Elements of monoidal topology
Lecture 7: Kleisli monoids

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Abstract

This lecture considers an alternative representation of the category $(\mathbb{T}, 2)$-**Cat** as the category $\mathbb{T}$-**Mon** of monoids in the hom-set of a Kleisli category that avoids explicit use of relations or lax extensions. As a motivating example serves an isomorphism $\mathbb{F}$-**Mon** $\cong$ **Top**, where $\mathbb{F}$ is the filter monad on the category **Set**.

1. A representation of topological spaces through neighborhood filters

1.1. The filter monad on the category **Set**

Definition 1. The *filter monad* $\mathbb{F} = (F, m, e)$ on the category **Set** of sets and maps is given by

(1) a functor $\mathbf{Set} \xrightarrow{F} \mathbf{Set}$, where $FX = \{\mathfrak{x} \mid \mathfrak{x} \text{ is a filter on } X\}$ for every set X, and $FX \xrightarrow{Ff} FY$ is defined by $Ff(\mathfrak{x}) = \{B \subseteq Y \mid f^{-1}(B) \in \mathfrak{x}\}$ for every map $X \xrightarrow{f} Y$;
(2) a natural transformation $1_{\mathbf{Set}} \xrightarrow{e} F$, where $X \xrightarrow{e_X} FX$ is defined by $e_X(x) = \dot{x} = \{A \subseteq X \mid x \in A\}$ (*principal filter*);
(3) a natural transformation $FF \xrightarrow{m} F$, where $FFX \xrightarrow{m_X} FX$ is defined by $m_X(\mathfrak{X}) = \Sigma\,\mathfrak{X}$ (*filtered sum* or *Kowalsky sum*), where $A \in \Sigma\,\mathfrak{X}$ iff $\{\mathfrak{x} \in FX \mid A \in \mathfrak{x}\} \in \mathfrak{X}$. ∎

Remark 2. There exists the *contravariant powerset functor* $\mathbf{Set}^{op} \xrightarrow{P^\bullet} \mathbf{Set}$ defined by $P^\bullet(X \xrightarrow{f} Y) = PY \xrightarrow{P^\bullet} PX$, where PX and PY are the powersets of the sets X and Y, respectively, and $P^\bullet(B) = f^{-1}(B) = \{x \in X \mid f(x) \in B\}$ for every subset $B \subseteq Y$. The functor $P^\bullet$ is *self-adjoint*, namely, there exists an adjoint situation $(P^\bullet)^{op} \dashv P^\bullet : \mathbf{Set}^{op} \to \mathbf{Set}$. This adjoint situation provides the *double-powerset monad* $\mathbb{P}^2 = (P^\bullet(P^\bullet)^{op}, m, e)$ on the category **Set**. Both the filter monad $\mathbb{F}$ and the ultrafilter monad β (recall Lecture 1) are restrictions of the above double-powerset monad $\mathbb{P}^2$. ∎

Definition 3. Given a set X, the set FX of filters on X can be partially ordered by the *refinement partial order*, i.e., for every $\mathfrak{x}, \mathfrak{y} \in FX$, $\mathfrak{x} \leqslant \mathfrak{y}$ iff $\mathfrak{x} \supseteq \mathfrak{y}$ (namely, given a subset $A \subseteq X$, if $A \in \mathfrak{y}$, then $A \in \mathfrak{x}$). A filter $\mathfrak{x}$ is *finer* than $\mathfrak{y}$ (or $\mathfrak{y}$ is *coarser* than $\mathfrak{x}$) provided that $\mathfrak{x} \supseteq \mathfrak{y}$. ∎

1.2. The Kleisli category of a monad

Definition 4. Given a monad $\mathbb{T} = (T, m, e)$ on a category $\mathbf{X}$, the *Kleisli category* $\mathbf{X}_{\mathbb{T}}$ associated to $\mathbb{T}$ is defined as follows. The objects of $\mathbf{X}_{\mathbb{T}}$ are those of $\mathbf{X}$. Given two $\mathbf{X}_{\mathbb{T}}$-objects X, Y, the hom-set $\mathbf{X}_{\mathbb{T}}(X, Y)$ is the hom-set $\mathbf{X}(X, TY)$ (the elements of which will be denoted $X \xrightharpoonup{f} Y$). Given two $\mathbf{X}_{\mathbb{T}}$-morphisms $X \xrightharpoonup{f} Y$, $Y \xrightharpoonup{g} Z$, their *Kleisli composition* in $\mathbf{X}_{\mathbb{T}}$ is defined via the composition in $\mathbf{X}$ as $g \circ f = m_Z \cdot Tg \cdot f$, i.e., as the $\mathbf{X}$-morphism $X \xrightarrow{f} TY \xrightarrow{Tg} TTZ \xrightarrow{m_Z} TZ$. The identity on an $\mathbf{X}_{\mathbb{T}}$-object X is the X-morphism $X \xrightarrow{e_X} TX$. ∎

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Preprint submitted to the Czech University of Life Sciences Prague (CZU) *February 23, 2022*

Example 5. Given the powerset monad $\mathbb{P} = (P, m, e)$ (recall Lecture 1), a $\mathbf{Set}_{\mathbb{P}}$-morphism $X \xrightharpoonup{f} Y$ is a map $X \xrightarrow{f} PY$, which can be considered as a relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!+\,} Y$ defined by $x\,r\,y$ iff $y \in f(x)$. Given two $\mathbf{Set}_{\mathbb{P}}$-morphisms $X \xrightharpoonup{f} Y$ and $Y \xrightharpoonup{g} Z$, the Kleisli composition $g \circ f$ is the composition $s \cdot r$ of the relations corresponding to g and f, respectively. Indeed, if t is the relation corresponding to $g \circ f$, then for every $x \in X$ and every $z \in Z$, $x\,t\,z$ iff $z \in g \circ f(x)$ iff $z \in m_Z \cdot Pg \cdot f(x) = m_Z(Pg \cdot f(x)) = \bigcup Pg \cdot f(x) = \bigcup Pg(f(x)) = \bigcup_{y \in f(x)} g(y)$ iff there exists $y \in f(x)$ such that $z \in g(y)$ iff there exists $y \in Y$ such that $y \in f(x)$ and $z \in g(y)$ iff there exists $y \in Y$ such that $x\,r\,y$ and $y\,s\,z$ iff $x\,(s \cdot r)\,z$. It follows that $\mathbf{Set}_{\mathbb{P}} = \mathbf{Rel}$. ▪

Remark 6. Given a Kleisli category $\mathbf{X}_{\mathbb{T}}$, there exists a functor $\mathbf{X}_{\mathbb{T}} \xrightarrow{G_{\mathbb{T}}} \mathbf{X}$, $G_{\mathbb{T}}(X \xrightharpoonup{f} Y) = TX \xrightarrow{m_Y \cdot Tf} TY = TX \xrightarrow{Tf} TTY \xrightarrow{m_Y} TY$. The functor $G_{\mathbb{T}}$ has a left adjoint $\mathbf{X} \xrightarrow{F_{\mathbb{T}}} \mathbf{X}_{\mathbb{T}}$, $F_{\mathbb{T}}(X \xrightarrow{f} Y) = X \xrightharpoonup{e_Y \cdot f} Y = X \xrightarrow{f} Y \xrightarrow{e_Y} TY$. The unit $1_{\mathbf{X}} \xrightarrow{\eta_{\mathbb{T}}} G_{\mathbb{T}} F_{\mathbb{T}}$ of this adjunction is e, and the co-unit $F_{\mathbb{T}} G_{\mathbb{T}} \xrightarrow{\varepsilon_{\mathbb{T}}} 1_{\mathbf{X}_{\mathbb{T}}}$ is given by $\mathbf{X}$-morphisms $TX \xrightarrow{1_{TX}} TX$. The monad associated to this adjunction gives back the original monad $\mathbb{T}$. ▪

