *if* $\mathcal{E}$ *is the class of* **C***-bimorphisms (i.e.,* **C***-morphisms which are both monomorphisms and epimorphisms), and* $\mathcal{M}$ *is the conglomerate of initial sources in* **C**, *then* $(\mathcal{E}, \mathcal{M})$ *is a factorization system for sources in* **C**.

Proposition 19.

(1) (T_0) *and* (T_1) *separation properties are closed under mono-sources in* $(\mathbb{T}, V)$-**Cat**. *Thus, the corresponding full subcategories are strongly epireflective in* $(\mathbb{T}, V)$-**Cat**.

(2) (R_0) *and* (R_1) *properties are closed under* U*-initial sources in* $(\mathbb{T}, V)$-**Cat** *for the forgetful functor* $(\mathbb{T}, V)$-**Cat** $\xrightarrow{U}$ **Set**. *Thus, the respective full subcategories are both mono- and epireflective in* $(\mathbb{T}, V)$-**Cat**.

PROOF.

(1) Take a mono-source $\mathcal{S} = ((X, a) \xrightarrow{f_i} (Y_i, b_i))_{i \in I}$ in $(\mathbb{T}, V)$-**Cat**. Since the forgetful functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{U}$ **Set** has a left adjoint (see Lecture 2), it preserves mono-sources, i.e., $U\mathcal{S} = (X \xrightarrow{f_i} Y_i)_{i \in I}$ is a mono-source in **Set**. By the results of Lecture 5, it then follows that $\bigwedge_{i \in I} f_i^\circ \cdot f_i = 1_X$.
If (Y_i, b_i) is (T_0) for every $i \in I$, then $(a \cdot e_X) \wedge (a \cdot e_X)^\circ \leqslant ((X, a) \xrightarrow{f_i} (Y_i, b_i)$ is a $(\mathbb{T}, V)$-continuous map for every $i \in I) \leqslant \bigwedge_{i,j \in I} (f_i^\circ \cdot b_i \cdot Tf_i \cdot e_X) \wedge (f_j^\circ \cdot b_j \cdot Tf_j \cdot e_X)^\circ \overset{\text{diagram (1.2)}}{=} \bigwedge_{i,j \in I} (f_i^\circ \cdot b_i \cdot e_{Y_i} \cdot f_i) \wedge (f_j^\circ \cdot b_j \cdot e_{Y_j} \cdot f_j)^\circ \leqslant \bigwedge_{i \in I} (f_i^\circ \cdot b_i \cdot e_{Y_i} \cdot f_i) \wedge (f_i^\circ \cdot b_i \cdot e_{Y_i} \cdot f_i)^\circ = \bigwedge_{i \in I} (f_i^\circ \cdot b_i \cdot e_{Y_i} \cdot f_i) \wedge (f_i^\circ \cdot (b_i \cdot e_{Y_i})^\circ \cdot f_i) = \bigwedge_{i \in I} f_i^\circ \cdot ((b_i \cdot e_{Y_i}) \wedge (b_i \cdot e_{Y_i})^\circ) \cdot f_i \overset{(Y_i, b_i) \text{ is } (T_0)}{\leqslant} \bigwedge_{i \in I} f_i^\circ \cdot f_i = 1_X$, i.e., $(a \cdot e_X) \wedge (a \cdot e_X)^\circ \leqslant 1_X$, which then implies that the $(\mathbb{T}, V)$-space (X, a) is (T_0).

If (Y_i, b_i) is (T_1) for every $i \in I$, then $a \cdot e_X \leqslant ((X,a) \xrightarrow{f_i} (Y_i, b_i)$ is a $(\mathbb{T}, V)$-continuous map for every $i \in I) \leqslant \bigwedge_{i\in I} f_i^\circ \cdot b_i \cdot Tf_i \cdot e_X \overset{\text{diagram (1.2)}}{=} \bigwedge_{i\in I} f_i^\circ \cdot b_i \cdot e_{Y_i} \cdot f_i \overset{(Y_i,b_i) \text{ is } (T_1)}{\leqslant} \bigwedge_{i\in I} f_i^\circ \cdot f_i = 1_X$, i.e., $(a \cdot e_X) \wedge (a \cdot e_X)^\circ \leqslant 1_X$, which implies that (X, a) is (T_1).
The last statement follows from the results of Lecture 5 on reflective subcategories.