Remark 7. Given sets X and Y, the hom-set $\mathbf{Set}_{\mathbb{F}}(X, Y)$ is partially ordered by the pointwise refinement partial order of Definition 3, namely, given $f, g \in \mathbf{Set}_{\mathbb{F}}(X, Y)$, $f \leqslant g$ iff $f(x) \leqslant g(x)$ for every $x \in X$ (recall that both f and g are maps $X \overset{f}{\underset{g}{\rightrightarrows}} FY$). Therefore, the partially ordered set $\mathbf{Set}_{\mathbb{F}}(X, Y)$ can be considered as a category, the objects of which are the elements of $\mathbf{Set}_{\mathbb{F}}(X, Y)$, and, for every two objects f and g, there exists precisely one morphism $f \to g$ provided that $f \leqslant g$. ▪

Lemma 8. *A partially ordered set S, considered as a category as in Remark 7, is a strict monoidal category (recall Lecture 4) precisely when it has a monoid structure whose multiplication $S \times S \to S$ is monotone.*

Proposition 9. $(\mathbf{Set}_{\mathbb{F}}(X, X), \circ, e_X)$ *is a strict monoidal category.*

Proof. In view of Lemma 8, it will be enough to show that the Kleisli composition preserves the refinement partial order. Thus, given $g_1, g_2, f_1, f_2 \in \mathbf{Set}_{\mathbb{F}}(X, X)$ such that $g_1 \leqslant g_2$ and $f_1 \leqslant f_2$, one has to show that $g_1 \circ f_1 \leqslant g_2 \circ f_2$, which is equivalent to $m_X \cdot Fg_1 \cdot f_1(x) \leqslant m_X \cdot Fg_2 \cdot f_2(x)$ for every $x \in X$, which, in its turn, is equivalent to $m_X \cdot Fg_1 \cdot f_1(x) \supseteq m_X \cdot Fg_2 \cdot f_2(x)$ for every $x \in X$.

Take an arbitrary element $x \in X$. Given $A \in m_X \cdot Fg_2 \cdot f_2(x)$, by Definition 1 (3), it follows that $\{\mathfrak{x} \in FX \mid A \in \mathfrak{x}\} \in Fg_2 \cdot f_2(x) = Fg_2(f_2(x))$, which implies by Definition 1 (1) that $g_2^{-1}(\{\mathfrak{x} \in FX \mid A \in \mathfrak{x}\}) \in f_2(x)$. Since $f_1 \leqslant f_2$, it follows that $f_1(x) \supseteq f_2(x)$, and then $g_2^{-1}(\{\mathfrak{x} \in FX \mid A \in \mathfrak{x}\}) \in f_1(x)$. Further, if $y \in g_2^{-1}(\{\mathfrak{x} \in FX \mid A \in \mathfrak{x}\})$, then $g_2(y) \in \{\mathfrak{x} \in FX \mid A \in \mathfrak{x}\}$, namely, $A \in g_2(y) \subseteq g_1(y)$ since $g_1 \leqslant g_2$. Thus, $A \in g_1(y)$ implies $g_1(y) \in \{\mathfrak{x} \in FX \mid A \in \mathfrak{x}\}$, which gives $y \in g_1^{-1}(\{\mathfrak{x} \in FX \mid A \in \mathfrak{x}\})$. As a consequence, one obtains that $g_2^{-1}(\{\mathfrak{x} \in FX \mid A \in \mathfrak{x}\}) \subseteq g_1^{-1}(\{\mathfrak{x} \in FX \mid A \in \mathfrak{x}\})$. Since $g_2^{-1}(\{\mathfrak{x} \in FX \mid A \in \mathfrak{x}\}) \in f_1(x)$ and $f_1(x)$ is a filter, it follows that $g_1^{-1}(\{\mathfrak{x} \in FX \mid A \in \mathfrak{x}\}) \in f_1(x)$, which implies $\{\mathfrak{x} \in FX \mid A \in \mathfrak{x}\} \in Fg_1 \cdot f_1(x)$, which finally gives $A \in m_X \cdot Fg_1 \cdot f_1(x)$. Therefore, $m_X \cdot Fg_1 \cdot f_1(x) \supseteq m_X \cdot Fg_2 \cdot f_2(x)$ as desired. □

1.3. Kleisli triples

Definition 10. A *Kleisli triple* on a category $\mathbf{X}$ consists of the following data:

- a function $\mathcal{O}_{\mathbf{X}} \xrightarrow{T} \mathcal{O}_{\mathbf{X}}$, which sends X to TX;
- an *extension operation* $(-)^{\mathbb{T}}$, which sends an $\mathbf{X}$-morphism $X \xrightarrow{f} TY$ to an $\mathbf{X}$-morphism $TX \xrightarrow{f^{\mathbb{T}}} TY$;
- an $\mathbf{X}$-morphism $X \xrightarrow{e_X} TX$ for every $\mathbf{X}$-object X;

such that

$$(g^{\mathbb{T}} \cdot f)^{\mathbb{T}} = g^{\mathbb{T}} \cdot f^{\mathbb{T}}, \quad e_X^{\mathbb{T}} = 1_{TX}, \quad f^{\mathbb{T}} \cdot e_X = f$$

for every $\mathbf{X}$-object X and every $\mathbf{X}$-morphisms $X \xrightarrow{f} TY$, $Y \xrightarrow{g} TZ$. If one defines $g \circ f = g^{\mathbb{T}} \cdot f$, then the above conditions are equivalent to this "Kleisli composition" being associative, and e_X being its identity, namely, given $\mathbf{X}$-morphisms $X \xrightarrow{f} TY$, $Y \xrightarrow{g} TZ$, and $Z \xrightarrow{h} TW$, $h \circ (g \circ f) = h \circ (g^{\mathbb{T}} \cdot f) = h^{\mathbb{T}} \cdot g^{\mathbb{T}} \cdot f = (h^{\mathbb{T}} \cdot g)^{\mathbb{T}} \cdot f = (h^{\mathbb{T}} \cdot g) \circ f = (h \circ g) \circ f$, $f \circ e_X = f^{\mathbb{T}} \cdot e_X = f$, and $e_Y \circ f = e_Y^{\mathbb{T}} \cdot f = 1_{TY} \cdot f = f$. ▪

Definition 11. A *Kleisli triple morphism* $(S, (-)^{\mathbb{S}}, d) \xrightarrow{\alpha} (T, (-)^{\mathbb{T}}, e)$ is given by a family of $\mathbf{X}$-morphisms $SX \xrightarrow{\alpha_X} TX$ for every $\mathbf{X}$-object X such that

$$\alpha_Y \cdot f^{\mathbb{S}} = (\alpha_Y \cdot f)^{\mathbb{T}} \cdot \alpha_X, \quad \alpha_X \cdot d_X = e_X$$

for every $\mathbf{X}$-morphism $X \xrightarrow{f} SY$. Observe that a Kleisli triple morphism is a family of $\mathbf{X}$-morphisms preserving the Kleisli composition and its identity, i.e., given $\mathbf{X}$-morphisms $X \xrightarrow{f} SY$ and $Y \xrightarrow{g} SZ$, $\alpha_Z \cdot (g \circ_{\mathbb{S}} f) = \alpha_Z \cdot g^{\mathbb{S}} \cdot f = (\alpha_Z \cdot g)^{\mathbb{T}} \cdot \alpha_Y \cdot f = (\alpha_Z \cdot g) \circ_{\mathbb{T}} (\alpha_Y \cdot f)$. ▪