(2) Given an U-initial source $((X,a) \xrightarrow{f_i} (Y_i, b_i))_{i\in I}$ in $(\mathbb{T}, V)$-**Cat**, $a = \bigwedge_{i\in I} f_i^\circ \cdot a_i \cdot Tf_i$ by Lecture 2.
If (Y_i, b_i) is (R_0) for every $i \in I$, then $(a \cdot e_X)^\circ \leqslant ((X,a) \xrightarrow{f_i} (Y_i, b_i)$ is a $(\mathbb{T}, V)$-continuous map for every $i \in I) \leqslant \bigwedge_{i\in I} (f_i^\circ \cdot b_i \cdot Tf_i \cdot e_X)^\circ \overset{\text{diagram (1.2)}}{=} \bigwedge_{i\in I} (f_i^\circ \cdot b_i \cdot e_{Y_i} \cdot f_i)^\circ = \bigwedge_{i\in I} f_i^\circ \cdot (b_i \cdot e_{Y_i})^\circ \cdot f_i \overset{(Y_i,b_i) \text{ is } (R_0)}{\leqslant} \bigwedge_{i\in I} f_i^\circ \cdot b_i \cdot e_{Y_i} \cdot f_i \overset{\text{diagram (1.2)}}{=} \bigwedge_{i\in I} f_i^\circ \cdot b_i \cdot Tf_i \cdot e_X = (\bigwedge_{i\in I} f_i^\circ \cdot b_i \cdot Tf_i) \cdot e_X = a \cdot e_X$, i.e., $(a \cdot e_X)^\circ \leqslant a \cdot e_X$, which then implies that (X, a) is (R_0).
If (Y_i, b_i) is (R_1) for every $i \in I$, then $a \cdot a^\circ \leqslant ((X,a) \xrightarrow{f_i} (Y_i, b_i)$ is a $(\mathbb{T}, V)$-continuous map for every $i \in I)$ $\leqslant \bigwedge_{i\in I} (f_i^\circ \cdot b_i \cdot Tf_i) \cdot \bigwedge_{j\in I} (f_j^\circ \cdot b_j \cdot Tf_j)^\circ \leqslant \bigwedge_{i\in I} (f_i^\circ \cdot b_i \cdot Tf_i) \cdot (f_i^\circ \cdot b_i \cdot Tf_i)^\circ = \bigwedge_{i\in I} f_i^\circ \cdot b_i \cdot Tf_i \cdot (Tf_i)^\circ \cdot b_i^\circ \cdot f_i \leqslant$ $(Tf_i \cdot (Tf_i)^\circ \leqslant 1_{TY_i}) \leqslant \bigwedge_{i\in I} f_i^\circ \cdot b_i \cdot b_i^\circ \cdot f_i \overset{(Y_i,b_i) \text{ is } (R_1)}{\leqslant} \bigwedge_{i\in I} f_i^\circ \cdot b_i \cdot e_{Y_i} \cdot f_i \overset{\text{diagram (1.2)}}{=} \bigwedge_{i\in I} f_i^\circ \cdot b_i \cdot Tf_i \cdot e_X = (\bigwedge_{i\in I} f_i^\circ \cdot b_i \cdot Tf_i) \cdot e_X = a \cdot e_X$, i.e., $a \cdot a^\circ \leqslant a \cdot e_X$, which implies that (X, a) is (R_1).
The last claim follows from the results of Lecture 5 on reflective subcategories and Proposition 18. □

3. Regular $(\mathbb{T}, V)$-spaces

Definition 20. A topological space (X, τ) is called *regular* provided that for every $x \in X$ and every closed subset $A \subseteq X$ such that $x \notin A$, there exist $U, V \in \tau$ such that $x \in U$, $A \subseteq V$ and $U \bigcap V = \varnothing$. ▪

Definition 21. A pair (X, a), where X is a set and $TX \overset{a}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} X$ is a V-relation is said to be a $(\mathbb{T}, V)$-*graph* provided that a is *reflexive*, i.e.,

$$\begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ & \underset{1_X}{\searrow} \;\; \leqslant & \big\downarrow a \\ & & X. \end{array}$$

▪

Lemma 22. *For a lax extension* $\hat{\mathbb{T}} = (\hat{T}, m, e)$ *to* V-**Rel** *of a* **Set**-*monad* $\mathbb{T} = (T, e, m)$, $\hat{T}1_X = \hat{T}(e_X^\circ) \cdot m_X^\circ$.

Proposition 23. *Given a* $(\mathbb{T}, V)$-*space* (X, a), *define* $TX \overset{\hat{a}}{\longrightarrow} TX = TX \overset{m_X^\circ}{\longrightarrow} TTX \overset{\hat{T}a}{\longrightarrow} TX$. *It then follows that* $(TX, \hat{a})$ *is a* V-*graph, but* $(X, a \cdot \hat{a})$ *and* $(X, a \cdot \hat{a}^\circ)$ *are* $(\mathbb{T}, V)$-*graphs. Moreover,* $a \cdot \hat{a} \leqslant a$ *is equivalent to the transitivity condition for* a.

Proof. Notice that $1_{TX} = T1_X \leqslant$ (properties of lax extensions of functors) $\leqslant \hat{T}1_X =$ (Lemma 22) $= \hat{T}(e_X^\circ) \cdot m_X^\circ \leqslant$ (Lemma 14) $\leqslant \hat{T}a \cdot m_X^\circ = \hat{a}$, i.e., $1_{TX} \leqslant \hat{a}$, which implies that $(TX, \hat{a})$ is a V-graph.

Further, $a \cdot \hat{a} \cdot e_X \geqslant ((TX, \hat{a})$ is a V-graph$) \geqslant a \cdot e_X \geqslant ((X, a)$ is a $(\mathbb{T}, V)$-space$) \geqslant 1_X$, i.e., $a \cdot \hat{a} \cdot e_X \geqslant 1_X$, which implies that $(X, a \cdot \hat{a})$ is a $(\mathbb{T}, V)$-graph.