Remark 12. A Kleisli triple $(T, (-)^{\mathbb{T}}, e)$ on a category $\mathbf{X}$ provides a monad $\mathbb{T} = (T, m, e)$ on $\mathbf{X}$ by setting $Tf = (e_Y \cdot f)^{\mathbb{T}}$ for every $\mathbf{X}$-morphism $X \xrightarrow{f} Y$, and $m_X = (1_{TX})^{\mathbb{T}}$ for every $\mathbf{X}$-object X. A Kleisli triple morphism $(S, (-)^{\mathbb{S}}, d) \xrightarrow{\alpha} (T, (-)^{\mathbb{T}}, e)$ provides then a morphism of the corresponding monads $\mathbb{S} \xrightarrow{\alpha} \mathbb{T}$. ▪

Remark 13. Given a monad $\mathbb{T} = (T, m, e)$ on a category $\mathbf{X}$, one gets a Kleisli triple $(T, (-)^{\mathbb{T}}, e)$ by setting $f^{\mathbb{T}} = m_Y \cdot Tf$ for every $\mathbf{X}$-morphism $X \xrightarrow{f} TY$. A monad morphism $\mathbb{S} \xrightarrow{\alpha} \mathbb{T}$ provides then a morphism $(S, (-)^{\mathbb{S}}, d) \xrightarrow{\alpha} (T, (-)^{\mathbb{T}}, e)$ of the corresponding Kleisli triples. ▪

Remark 14. The above passages from a Kleisli triple to a monad and from a monad to a Kleisli triple are inverse to each other, namely, both definitions describe the same structure on a category $\mathbf{X}$ (and the respective two definitions of Kleisli composition then correspond). ▪

1.4. Monoids in monoidal categories

Definition 15. Let $\mathbf{C}$ be a monoidal category (see Lecture 4). A *monoid* M in $\mathbf{C}$ is a $\mathbf{C}$-object together with two $\mathbf{C}$-morphisms $M \otimes M \xrightarrow{m} M$ and $E \xrightarrow{e} M$ such that the following two diagrams commute:

$$\begin{tikzcd}
M \otimes (M \otimes M) \arrow[r, "\alpha_{M,M,M}"] \arrow[d, "1_M \otimes m"'] & (M \otimes M) \otimes M \arrow[r, "m \otimes 1_M"] & M \otimes M \arrow[d, "m"] \\
M \otimes M \arrow[rr, "m"'] & & M
\end{tikzcd}$$

$$\begin{tikzcd}
E \otimes M \arrow[r, "e \otimes 1_M"] \arrow[dr, "\lambda_M"'] & M \otimes M \arrow[d, "m"] & M \otimes E \arrow[l, "1_M \otimes e"'] \arrow[dl, "\rho_M"] \\
& M. &
\end{tikzcd}$$

A *homomorphism of monoids* $(M, m, e) \xrightarrow{f} (N, n, d)$ is a $\mathbf{C}$-morphism $M \xrightarrow{f} N$ such that the following two diagrams commute:

$$\begin{tikzcd}
M \otimes M \arrow[r, "f \otimes f"] \arrow[d, "m"'] & N \otimes N \arrow[d, "n"] \\
M \arrow[r, "f"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
& E \arrow[dl, "e"'] \arrow[dr, "d"] & \\
M \arrow[rr, "f"'] & & N.
\end{tikzcd}$$

$\mathbf{Mon_C}$ stands for the category of monoids in $\mathbf{C}$ and their homomorphisms. ▪

Example 16.

(1) The category **Set** is monoidal w.r.t. cartesian product of sets. The category $\mathbf{Mon}_{\mathbf{Set}}$ is exactly the category **Mon** of monoids and their homomorphisms in the sense of universal algebra.
(2) The category **Sup** of $\bigvee$-semilattices and $\bigvee$-preserving maps is monoidal w.r.t. the usual tensor product. The category **Quant** of quantales and their homomorphisms is exactly the category $\mathbf{Mon}_{\mathbf{Sup}}$.
(3) Given a partially ordered set $(S, \leqslant)$, considered as a monoidal category $(S, \leqslant, \otimes, k)$ as in Lemma 8, a monoid in S is an element $s \in S$ such that $s \otimes s \leqslant s$ and $k \leqslant s$. Observe that if s is a monoid in S, then $k \leqslant s$ implies $s = s \otimes k \leqslant s \otimes s$. Since $s \otimes s \leqslant s$, it follows that $s \otimes s = s$. ∎

1.5. Topological spaces via neighborhood filters

Theorem 17. *The category* **Top** *of topological spaces and continuous maps is (concretely) isomorphic to the category* $\mathbb{F}$-**Mon**, *the objects of which are pairs* (X, ν) *such that* $X \xrightarrow{\nu} FX$ *is a monoid in* $\mathbf{Set}_{\mathbb{F}}(X, X)$ *(i.e.,* $\nu \circ \nu \leqslant \nu$ *and* $e_X \leqslant \nu$*), and whose morphisms* $(X, \nu) \xrightarrow{f} (Y, \mu)$ *are maps* $X \xrightarrow{f} Y$ *such that* $f_{\natural} \circ \nu \leqslant \mu \circ f_{\natural}$, *where* $f_{\natural} = e_Y \cdot f$ *is the image of the map* f *under the left adjoint functor* $\mathbf{Set} \to \mathbf{Set}_{\mathbb{F}}$ *of Remark 6.*

Proof. Given a topological space (X, τ), where τ is a topology on a set X, define a map $X \xrightarrow{\nu} FX$ by $\nu(x) = \{A \subseteq X \mid \text{there exists } U \in \tau \text{ such that } x \in U \subseteq A\}$ (*neighborhood filter* of x). It is easy to see that $\nu(x)$ is contained in the principal filter $e_X(x) = \dot{x}$ for every $x \in X$. Therefore, $e_X \leqslant \nu$ in the pointwise refinement partial order. To show that $\nu \circ \nu \leqslant \nu$, we will need the following simple lemma.

Lemma 18. *For* $x \in X$ *and* $A \subseteq X$, $A \in \nu(x)$ *iff there exists* $B \in \nu(x)$ *such that* $A \in \nu(y)$ *for every* $y \in B$.