Lastly, $a \cdot \hat{a}^\circ \cdot e_X \geqslant ((TX, \hat{a})$ is a V-graph$) \geqslant a \cdot (1_{TX})^\circ \cdot e_X = a \cdot 1_{TX} \cdot e_X = a \cdot e_X \geqslant ((X, a)$ is a $(\mathbb{T}, V)$-space$) \geqslant 1_X$, i.e., $a \cdot \hat{a}^\circ \cdot e_X \geqslant 1_X$, which implies that $(X, a \cdot \hat{a}^\circ)$ is a $(\mathbb{T}, V)$-graph.

Lastly, $a \cdot \hat{a} \leqslant a$ iff $a \cdot \hat{T}a \cdot m_X^\circ \leqslant a$ iff $a \cdot \hat{T}a \leqslant a \cdot m_X$ by Proposition 7. □

Definition 24.

(1) A $(\mathbb{T}, V)$-space (X, a) is called *regular* provided that $a \cdot \hat{a}^\circ \leqslant a$, i.e., $a \cdot m_X \cdot (\hat{T}a)^\circ \leqslant a$, or, in pointwise notation, $\hat{T}a(\mathfrak{Y}, \mathfrak{x}) \otimes a(m_X(\mathfrak{Y}), x) \leqslant a(\mathfrak{x}, x)$ for every $\mathfrak{Y} \in TTX$, $\mathfrak{x} \in TX$, and every $x \in X$.
(2) The full subcategory of $(\mathbb{T}, V)$-**Cat** of regular spaces is denoted $(\mathbb{T}, V)$-$\mathbf{Cat}_{\text{reg}}$. ▪

Proposition 25. *A V-category (X, a) is regular iff $a = a^\circ$.*

Proof. Given a V-category (X, a), it follows that $\hat{a} = a$. Thus, regularity is equivalent to $a \cdot a^\circ \leqslant a$, which is exactly (R_1). By Corollary 16, (R_1) is equivalent to $a = a^\circ$. □

Example 26.

(1) In the categories 2-**Cat** $\cong$ **Prost** and P_+-**Cat** $\cong$ **QPMet**, by Proposition 25, preordered sets and quasi-pseudo-metric spaces are regular exactly when they are symmetric (recall Example 17).
(2) In the category **Top** $\cong (\beta, 2)$-**Cat**, regular topological spaces are precisely the regular spaces in the sense of general topology of Definition 20.
(3) In the category **App** $\cong (\beta, \mathsf{P}_+)$-**Cat**, an approach space (X, a) is regular precisely when for every $\mathfrak{x}, \mathfrak{y} \in \beta X$ and every $x \in X$, it follows that $a(\mathfrak{x}, x) \leqslant \hat{a}(\mathfrak{y}, \mathfrak{x}) + a(\mathfrak{y}, x)$, where $\hat{a}(\mathfrak{y}, \mathfrak{x}) = \inf\{u \in [0, \infty] \mid A^{(u)} \in \mathfrak{x}$ for every $A \in \mathfrak{y}\}$ (recall Lecture 1 for the notation $A^{(u)}$). ▪

Proposition 27.

(1) *If V is lean and strictly two-sided, and $\hat{T}$ is flat, then every compact Hausdorff $(\mathbb{T}, V)$-space is regular.*
(2) *The subcategory $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{reg}}$ is closed in the category $(\mathbb{T}, V)$-**Cat** under U-initial sources for the forgetful functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{U}$ **Set**, and, therefore, is both mono- and epireflective in $(\mathbb{T}, V)$-**Cat**.*

Proof.

(1) Given a compact Hausdorff $(\mathbb{T}, V)$-space (X, a), if V is a lean and strictly two-sided quantale, then the V-relation $TX \overset{a}{\longrightarrow\!\!\!\!\!\!\!+\;\;} X$ is a map, and, moreover, it follows that $a \cdot Ta = a \cdot m_X$ (see Lecture 5). It then follows that $a \cdot \hat{a}^\circ = a \cdot (\hat{T}a \cdot m_X^\circ)^\circ = a \cdot m_X \cdot (\hat{T}a)^\circ \overset{T \text{ is flat}}{=} a \cdot m_X \cdot (Ta)^\circ \overset{a \cdot m_X = a \cdot Ta}{=} a \cdot Ta \cdot (Ta)^\circ \leqslant$ $(Ta \cdot (Ta)^\circ \leqslant 1_{TX}) \leqslant a$, i.e., $a \cdot \hat{a}^\circ \leqslant a$, i.e., (X, a) is regular.
(2) Given an U-initial source $((X, a) \xrightarrow{f_i} (Y_i, b_i))_{i \in I}$ in $(\mathbb{T}, V)$-**Cat**, $a = \bigwedge_{i \in I} f_i^\circ \cdot b_i \cdot Tf_i$ by Lecture 2.
If (Y_i, b_i) is regular for every $i \in I$, then for every $i \in I$, it follows that $a \cdot \hat{a}^\circ = a \cdot (\hat{T}a \cdot m_X^\circ)^\circ = a \cdot m_X \cdot (\hat{T}a)^\circ \leqslant ((X, a) \xrightarrow{f_i} (Y_i, b_i)$ is a $(\mathbb{T}, V)$-continuous map$) \leqslant f_i^\circ \cdot b_i \cdot Tf_i \cdot m_X \cdot (\hat{T}(f_i^\circ \cdot b_i \cdot Tf_i))^\circ =$ (for every map $X \xrightarrow{f} Y$ and every V-relations $Y \overset{s}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Z$, $Z \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$, $\hat{T}(s \cdot f) = \hat{T}s \cdot Tf$ and $\hat{T}(f^\circ \cdot r) = (Tf)^\circ \cdot \hat{T}r$ by Lecture 2) $= f_i^\circ \cdot b_i \cdot Tf_i \cdot m_X \cdot ((Tf_i)^\circ \cdot \hat{T}b_i \cdot TTf_i)^\circ = f_i^\circ \cdot b_i \cdot Tf_i \cdot m_X \cdot (TTf_i)^\circ \cdot (\hat{T}b_i)^\circ \cdot Tf_i =$ (since $TT \xrightarrow{m} T$ is a natural transformation, the following diagram

$$\begin{array}{ccc} TTX & \xrightarrow{m_X} & TX \\ {\scriptstyle TTf_i}\downarrow & & \downarrow{\scriptstyle Tf_i} \\ TTY_i & \xrightarrow[m_{Y_i}]{} & TY_i \end{array}$$

commutes, i.e., $Tf_i \cdot m_X = m_{Y_i} \cdot TTf_i) = f_i^\circ \cdot b_i \cdot m_{Y_i} \cdot TTf_i \cdot (TTf_i)^\circ \cdot (\hat{T}b_i)^\circ \cdot Tf_i \leqslant (TTf_i \cdot (TTf_i)^\circ \leqslant 1_{TTY_i})$ $\leqslant f_i^\circ \cdot b_i \cdot m_{Y_i} \cdot (\hat{T}b_i)^\circ \cdot Tf_i \overset{(Y_i, b_i) \text{ is regular}}{\leqslant} f_i^\circ \cdot b_i \cdot Tf_i$. As a consequence, it follows that $a \cdot \hat{a}^\circ \leqslant \bigwedge_{i \in I} f_i^\circ \cdot b_i \cdot Tf_i = a$, i.e., $a \cdot \hat{a}^\circ \leqslant a$, which implies that (X, a) is regular.
The last claim follows from the results of Lecture 5 on reflective subcategories and Proposition 18. □

Remark 28. A regular $(\mathbb{T}, V)$-space may not be Hausdorff (or even separated). This can be seen, e.g., for V-spaces: Hausdorffness means discreteness (Lecture 5), and regularity means symmetry (Proposition 25). ▪

Definition 29. Given a functor **Set** $\xrightarrow{T}$ **Set**, a lax extension V-**Rel** $\xrightarrow{\hat{T}}$ V-**Rel** of T to V-**Rel** is said to be *symmetric* provided that $\hat{T}(r^\circ) = (\hat{T}r)^\circ$ for every V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$. ▪

Proposition 30. *Given a morphism of symmetric lax extensions of monads $\hat{\mathbb{S}} \xrightarrow{\alpha} \hat{\mathbb{T}}$, the respective algebraic functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{A_\alpha}$ $(\mathbb{S}, V)$-**Cat**, $A_\alpha((X, a) \xrightarrow{f} (Y, b)) = (X, a \cdot \alpha_X) \xrightarrow{f} (Y, b \cdot \alpha_Y)$ preserves regularity.*

Proof. Suppose that $\mathbb{S} = (S, n, d)$ and take a regular $(\mathbb{T}, V)$-space (X, a). To show that the $(\mathbb{S}, V)$-space $(X, a \cdot \alpha_X)$ is regular, notice that $a \cdot \alpha_X \cdot \widehat{a \cdot \alpha_X}^\circ = a \cdot \alpha_X \cdot (\hat{S}(a \cdot \alpha_X) \cdot n_X^\circ)^\circ =$ (for every map $X \xrightarrow{f} Y$ and every V-relation $Y \overset{s}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} Z$, $\hat{S}(s \cdot f) = \hat{S}s \cdot Sf$ by Lecture 2) $= a \cdot \alpha_X \cdot (\hat{S}a \cdot S\alpha_X \cdot n_X^\circ)^\circ =$ $a \cdot \alpha_X \cdot n_X \cdot (S\alpha_X)^\circ \cdot (\hat{S}a)^\circ \overset{\hat{S}\text{ is symmetric}}{=} a \cdot \alpha_X \cdot n_X \cdot (S\alpha_X)^\circ \cdot \hat{S}(a^\circ) =$ (by Lecture 2, since $\mathbb{S} \xrightarrow{\alpha} \mathbb{T}$ is a morphism of monads, the following diagram

$$\begin{array}{ccc} SS & \xrightarrow{\alpha\circ\alpha} & TT \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle m} \\ S & \xrightarrow[\alpha]{} & T \end{array}$$

commutes, where $\alpha \circ \alpha$ is defined by the diagonal of the commutative diagram

$$\begin{array}{ccc} SS & \xrightarrow{S\alpha} & ST \\ {\scriptstyle \alpha S}\downarrow & \searrow & \downarrow{\scriptstyle \alpha T} \\ TS & \xrightarrow[T\alpha]{} & TT, \end{array} \tag{3.1}$$