Proof.
$\Rightarrow$: If $A \in \nu(x)$, then there exists $U \in \tau$ such that $x \in U \subseteq A$. Put $B = U$ and notice that, first, $B \in \nu(x)$ and, second, $A \in \nu(y)$ for every $y \in B$ since $U \in \tau$.
$\Leftarrow$: Given $y \in B$, it follows that $A \in \nu(y)$, i.e., there exists $V_y \in \tau$ such that $y \in V_y \subseteq A$. Thus, $B \subseteq \bigcup_{y \in B} V_y \subseteq A$, which implies $A \in \nu(x)$, since $B \in \nu(x)$ and $\nu(x)$ is a filter. □

Given an element $x \in X$ and a subset $A \subseteq X$, define a set $A^{\mathbb{F}} = \{\mathfrak{x} \in FX \mid A \in \mathfrak{x}\}$ (the set of filters containing A). Then $A \in \nu \circ \nu(x)$ iff $A \in m_X \cdot F\nu \cdot \nu(x)$ iff $A^{\mathbb{F}} \in F\nu \cdot \nu(x) = F\nu(\nu(x))$ iff $\nu^{-1}(A^{\mathbb{F}}) \in \nu(x)$ iff there exists $B \in \nu(x)$ such that $B \subseteq \nu^{-1}(A^{\mathbb{F}})$ iff there exists $B \in \nu(x)$ such that $\nu(y) \in A^{\mathbb{F}}$ for every $y \in B$ iff there exists $B \in \nu(x)$ such that $A \in \nu(y)$ for every $y \in B$ iff (Lemma 18) $A \in \nu(x)$. As a consequence, one obtains $\nu \circ \nu(x) = \nu(x)$. By Example 16 (3) and the above two properties ($\nu \circ \nu \leqslant \nu$ and $e_X \leqslant \nu$), a topological space (X, τ) provides a monoid ν in the monoidal category $\mathbf{Set}_{\mathbb{F}}(X, X)$.

Consider a continuous map $(X, \tau) \xrightarrow{f} (Y, \sigma)$, and let ν and μ be the monoids corresponding to the spaces (X, τ) and (Y, μ), respectively. First, we show that $Ff \cdot \nu \leqslant \mu \cdot f$. Indeed, given $x \in X$ and $B \subseteq Y$, $B \in \mu \cdot f(x) = \mu(f(x))$ iff there exists $V \in \sigma$ such that $f(x) \in V \subseteq B$, which implies (since f is continuous) $f^{-1}(V) \in \tau$ and $x \in f^{-1}(V) \subseteq f^{-1}(B)$, which results in $f^{-1}(B) \in \nu(x)$, which is equivalent to $B \in Ff(\nu(x)) = Ff \cdot \nu(x)$. As a consequence, one gets $Ff \cdot \nu(x) \supseteq \mu \cdot f(x)$ or $Ff \cdot \nu(x) \leqslant \mu \cdot f(x)$. Second, since $\mathbb{F} = (F, m, e)$ is a monad on **Set**, the following two diagrams commute:

$$
\begin{tikzcd}
FY \arrow[r, "Fe_Y"] \arrow[dr, "1_{FY}"'] & FFY \arrow[d, "m_Y"] \\
& FY
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "e_Y"] \arrow[d, "\mu"'] & FY \arrow[d, "F\mu"] \\
FY \arrow[r, "e_{FY}"] \arrow[dr, "1_{FY}"'] & FFY \arrow[d, "m_Y"] \\
& FY.
\end{tikzcd}
\tag{1.1}
$$

Thus, $Ff \cdot \nu = 1_{FY} \cdot Ff \cdot \nu = m_Y \cdot Fe_Y \cdot Ff \cdot \nu = m_Y \cdot F(e_Y \cdot f) \cdot \nu = m_Y \cdot Ff_{\natural} \cdot \nu = f_{\natural} \circ \nu$ by the left-hand side of diagram (1.1), and $\mu \cdot f = 1_{FY} \cdot \mu \cdot f = m_Y \cdot F\mu \cdot e_Y \cdot f = m_Y \cdot F\mu \cdot f_{\natural} = \mu \circ f_{\natural}$ by the right-hand side of diagram (1.1). As a result, one obtains $f_{\natural} \circ \nu \leqslant \mu \circ f_{\natural}$.

The above constructions define a functor $\mathbf{Top} \xrightarrow{G} \mathbb{F}\text{-}\mathbf{Mon}$ by $G((X,\tau) \xrightarrow{f} (Y,\sigma)) = (X,\nu) \xrightarrow{f} (Y,\mu)$. To obtain a functor in the opposite direction, one proceeds as follows.

Given an $\mathbb{F}$-**Mon**-object (X,ν), define $\tau = \{U \subseteq X \mid \text{for every } x \in X, \text{ if } x \in U, \text{ then } U \in \nu(x)\}$. To show that τ is a topology on the set X, one notices the following.

- Since the set X is an element of every filter on X, $X \in \tau$. Since the empty set $\varnothing$ clearly satisfies the condition on the elements of τ, $\varnothing \in \tau$.
- Given $U, V \in \tau$, if $x \in U \bigcap V$, then $U \in \nu(x)$ and $V \in \nu(x)$, which implies $U \bigcap V \in \nu(x)$, since $\nu(x)$ is a filter. As a consequence, one obtains that $U \bigcap V \in \nu(x)$.
- Given $U_i \in \tau$ for every $i \in I$, if $x \in \bigcup_{i\in I} U_i$, then $x \in U_{i_0}$ for some $i_0 \in I$, which implies $U_{i_0} \in \nu(x)$. Since $U_{i_0} \subseteq \bigcup_{i\in I} U_i$ and $\nu(x)$ is a filter, $\bigcup_{i\in I} U_i \in \nu(x)$. As a result, one obtains that $\bigcup_{i\in I} U_i \in \nu(x)$.

Given an $\mathbb{F}$-**Mon**-morphism $(X,\nu) \xrightarrow{f} (Y,\mu)$, to show that the map $X \xrightarrow{f} Y$ provides a continuous map $(X,\tau) \xrightarrow{f} (Y,\sigma)$ (where τ and σ are obtained from ν and μ, respectively), notice that given $V \in \sigma$, for every $x \in f^{-1}(V)$, $f^{-1}(V) \in \nu(x)$ iff $V \in Ff(\nu(x)) = Ff \cdot \nu(x)$. Since $V \in \sigma$ and $f(x) \in V$, it follows that $V \in \mu(f(x)) = \mu \cdot f(x)$. Since f is an $\mathbb{F}$-**Mon**-morphism, $Ff \cdot \nu(x) \supseteq \mu \cdot f(x)$, and, therefore, $V \in Ff \cdot \nu(x)$. As a consequence, one obtains that $f^{-1}(V) \in \tau$, i.e., the map $X \xrightarrow{f} Y$ is continuous.

The above constructions define a functor $\mathbb{F}\text{-}\mathbf{Mon} \xrightarrow{H} \mathbf{Top}$ by $H((X,\nu) \xrightarrow{f} (Y,\mu)) = (X,\tau) \xrightarrow{f} (Y,\sigma)$. Straightforward calculations show that the functors G and H are inverse to each other and, moreover, commute with the respective forgetful functors of the constructs $(\mathbf{Top}, |-|)$ and $(\mathbb{F}\text{-}\mathbf{Mon}, |-|)$. $\square$

2. Power-enriched monads

Remark 19. Given the powerset monad $\mathbb{P}$ on the category **Set**, the Eilenberg-Moore category $\mathbf{Set}^{\mathbb{P}}$ of $\mathbb{P}$ (see Lecture 1) is isomorphic to the category **Sup**. Indeed, given a $\mathbb{P}$-algebra (X,a), one defines an operation $PX \xrightarrow{\bigvee} X$ by $\bigvee S = a(S)$ providing thus a $\bigvee$-semilattice $(X,\bigvee)$. A $\mathbb{P}$-homomorphism $(X,a) \xrightarrow{f} (Y,b)$ results then in a $\bigvee$-preserving map $(X,\bigvee) \xrightarrow{f} (Y,\bigvee)$. Conversely, given a $\bigvee$-semilattice $(X,\bigvee)$, the map $PX \xrightarrow{a} X$ defined by $a(S) = \bigvee S$ provides a $\mathbb{P}$-algebra (X,a). A $\bigvee$-preserving map $(X,\bigvee) \xrightarrow{f} (Y,\bigvee)$ results then in a $\mathbb{P}$-homomorphism $(X,a) \xrightarrow{f} (Y,b)$. Altogether, one obtains a concrete isomorphism $\mathbf{Set}^{\mathbb{P}} \cong \mathbf{Sup}$. ▪

Remark 20. Given the Eilenberg-Moore category $\mathbf{X}^{\mathbb{T}}$ of a monad $\mathbb{T}$ on a category $\mathbf{X}$, there exists a functor $\mathbf{X}^{\mathbb{T}} \xrightarrow{G^{\mathbb{T}}} \mathbf{X}$, $G^{\mathbb{T}}((X,a) \xrightarrow{f} (Y,b)) = X \xrightarrow{f} Y$. The functor $G^{\mathbb{T}}$ has a left adjoint $\mathbf{X} \xrightarrow{F^{\mathbb{T}}} \mathbf{X}^{\mathbb{T}}$, $F^{\mathbb{T}}(X \xrightarrow{f} Y) = (TX, m_X) \xrightarrow{Tf} (TY, m_Y)$, where (TX, m_X) is the so-called *free* $\mathbb{T}$*-algebra* on a given set X. The unit $1_{\mathbf{X}} \xrightarrow{\eta^{\mathbb{T}}} G^{\mathbb{T}}F^{\mathbb{T}}$ of this adjunction is e, and the co-unit $F^{\mathbb{T}}G^{\mathbb{T}} \xrightarrow{\varepsilon^{\mathbb{T}}} 1_{\mathbf{X}^{\mathbb{T}}}$ is given by $\mathbb{T}$-homomorphisms $(TX, m_X) \xrightarrow{\varepsilon^{\mathbb{T}}_{(X,a)} = a} (X,a)$. The monad associated to this adjunction gives back the original monad $\mathbb{T}$. ▪

Remark 21. Given a monad $\mathbb{T} = (T, m, e)$ on a category $\mathbf{X}$, there exists a full and faithful *comparison functor* $\mathbf{X}_{\mathbb{T}} \xrightarrow{K} \mathbf{X}^{\mathbb{T}}$ defined by $K(X \xrightharpoonup{f} Y) = (TX, m_X) \xrightarrow{m_Y \cdot Tf} (TY, m_Y)$. ▪

Proposition 22. *Given a monad* $\mathbb{T} = (T, m, e)$ *on* **Set***, there exists a one-to-one correspondence between*

(1) *monad morphisms* $\mathbb{P} \xrightarrow{\tau} \mathbb{T}$ *(recall Lecture 2), where* $\mathbb{P} = (P, n, d)$ *is the powerset monad on* **Set***;*

(2) *extensions* E *of the functor* $\mathbf{Set} \xrightarrow{F_{\mathbb{T}}} \mathbf{Set}_{\mathbb{T}}$ *along the functor* $\mathbf{Set} \xrightarrow{(-)_\circ} \mathbf{Rel}$ *(recall Lecture 1):*

$$\begin{tikzcd} \mathbf{Rel} \arrow[r, dotted, "E"] & \mathbf{Set}_{\mathbb{T}} \\ \mathbf{Set} \arrow[u, "(-)_\circ"] \arrow[ur, "F_{\mathbb{T}}"'] & \end{tikzcd}$$

(3) *liftings L of the functor $\mathbf{Set}_{\mathbb{T}} \xrightarrow{G_{\mathbb{T}}} \mathbf{Set}$ along the forgetful functor $\mathbf{Sup} \xrightarrow{|-|} \mathbf{Set}$:*

$$\begin{tikzcd} \mathbf{Set}_{\mathbb{T}} \arrow[r, dotted, "L"] \arrow[dr, "G_{\mathbb{T}}"'] & \mathbf{Sup} \arrow[d, "|-|"] \\ & \mathbf{Set} \end{tikzcd}$$

(4) *$\bigvee$-semilattice structures on the set TX such that the maps $TX \xrightarrow{Tf} TY$ and $TTX \xrightarrow{m_X} TX$ are $\bigvee$-preserving for every map $X \xrightarrow{f} Y$ and every set X.*

Proof. In view of Example 5 and Remarks 19, 20, one can identify the category **Rel** with $\mathbf{Set}_{\mathbb{P}}$, the category **Sup** with $\mathbf{Set}^{\mathbb{P}}$, and the forgetful functor $\mathbf{Sup} \xrightarrow{|-|} \mathbf{Set}$ with $\mathbf{Set}^{\mathbb{P}} \xrightarrow{G^{\mathbb{P}}} \mathbf{Set}$.

(1) $\Leftrightarrow$ (2): Given a monad morphism $\mathbb{P} \xrightarrow{\tau} \mathbb{T}$, one defines a functor $\mathbf{Set}_{\mathbb{P}} \xrightarrow{E} \mathbf{Set}_{\mathbb{T}}$ by $E(X \xrightharpoonup{f} Y) = X \xrightharpoonup{\tau_Y \cdot f} Y$. Given now a map $X \xrightarrow{g} Y$, $E(-)_\circ(X \xrightarrow{g} Y) = X \xrightharpoonup{\tau_Y \cdot s} Y$, where $X \xrightarrow{s} PY$ is defined by $s(x) = \{f(x)\}$, and $F_{\mathbb{T}}(X \xrightarrow{g} Y) = X \xrightharpoonup{e_Y \cdot f} Y$. For every $x \in X$, $\tau_Y \cdot s(x) = \tau_Y(\{f(x)\}) = \tau_Y \cdot d_Y(f(x)) \overset{\tau_Y \cdot d_Y = e_Y}{=} e_Y(f(x)) = e_Y \cdot f(x)$. Thus, $\tau_Y \cdot s = e_Y \cdot f$, i.e., the required triangle commutes.

Conversely, given an extension $\mathbf{Set}_{\mathbb{P}} \xrightarrow{E} \mathbf{Set}_{\mathbb{T}}$, define a monad morphism $\mathbb{P} \xrightarrow{\tau} \mathbb{T}$ by $PX \xrightarrow{\tau_X} TX = PX \xrightarrow{E1_{PX}} TX$. Diagram chasing shows that τ satisfies all the required properties.

(1) $\Leftrightarrow$ (3): Given a monad morphism $\mathbb{P} \xrightarrow{\tau} \mathbb{T}$, one defines a functor $\mathbf{Set}_{\mathbb{T}} \xrightarrow{L} \mathbf{Set}^{\mathbb{P}}$ by $L(X \xrightharpoonup{f} Y) = (TX, m_X \cdot \tau_{TX}) \xrightarrow{m_Y \cdot Tf} (TY, m_Y \cdot \tau_{TY})$ (cf. Remark 21). Notice that $G^{\mathbb{P}} L(X \xrightharpoonup{f} Y) = TX \xrightarrow{m_Y \cdot Tf} TY = G_{\mathbb{T}}(X \xrightharpoonup{f} Y)$, namely, the required triangle commutes.

Conversely, given a lifting $\mathbf{Set}_{\mathbb{T}} \xrightarrow{L} \mathbf{Set}^{\mathbb{P}}$, define a monad morphism $\mathbb{P} \xrightarrow{\tau} \mathbb{T}$ by $PX \xrightarrow{\tau_X} TX = PX \xrightarrow{Pe_X} PTX \xrightarrow{a} TX$, where a is the structure map of the Eilenberg-Moore algebra $LX = (TX, a)$ (recall that $G^{\mathbb{P}} LX = G_{\mathbb{T}} X = TX$). Diagram chasing shows that τ satisfies all the required properties.