i.e., $\alpha \circ \alpha = T\alpha \cdot \alpha S = \alpha T \cdot S\alpha) = a \cdot m_X \cdot T\alpha_X \cdot \alpha_{SX} \cdot (S\alpha_X)^\circ \cdot \hat{S}(a^\circ) \leqslant$ (diagram (3.1) implies $T\alpha_X \cdot \alpha_{SX} = \alpha_{TX} \cdot S\alpha_X$, which gives $\alpha_{SX} \cdot (S\alpha_X)^\circ \leqslant (T\alpha_X)^\circ \cdot T\alpha_X \cdot \alpha_{SX} \cdot (S\alpha_X)^\circ = (T\alpha_X)^\circ \cdot \alpha_{TX} \cdot S\alpha_X \cdot (S\alpha_X)^\circ \leqslant (T\alpha_X)^\circ \cdot \alpha_{TX}$, since $1_{TSX} \leqslant (T\alpha_X)^\circ \cdot T\alpha_X$ and $S\alpha_X \cdot (S\alpha_X)^\circ \leqslant 1_{STX}$) $\leqslant a \cdot m_X \cdot T\alpha_X \cdot (T\alpha_X)^\circ \cdot \alpha_{TX} \cdot \hat{S}(a^\circ) \leqslant$ $(T\alpha_X \cdot (T\alpha_X)^\circ \leqslant 1_{TTX}) \leqslant a \cdot m_X \cdot \alpha_{TX} \cdot \hat{S}(a^\circ) \leqslant$ (since α is a morphism of lax extensions of functors,

$$\begin{array}{ccc} SX & \xrightarrow{\alpha_X} & TX \\ {\scriptstyle \hat{S}(a^\circ)}\,\big\downarrow\!\!\!\!+ & \leqslant & \big\downarrow\!\!\!\!+\,{\scriptstyle \hat{T}(a^\circ)} \\ STX & \xrightarrow[\alpha_{TX}]{} & TTX, \end{array}$$

i.e., $\alpha_{TX} \cdot \hat{S}(a^\circ) \leqslant \hat{T}(a^\circ) \cdot \alpha_X) \leqslant a \cdot m_X \cdot \hat{T}(a^\circ) \cdot \alpha_X \overset{\hat{T}\text{ is symmetric}}{=} a \cdot m_X \cdot (\hat{T}a)^\circ \cdot \alpha_X = a \cdot (\hat{T}a \cdot m_X^\circ)^\circ \cdot \alpha_X =$ $a \cdot \hat{a}^\circ \cdot \alpha_X \overset{(X,a)\text{ is regular}}{\leqslant} a \cdot \alpha_X$, i.e., $a \cdot \alpha_X \cdot \widehat{a \cdot \alpha_X}^\circ \leqslant a \cdot \alpha_X$. □

Remark 31. Given a lax extension $\hat{\mathbb{T}}$ of a monad $\mathbb{T} = (T, m, e)$ on **Set**, $\mathbb{I} \xrightarrow{e} \hat{\mathbb{T}}$ is a morphism of lax extensions of monads, where $\mathbb{I} = (1_{\mathbf{Set}}, 1, 1)$ is the identity monad on **Set**. ▪

Corollary 32. *Given a symmetric lax extension $\hat{\mathbb{T}}$ of a monad $\mathbb{T} = (T, m, e)$ on **Set**, the algebraic functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{A_e}$ V-**Cat**, $A_\alpha((X, a) \xrightarrow{f} (Y, b)) = (X, a \cdot e_X) \xrightarrow{f} (Y, b \cdot e_Y)$ preserves regularity.*

Proof. The claim follows from Remark 31 and Proposition 30. □

Remark 33. If $(\mathbb{T}, V) = (\beta, \mathsf{P}_+)$ (the symmetricity condition is satisfied for the lax extension of β to P_+-**Rel** of Lecture 1), then Corollary 32 says that the underlying metric of a regular approach space is symmetric. ▪

4. Normal and extremally disconnected $(\mathbb{T}, V)$-spaces

Definition 34.

(1) A topological space (X,τ) is said to be *normal* provided that for every disjoint closed subsets $A, B \subseteq X$, there exist disjoint elements $U, V \in \tau$ such that $A \subseteq U$ and $B \subseteq V$.
(2) A topological space (X,τ) is *extremally disconnected* if the closure of every open subset of X is open. ▪

Proposition 35. *For every topological space (X,τ) represented as a $(\beta, 2)$-space (X,a), equivalent are:*

(1) *(X,τ) is a normal topological space;*
(2) $\hat{a} \cdot \hat{a}^\circ \leqslant \hat{a}^\circ \cdot \hat{a}$.

Definition 36. A $(\mathbb{T}, V)$-space (X,a) is called *normal* provided that $\hat{a}\cdot\hat{a}^\circ \leqslant \hat{a}^\circ\cdot\hat{a}$, or, in pointwise notation, $a(\mathfrak{x},\mathfrak{y}) \otimes a(\mathfrak{x},\mathfrak{z}) \leqslant \bigvee_{\mathfrak{s}\in TX} a(\mathfrak{y},\mathfrak{s}) \otimes a(\mathfrak{z},\mathfrak{s})$ for every $\mathfrak{x},\mathfrak{y},\mathfrak{z} \in TX$. ▪