(3) $\Leftrightarrow$ (4): Given a map $X \xrightarrow{f} Y$, one obtains a $\mathbf{Set}_{\mathbb{T}}$-morphism $X \xrightharpoonup{e_Y \cdot f} Y$. Since $G^{\mathbb{P}} L(X \xrightharpoonup{e_Y \cdot f} Y) = G_{\mathbb{T}}(X \xrightharpoonup{e_Y \cdot f} Y) = TX \xrightarrow{m_Y \cdot T(e_Y \cdot f)} TY$ and $m_Y \cdot T(e_Y \cdot f) = m_Y \cdot Te_Y \cdot Tf = (m_Y \cdot Te_Y) \cdot Tf = 1_{TY} \cdot Tf = Tf$, it follows that the functor L sends a $\mathbf{Set}_{\mathbb{T}}$-morphism $X \xrightharpoonup{e_Y \cdot f} Y$ to a $\bigvee$-preserving map $TX \xrightarrow{Tf} TY$. Moreover, since $TX \xrightharpoonup{1_{TX}} X$ is a $\mathbf{Set}_{\mathbb{T}}$-morphism, $G^{\mathbb{P}} L(TX \xrightharpoonup{1_{TX}} X) = G_{\mathbb{T}}(TX \xrightharpoonup{1_{TX}} X) = TTX \xrightarrow{m_X \cdot T1_{TX}} TX$ and $m_X \cdot T1_{TX} = m_X \cdot 1_{TTX} = m_X$ together imply that the functor L sends a $\mathbf{Set}_{\mathbb{T}}$-morphism $TX \xrightharpoonup{1_{TX}} X$ to a $\bigvee$-preserving map $TTX \xrightarrow{m_X} TX$. As a consequence, it follows that the conditions of item (4) are just pointwise restatements of the condition of item (3). $\square$

Remark 23.

(1) Given a morphism $\mathbb{P} \xrightarrow{\tau} \mathbb{T}$ of monads on **Set**, Proposition 22 (3) equips the underlying set TX of a free $\mathbb{T}$-algebra with a partial order given by

$$\mathfrak{x} \leqslant \mathfrak{y} \text{ iff } m_X \cdot \tau_{TX}(\{\mathfrak{x}, \mathfrak{y}\}) = \mathfrak{y} \tag{2.1}$$

for every $\mathfrak{x}, \mathfrak{y} \in TX$ (cf. Remark 19).

(2) For every set X, the map $PX \xrightarrow{\tau_X} TX$ is monotone, since given $A, B \in PX$ with $A \subseteq B$, the diagram

$$\begin{tikzcd} PPX \arrow[r, "P\tau_X"] \arrow[d, "n_X"'] & PTX \arrow[r, "\tau_{TX}"] & TTX \arrow[d, "m_X"] \\ PX \arrow[rr, "\tau_X"'] & & TX \end{tikzcd}$$

commutes (τ is a morphism of monads). As a consequence, $m_X \cdot \tau_{TX}(\{\tau_X(A), \tau_X(B)\}) = m_X \cdot \tau_{TX} \cdot P\tau_X(\{A, B\}) = \tau_X \cdot n_X(\{A, B\}) = \tau_X(A \bigcup B) = \tau_X(B)$, namely, $\tau_X(A) \leqslant \tau_X(B)$.

(3) The hom-sets $\mathbf{Set}_{\mathbb{T}}(X,Y)$ become partially ordered by the respective pointwise order, i.e., for every $\mathbf{X}$-morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} TY$, $f \leqslant g$ iff $f(x) \leqslant g(x)$ for every $x \in X$.

(4) Given $f, g \in \mathbf{Set}_{\mathbb{T}}(X,Y)$ and $h \in \mathbf{Set}_{\mathbb{T}}(Y,Z)$, if $f \leqslant g$, then $h \circ f = m_Z \cdot Th \cdot f \leqslant m_Z \cdot Th \cdot g = h \circ g$, since Th, m_Z are monotone by Proposition 22 (4), i.e., composition on the right is monotone. Composition on the left $\mathbf{Set}_{\mathbb{T}}(Y,Z) \xrightarrow{(-)^{\mathbb{T}} \cdot f} \mathbf{Set}_{\mathbb{T}}(X,Z)$ for an $\mathbf{X}$-morphism $X \xrightarrow{f} TY$ may though fail to be monotone.

(5) To make $\mathbf{Set}_{\mathbb{T}}$ a partially ordered category (recall Lecture 4), it is enough $(-)^{\mathbb{T}}$ to be order-preserving, i.e., $f \leqslant g$ implies $f^{\mathbb{T}} \leqslant g^{\mathbb{T}}$ for every $\mathbf{X}$-morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} TY$. If this condition is satisfied, then the functors $\mathbf{Rel} \xrightarrow{E} \mathbf{Set}_{\mathbb{T}}$ and $\mathbf{Set}_{\mathbb{T}} \xrightarrow{L} \mathbf{Sup}$ of Proposition 22 become functors between partially ordered categories, i.e., preserve the partial order on hom-sets (notice that $Lf = f^{\mathbb{T}}$ for every $\mathbf{Set}_{\mathbb{T}}$-morphism $X \overset{f}{\rightharpoondown} Y$; and $Ef = \tau_Y \cdot f$ for every $\mathbf{Set}_{\mathbb{P}}$-morphism $X \overset{f}{\rightharpoondown} Y$, where the map τ_Y is monotone). ■

Definition 24. A *power-enriched monad* is a pair $(\mathbb{T}, \tau)$, where $\mathbb{T}$ is a monad on $\mathbf{Set}$ and $\mathbb{P} \xrightarrow{\tau} \mathbb{T}$ is a monad morphism such that $f \leqslant g$ implies $f^{\mathbb{T}} \leqslant g^{\mathbb{T}}$ for every $\mathbf{Set}$-morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} TY$. A *morphism* $(\mathbb{S}, \sigma) \xrightarrow{\alpha} (\mathbb{T}, \tau)$ of power-enriched monads is a monad morphism $\mathbb{S} \xrightarrow{\alpha} \mathbb{T}$ such that the next triangle commutes

$$\begin{tikzcd} & \mathbb{P} \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\ \mathbb{S} \arrow[rr, "\alpha"'] & & \mathbb{T}. \end{tikzcd}$$

■

Example 25.

(1) There exist exactly two trivial monads on $\mathbf{Set}$ (admitting only trivial $\mathbb{T}$-algebras), i.e., the monad sending every set to a singleton $1 = \{*\}$, and the monad sending the empty set to itself and all the other sets to 1 (recall Lecture 5). The first one, denoted $\mathbb{1}$, is clearly power-enriched, where the unique monad morphism $\mathbb{P} \xrightarrow{\tau} \mathbb{1}$ is given by the unique maps $PX \xrightarrow{!_X} 1$ for every set X. The second one, say $\mathbb{T}$, is clearly not power-enriched, since there exists no map $P\varnothing = 1 \to \varnothing = T\varnothing$.

(2) The powerset monad $\mathbb{P}$ with the identity monad morphism $\mathbb{P} \xrightarrow{1_{\mathbb{P}}} \mathbb{P}$ is power-enriched. The partial order on the sets PX induced by condition (2.1) is the usual inclusion of subsets, since $\bigvee$ is the union of sets.

(3) The filter monad $\mathbb{F}$ is power-enriched, since the principal filter natural transformation τ defined on a set X by $PX \xrightarrow{\tau_X} FX$, $\tau_X(A) = \dot{A} = \{B \subseteq X \mid A \subseteq B\}$ (*principal filter*) provides a monad morphism $\mathbb{P} \xrightarrow{\tau} \mathbb{F}$. The partial order on FX induced by condition (2.1) is the refinement partial order of Definition 3, and the operation $\bigvee$ on FX is given by the intersection of filters. For the latter statement, observe that given a subset $\{\mathfrak{x}_s \mid s \in S\} \subseteq FX$, $\bigvee\{\mathfrak{x}_s \mid s \in S\} = m_X \cdot \tau_{FX}(\{\mathfrak{x}_s \mid s \in S\})$. Therefore, given $A \subseteq X$, $A \in \bigvee\{\mathfrak{x}_s \mid s \in S\}$ iff $A \in m_X \cdot \tau_{FX}(\{\mathfrak{x}_s \mid s \in S\})$ iff $\{\mathfrak{z} \in FX \mid A \in \mathfrak{z}\} \in \tau_{FX}(\{\mathfrak{x}_s \mid s \in S\})$ iff $\{\mathfrak{z} \in FX \mid A \in \mathfrak{z}\} \in \{B \subseteq FX \mid \{\mathfrak{x}_s \mid s \in S\} \subseteq B\}$ iff $\{\mathfrak{x}_s \mid s \in S\} \subseteq \{\mathfrak{z} \in FX \mid A \in \mathfrak{z}\}$ iff $A \in \mathfrak{x}_s$ for every $s \in S$ iff $A \in \bigcap_{s \in S} \mathfrak{x}_s$. The former statement follows then from the latter, since given $\mathfrak{x}, \mathfrak{y} \in FX$, $\mathfrak{x} \leqslant \mathfrak{y}$ iff $m_X \cdot \tau_{FX}(\{\mathfrak{x}, \mathfrak{y}\}) = \mathfrak{y}$ iff $\mathfrak{x} \bigcap \mathfrak{y} = \mathfrak{y}$ iff $\mathfrak{x} \supseteq \mathfrak{y}$.

(4) The ultrafilter monad β is not power-enriched, since $\beta\varnothing = \varnothing$ (recall from Lecture 1 that an ultrafilter cannot contain the empty set), which is not a $\bigvee$-semilattice (observe that every $\bigvee$-semilattice contains a distinguished element $\bigvee \varnothing$, i.e., the underlying set of every $\bigvee$-semilattice is non-empty). ■

3. Kleisli monoids

Definition 26. Given a monad $\mathbb{T} = (T, m, e)$ on a category $\mathbf{X}$ such that the respective Kleisli category $\mathbf{X}_{\mathbb{T}}$ is a partially ordered category, $\mathbb{T}$-$\mathbf{Mon}$ is the category of $\mathbb{T}$-*monoids* (or *Kleisli monoids*), whose objects are pairs (X,ν), where X is an **X**-object, and $X \overset{\nu}{\rightharpoonup} X$ is an $\mathbf{X}_{\mathbb{T}}$-morphism, which is *reflexive* ($e_X \leqslant \nu$) and *transitive* ($\nu \circ \nu \leqslant \nu$), where $\circ$ is the Kleisli composition in the category $\mathbf{X}_{\mathbb{T}}$; and whose morphisms $(X,\nu) \xrightarrow{f} (Y,\mu)$ are **X**-morphisms $X \xrightarrow{f} Y$ such that $Tf \cdot \nu \leqslant \mu \cdot f$, i.e.,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\nu"'] \arrow[dr, phantom, "\leqslant"] & Y \arrow[d, "\mu"] \\ TX \arrow[r, "Tf"'] & TY \end{tikzcd}$$

or equivalently $f_\natural \circ \nu \leqslant \mu \circ f_\natural$, where $f_\natural = e_Y \cdot f$, i.e.,

$$\begin{tikzcd} X \arrow[r, "f_\natural"] \arrow[d, "\nu"'] \arrow[dr, phantom, "\leqslant"] & Y \arrow[d, "\mu"] \\ X \arrow[r, "f_\natural"'] & Y. \end{tikzcd}$$

If $\mathbb{T} = (T,\tau)$ is a power-enriched monad, then the partial order on the hom-sets of $\mathbf{X}_{\mathbb{T}}$ depends on τ. ■

Remark 27. Given a $\mathbb{T}$-monoid (X,ν), $\nu = \nu \circ e_X \leqslant \nu \circ \nu \leqslant \nu$ implies $\nu \circ \nu = \nu$. ■

Remark 28. Given a $\mathbb{T}$-monoid (X,ν), the functor $\mathbf{X}_{\mathbb{T}} \xrightarrow{G_{\mathbb{T}}=(-)^{\mathbb{T}}} \mathbf{X}$ has the following property (preservation of idempotency): $\nu^{\mathbb{T}} = (\nu \circ \nu)^{\mathbb{T}} = (m_X \cdot T\nu \cdot \nu)^{\mathbb{T}} = m_X \cdot T(m_X \cdot T\nu \cdot \nu) = m_X \cdot Tm_X \cdot TT\nu \cdot T\nu \overset{(\dagger)}{=} m_X \cdot T\nu \cdot m_X \cdot T\nu = \nu^{\mathbb{T}} \cdot \nu^{\mathbb{T}}$, where $(\dagger)$ relies on commutativity of the following diagram

$$\begin{tikzcd} TTX \arrow[r, "TT\nu"] \arrow[d, "m_X"'] & TTTX \arrow[r, "Tm_X"] \arrow[d, "m_{TX}"] & TTX \arrow[d, "m_X"] \\ TX \arrow[r, "T\nu"'] & TTX \arrow[r, "m_X"'] & TX. \end{tikzcd}$$

■

Example 29.

(1) If $\mathbb{T}$ is the trivial monad $\mathbb{1}$ on the category **Set** of Example 25 (1), then the respective Kleisli monoids are pairs $(X, X \xrightarrow{!_X} \{*\})$, and the respective morphisms are maps $X \xrightarrow{f} Y$, i.e., $\mathbb{1}$-**Mon** $\cong$ **Set**.

(2) If $\mathbb{T}$ is the powerset monad $\mathbb{P}$ with the identity monad morphism $\mathbb{P} \xrightarrow{1_{\mathbb{P}}} \mathbb{P}$, then $\mathbb{P}$-**Mon** is the category **Prost** of preordered sets and monotone maps that can be seen as follows. First, given a set X, the partial order on PX is the inclusion of sets. Second, a map $X \xrightarrow{\nu} PX$ induces a relation $\leqslant$ on X by $x \leqslant y$ iff $x \in \nu(y)$. If ν is reflexive ($e_X \leqslant \nu$), then given $x \in X$, $e_X(x) = \{x\} \subseteq \nu(x)$ implies $x \in \nu(x)$ implies $x \leqslant x$, i.e., $\leqslant$ is a reflexive relation. If ν is transitive ($\nu \circ \nu \leqslant \nu$), then given $z \in X$, $\nu \circ \nu(z) \leqslant \nu(z)$ implies $m_X \cdot P\nu \cdot \nu(z) \subseteq \nu(z)$ implies $\bigcup P\nu(\nu(z)) \subseteq \nu(z)$ implies $\bigcup_{y \in \nu(z)} \nu(y) \subseteq \nu(z)$. Thus, given $x,y,z \in X$ such that $x \leqslant y$ and $y \leqslant z$, $x \in \nu(y)$ and $y \in \nu(z)$ implies $x \in \bigcup_{y \in \nu(z)} \nu(y) \subseteq \nu(z)$ implies $x \in \nu(z)$ implies $x \leqslant z$, i.e., $\leqslant$ is a transitive relation. Third, given a $\mathbb{P}$-monoid morphism $(X,\nu) \xrightarrow{f} (Y,\mu)$, $x_1 \leqslant x_2$ implies $x_1 \in \nu(x_2)$ implies $f(x_1) \in f(\nu(x_2)) = Pf \cdot \nu(x_2) \subseteq \mu \cdot f(x_2) = \mu(f(x_2))$ implies $f(x_1) \leqslant f(x_2)$, i.e., the map $X \xrightarrow{f} Y$ is monotone. Fourth, the above-mentioned arguments are reversible.