Definition 37. A lax extension $\hat{\mathbb{T}}$ to V-**Rel** of a monad $\mathbb{T} = (T, m, e)$ on **Set** is *associative* provided that $t \cdot \hat{T}(s \cdot \hat{T}r \cdot m_X^\circ) \cdot m_X^\circ = t \cdot \hat{T}s \cdot m_Y^\circ \cdot \hat{T}r \cdot m_X^\circ$ for all unitary V-relations $TX \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$, $TY \overset{s}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Z$, and $TZ \overset{t}{\longrightarrow\!\!\!\!\!\!\!+\;\;} W$, where a V-relation $TX \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$ is *unitary* provided that $r \cdot \hat{T}1_X \leqslant r$ and $e_Y^\circ \cdot \hat{T}r \cdot m_X^\circ \leqslant r$. ▪

Proposition 38. *For every lax extension $\hat{\mathbb{T}}$ to V-**Rel** of a monad $\mathbb{T} = (T, m, e)$ on **Set**, equivalent are:*

(1) *$\hat{\mathbb{T}}$ is associative;*
(2) *V-**Rel** $\xrightarrow{\hat{T}}$ V-**Rel** preserves composition and $\hat{T} \xrightarrow{m^\circ} \hat{T}\hat{T}$ is natural.*

Proposition 39. *If $\hat{\mathbb{T}}$ is associative, then for every $(\mathbb{T}, V)$-space (X,a), equivalent are:*

(1) *(X,a) is normal;*
(2) *$(TX,\hat{a})$ is a normal V-space;*
(3) *$(TX, \hat{a}^\circ \cdot \hat{a})$ is a V-space.*

Proof.

"(1) ⇔ (2)": Notice that given a V-space (Y,b), it follows that $\hat{b} = b$. Thus, $(TX,\hat{a})$ is a normal V-space iff $\hat{a}\cdot\hat{a}^\circ \leqslant \hat{a}^\circ\cdot\hat{a}$ iff (X,a) is a normal $(\mathbb{T},V)$-space. It remains to show that if (X,a) is a $(\mathbb{T},V)$-space, then $(TX,\hat{a})$ is a V-space. By Proposition 23, $(TX,\hat{a})$ is a V-graph, i.e., $1_{TX} \leqslant \hat{a}$, which proves reflexivity. To show transitivity, notice that $\hat{a}\cdot\hat{a} = \hat{T}a \cdot m_X^\circ \cdot \hat{T}a \cdot m_X^\circ =$ (since $\hat{\mathbb{T}}$ is associative, $\hat{T} \xrightarrow{m^\circ} \hat{T}\hat{T}$ is natural by Proposition 38, and, therefore, the following diagram

$$\begin{array}{ccc} TTX & \xrightarrow{m_{TX}^\circ} & TTTX \\ {\scriptstyle \hat{T}a}\downarrow & & \downarrow{\scriptstyle \hat{T}\hat{T}a} \\ TX & \xrightarrow[m_X^\circ]{} & TTX, \end{array}$$

commutes, i.e., $m_X^\circ \cdot \hat{T}a = \hat{T}\hat{T}a \cdot m_{TX}^\circ) = \hat{T}a \cdot \hat{T}\hat{T}a \cdot m_{TX}^\circ \cdot m_X^\circ = \hat{T}a \cdot \hat{T}\hat{T}a \cdot (m_X \cdot m_{TX})^\circ =$ (since $\mathbb{T} = (T,m,e)$ is a monad, the following diagram

$$\begin{array}{ccc} TTT & \xrightarrow{Tm} & TT \\ {\scriptstyle mT}\downarrow & & \downarrow{\scriptstyle m} \\ TT & \xrightarrow[m]{} & T \end{array}$$

commutes, i.e., $m_X \cdot m_{TX} = m_X \cdot Tm_X) = \hat{T}a \cdot \hat{T}\hat{T}a \cdot (m_X \cdot Tm_X)^\circ = \hat{T}a \cdot \hat{T}\hat{T}a \cdot (Tm_X)^\circ \cdot m_X^\circ \leqslant$ ($(Tm_X)^\circ \leqslant \hat{T}(m_X^\circ)$ by the properties of lax extensions of monads of Lecture 2) $\leqslant \hat{T}a \cdot \hat{T}\hat{T}a \cdot \hat{T}(m_X^\circ) \cdot m_X^\circ \leqslant$ (properties of lax extensions of monads of Lecture 2) $\leqslant \hat{T}(a \cdot \hat{T}a \cdot m_X^\circ) \cdot m_X^\circ \leqslant$ (since (X,a) is a $(\mathbb{T},V)$-category, $a \cdot \hat{T}a \leqslant a \cdot m_X$, which implies $a \cdot \hat{T}a \cdot m_X^\circ \leqslant a$ by Proposition 7) $\leqslant \hat{T}a \cdot m_X^\circ = \hat{a}$.

"(2) ⇒ (3)": By Proposition 23, $(TX, \hat{a})$ is a V-graph, i.e., $1_{TX} \leqslant \hat{a}$, which implies $1_{TX} = (1_{TX})^\circ \leqslant \hat{a}^\circ$, and, therefore, $1_{TX} = 1_{TX} \cdot 1_{TX} \leqslant \hat{a}^\circ \cdot \hat{a}$, which proves reflexivity. To show transitivity, notice that $\hat{a}^\circ \cdot \hat{a} \cdot \hat{a}^\circ \cdot \hat{a} \overset{(X,a) \text{ is normal}}{\leqslant} \hat{a}^\circ \cdot \hat{a}^\circ \cdot \hat{a} \cdot \hat{a} = (\hat{a} \cdot \hat{a})^\circ \cdot \hat{a} \cdot \hat{a} \overset{(X,a) \text{ is a } V\text{-space}}{\leqslant} \hat{a}^\circ \cdot \hat{a}$.

"(3) ⇒ (1)": $\hat{a} \cdot \hat{a}^\circ \leqslant$ ($(TX, \hat{a}^\circ \cdot \hat{a})$ is a V-space) $\leqslant \hat{a} \cdot \hat{a}^\circ \cdot \hat{a} \cdot \hat{a}^\circ \leqslant$ ($(TX, \hat{a})$ is a V-graph implies $1_{TX} \leqslant \hat{a}$ implies $1_{TX} \leqslant \hat{a}^\circ$) $\leqslant \hat{a}^\circ \cdot \hat{a}$, i.e., $\hat{a} \cdot \hat{a}^\circ \leqslant \hat{a}^\circ \cdot \hat{a}$, which proves normality of (X, a). □

Proposition 40. *For every topological space (X, τ) represented as a $(\beta, 2)$-space (X, a), equivalent are:*

(1) *(X, τ) is extremally disconnected;*
(2) $\hat{a}^\circ \cdot \hat{a} \leqslant \hat{a} \cdot \hat{a}^\circ$.

Definition 41. A $(\mathbb{T}, V)$-space is called *extremally disconnected* provided that $\hat{a}^\circ \cdot \hat{a} \leqslant \hat{a} \cdot \hat{a}^\circ$. ▪

Proposition 42. *A V-space (X, a) is normal iff (X, a°) is extremally disconnected.*

Proof. Given a V-space (X, a), (X, a°) is a V-space by the results of Lecture 4. Moreover, since $\hat{a} = a$, (X, a) is normal iff $a \cdot a^\circ \leqslant a^\circ \cdot a$ iff $(a^\circ)^\circ \cdot a^\circ \leqslant a^\circ \cdot (a^\circ)^\circ$ iff (X, a°) is extremally disconnected. □

Proposition 43. *If $\hat{\mathbb{T}}$ is associative, then for every $(\mathbb{T}, V)$-space (X, a), equivalent are:*

(1) *(X, a) is extremally disconnected;*
(2) *$(TX, \hat{a})$ is an extremally disconnected V-space;*
(3) *$(TX, \hat{a}^\circ)$ is a normal V-space;*
(4) *$(TX, \hat{a} \cdot \hat{a}^\circ)$ is a V-space.*

Proof.
"(1) ⇔ (2)": See the respective item of the proof of Proposition 39.
"(2) ⇔ (3)": Follows from Proposition 42.
"(3) ⇔ (4)": Follows from "(2) ⇔ (3)" of Proposition 39. □

Definition 44. Given a preordered set $(X, \leqslant)$, the preorder $\leqslant$ is said to be *confluent* provided that for every $x, y, z \in X$, if $x \leqslant y$ and $x \leqslant z$, then there exists $s \in X$ such that $y \leqslant s$ and $z \leqslant s$. *Co-confluence* is defined dually: for every $x, y, z \in X$, if $y \leqslant x$ and $z \leqslant x$, then there is $s \in X$ such that $s \leqslant y$ and $s \leqslant z$. ▪

Remark 45. Given a preordered set $(X, \leqslant)$, if the preorder $\leqslant$ is symmetric (i.e., for every $x, y \in X$, if $x \leqslant y$, then $y \leqslant x$), then $\leqslant$ is both confluent and co-confluent. ▪

Example 46.

(1) A V-space (X, a) is normal iff for every $x, y, z \in X$, it follows that $a(x, y) \otimes a(x, z) \leqslant \bigvee_{s \in X} a(y, s) \otimes a(z, s)$. Moreover, (X, a) is extremally disconnected iff for every $x, y, z \in X$, it follows that $a(y, x) \otimes a(z, x) \leqslant \bigvee_{s \in X} a(s, y) \otimes a(s, z)$. In particular, a preordered set $(X, \leqslant)$ considered as a 2-category is normal iff the preorder $\leqslant$ is confluent. Moreover, $(X, \leqslant)$ is extremally disconnected iff the preorder $\leqslant$ is co-confluent. Thus, a normal $(\mathbb{T}, V)$-space is not necessarily regular. However, a regular V-space (X, a), i.e., a symmetric V-space ($a = a^\circ$ by Proposition 25), is both normal and extremally disconnected.
(2) A topological space considered as a $(\beta, 2)$-category (X, a) is normal or extremally disconnected iff it is normal or extremally disconnected in the sense of general topology (Propositions 35, 40). Moreover, by Proposition 39, (X, a) is normal iff the preorder $\preceq$ (equal to $\hat{a}$) on βX is confluent.
(3) In the category **QPMet** $\cong \mathsf{P}_+$-**Cat**, a quasi-pseudo-metric space (X, a) is normal iff for every $x, y, z \in X$, it follows that $a(x, y) + a(x, z) \geqslant \inf_{s \in X} a(y, s) + a(z, s)$.
(4) In the category **App** $\cong (\beta, \mathsf{P}_+)$-**Cat**, an approach space considered as a (β, P_+)-space (X, a) is normal iff for every $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \beta X$, it follows that $\hat{a}(\mathfrak{x}, \mathfrak{y}) + \hat{a}(\mathfrak{x}, \mathfrak{z}) \geqslant \inf_{\mathfrak{s} \in \beta X} \hat{a}(\mathfrak{y}, \mathfrak{s}) + \hat{a}(\mathfrak{z}, \mathfrak{s})$, where $\hat{a}(\mathfrak{x}, \mathfrak{y}) = \inf\{u \in [0, \infty] \mid A^{(u)} \in \mathfrak{y} \text{ for every } A \in \mathfrak{x}\}$ (recall Lecture 1 for the notation $A^{(u)}$). ▪

Proposition 47. *If $\hat{\mathbb{T}}$ is associative and flat, then every $\mathbb{T}$-algebra is a normal $(\mathbb{T}, V)$-space.*

Proof. Given a $\mathbb{T}$-algebra (X, a), by Proposition 39, it is enough to show that $(TX, \hat{a})$ is a normal V-space. Since a is a map $TX \xrightarrow{a} X$, for every $\mathfrak{x}, \mathfrak{y} \in TX$, it follows that

$$\begin{aligned}\hat{a}(\mathfrak{x}, \mathfrak{y}) = (\hat{T}a \cdot m_X^\circ)(\mathfrak{x}, \mathfrak{y}) \overset{\mathbb{T}\text{ is flat}}{=} (Ta \cdot m_X^\circ)(\mathfrak{x}, \mathfrak{y}) = \bigvee_{\mathfrak{Z} \in TTX} m_X^\circ(\mathfrak{x}, \mathfrak{Z}) \otimes Ta(\mathfrak{Z}, \mathfrak{y}) = \\ \bigvee_{\mathfrak{Z} \in TTX} m_X(\mathfrak{Z}, \mathfrak{x}) \otimes Ta(\mathfrak{Z}, \mathfrak{y}) = \begin{cases} k, & \exists \mathfrak{Z} \in TTX : m_X(\mathfrak{Z}) = \mathfrak{x}, Ta(\mathfrak{Z}) = \mathfrak{y} \\ \bot_V, & \text{otherwise,} \end{cases}\end{aligned} \tag{4.1}$$

i.e., $\hat{a}$ is completely determined by its induced preorder $\leqslant$ on TX of Definition 2. Thus, to show that $(TX, \hat{a})$ is a normal V-space, by Example 46 (1), one has to verify that the induced preorder $\leqslant$ on TX is confluent.

Given $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in TX$ such that $\mathfrak{x} \leqslant \mathfrak{y}$ and $\mathfrak{x} \leqslant \mathfrak{z}$, in view of formula (4.1), there exist $\mathfrak{Y}, \mathfrak{Z} \in TTX$ such that $m_X(\mathfrak{Y}) = \mathfrak{x} = m_X(\mathfrak{Z})$ and $Ta(\mathfrak{Y}) = \mathfrak{y}$, $Ta(\mathfrak{Z}) = \mathfrak{z}$. Since (X, a) is a $\mathbb{T}$-algebra, one obtains $a \cdot Ta = a \cdot m_X$, and, therefore, for $y = a(\mathfrak{y})$ and $z = a(\mathfrak{z})$, it follows that $y = a(\mathfrak{y}) = a(Ta(\mathfrak{Y})) = a \cdot Ta(\mathfrak{Y}) = a \cdot m_X(\mathfrak{Y}) = a(\mathfrak{x}) = a \cdot m_X(\mathfrak{Z}) = a \cdot Ta(\mathfrak{Z}) = a(Ta(\mathfrak{Z})) = a(\mathfrak{z}) = z$. We now show that $\mathfrak{y} \leqslant e_X(y)$ and $\mathfrak{z} \leqslant e_X(z)$, which will finish the proof, since $y = z$ implies $e_X(y) = e_X(z)$.

For $\mathfrak{y} \leqslant e_X(y)$, notice that for $\mathfrak{W} = e_{TX}(\mathfrak{y})$, $m_X(\mathfrak{W}) = m_X(e_{TX}(\mathfrak{y})) = m_X \cdot e_{TX}(\mathfrak{y}) = (m \cdot eT = 1_T$, since $\mathbb{T}$ is a monad$) = \mathfrak{y}$ and $Ta(\mathfrak{W}) = Ta(e_{TX}(\mathfrak{y})) = Ta \cdot e_{TX}(\mathfrak{y}) =$ (since $1_{\mathbf{Set}} \xrightarrow{e} T$ is a natural transformation, the following diagram

$$\begin{array}{ccc} TX & \xrightarrow{e_{TX}} & TTX \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle Ta} \\ X & \xrightarrow[e_X]{} & TX \end{array}$$

commutes, i.e., $Ta \cdot e_{TX} = e_X \cdot a) = e_X \cdot a(\mathfrak{y}) = e_X(a(\mathfrak{y})) = e_X(y)$. The case $\mathfrak{z} \leqslant e_X(z)$ is similar. □

Corollary 48. *If the quantale V is strictly two-sided and lean, and $\hat{\mathbb{T}}$ is associative and flat, then every compact Hausdorff $(\mathbb{T}, V)$-space is normal.*

Proof. The claim follows from Proposition 47 and the fact that if V is a strictly two-sided and lean quantale, and $\hat{T}$ is flat, then $(\mathbb{T}, V)$-$\mathbf{Cat}_{\text{CompHaus}} = \mathbf{Set}^{\mathbb{T}}$ (see Lecture 5). □

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