(3) The filter monad $\mathbb{F}$ with the principal filter natural transformation $\mathbb{P} \xrightarrow{\tau} \mathbb{F}$ provides the category $\mathbb{F}$-**Mon**, which is isomorphic to the category **Top** of topological spaces and continuous maps by Theorem 17. ■

Proposition 30. *A morphism of power-enriched monads* $(\mathbb{S} = (S,n,d),\sigma) \xrightarrow{\alpha} (\mathbb{T} = (T,m,e),\tau)$ *provides a concrete functor* $\mathbb{S}$-**Mon** $\xrightarrow{F_\alpha}$ $\mathbb{T}$-**Mon** *defined by* $F_\alpha((X,\nu) \xrightarrow{f} (Y,\mu)) = (X, \alpha_X \cdot \nu) \xrightarrow{f} (Y, \alpha_Y \cdot \mu)$.

Proof. First, observe that there exists a functor $\mathbf{Set}_{\mathbb{S}} \xrightarrow{\mathbf{Set}_\alpha} \mathbf{Set}_{\mathbb{T}}$ defined by $\mathbf{Set}_\alpha(X \xrightharpoonup{f} Y) = X \xrightharpoonup{\alpha_Y \cdot f} Y$. To show that $\mathbf{Set}_\alpha$ preserves the Kleisli composition, notice that given $\mathbf{Set}_{\mathbb{S}}$-morphisms $X \xrightharpoonup{f} Y$ and $Y \xrightharpoonup{g} Z$, $\mathbf{Set}_\alpha(g \circ f) = \alpha_Z \cdot (g \circ f) = \alpha_Z \cdot n_Z \cdot Sg \cdot f \overset{(\dagger)}{=} m_Z \cdot T\alpha_Z \cdot Tg \cdot \alpha_Y \cdot f = m_Z \cdot T(\alpha_Z \cdot g) \cdot \alpha_Y \cdot f = (\alpha_Z \cdot g) \circ (\alpha_Y \cdot f) = \mathbf{Set}_\alpha g \circ \mathbf{Set}_\alpha f$, where $(\dagger)$ relies on commutativity of the following diagram

$$\begin{tikzcd}
SY \arrow[r, "Sg"] \arrow[d, "\alpha_Y"'] & SSZ \arrow[rr, "n_Z"] \arrow[d, "\alpha_{SZ}"] & & SZ \arrow[d, "\alpha_Z"] \\
TY \arrow[r, "Tg"'] & TSZ \arrow[r, "T\alpha_Z"'] & TTZ \arrow[r, "m_Z"'] & TZ.
\end{tikzcd}$$

Second, notice that given a set X, the map $SX \xrightarrow{\alpha_X} TX$ is $\bigvee$-preserving, which follows from the next commutative diagram

$$\begin{tikzcd}
PSX \arrow[rr, "P\alpha_X"] \arrow[d, "\sigma_{SX}"'] & & PTX \arrow[dl, "\sigma_{TX}"'] \arrow[d, "\tau_{TX}"] \\
SSX \arrow[r, "S\alpha_X"] \arrow[d, "n_X"'] & STX \arrow[r, "\alpha_{TX}"] & TTX \arrow[d, "m_X"] \\
SX \arrow[rr, "\alpha_X"'] & & TX
\end{tikzcd}$$

and the definition of $\bigvee$ on the sets SX and TX. In particular, it follows that the map α_X is monotone.

Third, observe that the functor F_α is correct on objects, since given an $\mathbb{S}$-monoid (X, ν), $d_X \leqslant \nu$ implies $e_X = \alpha_X \cdot d_X \leqslant \alpha_X \cdot \nu$ (since α is a monad morphism, whose components are monotone), and $\nu \circ \nu \leqslant \nu$ implies $\nu \circ \nu = \nu$ (by Remark 27) implies $(\alpha_X \cdot \nu) \circ (\alpha_X \cdot \nu) = \alpha_X \cdot (\nu \circ \nu) = \alpha_X \cdot \nu$ (since $\mathbf{Set}_\alpha$ is a functor).

Fourth, notice that the functor F_α is correct on morphisms, since given an $\mathbb{S}$-monoid morphism $(X, \nu) \xrightarrow{f} (Y, \mu)$, $Sf \cdot \nu \leqslant \mu \cdot f$ implies $\alpha_Y \cdot Sf \cdot \nu \leqslant \alpha_Y \cdot \mu \cdot f$ (since α_Y is monotone) implies $Tf \cdot \alpha_X \cdot \nu \leqslant \alpha_Y \cdot \mu \cdot f$ by commutativity of the following diagram

$$\begin{tikzcd}
SX \arrow[r, "\alpha_X"] \arrow[d, "Sf"'] & TX \arrow[d, "Tf"] \\
SY \arrow[r, "\alpha_Y"'] & TY.
\end{tikzcd}$$

Fifth, the functor F_α is concrete, since it does not change the underlying sets of Kleisli monoids. ☐

4. The Kleisli extension

Definition 31. Define a functor $\mathbf{Rel}^{op} \xrightarrow{(-)^\flat} \mathbf{Set}_{\mathbb{P}}$ by $(X \overset{r}{\longrightarrow\!\!\!\!\!+\,} Y)^\flat = Y \xrightharpoonup{r^\flat} X$, where the map $Y \xrightarrow{r^\flat} PX$ is given by $x \in r^\flat(y)$ iff $x\,r\,y$ (representing the opposite relation $Y \overset{r^\circ}{\longrightarrow\!\!\!\!\!+\,} X$; cf. Example 5). ▪

Definition 32. The functors of Definition 31 and Proposition 22 provide a functor $\mathbf{Rel}^{op} \xrightarrow{(-)^\tau} \mathbf{Set}^{\mathbb{P}} = \mathbf{Rel}^{op} \xrightarrow{(-)^\flat} \mathbf{Set}_{\mathbb{P}} \xrightarrow{E} \mathbf{Set}_{\mathbb{T}} \xrightarrow{L} \mathbf{Set}^{\mathbb{P}}$, $(X \overset{r}{\longrightarrow\!\!\!\!\!+\,} Y)^\tau = TY \xrightarrow{r^\tau} TX$, where $r^\tau = m_X \cdot T(\tau_X \cdot r^\flat) = (\tau_X \cdot r^\flat)^{\mathbb{T}}$. ▪

Definition 33. Given a power-enriched monad $(\mathbb{T}, \tau)$, the *Kleisli extension* $\check{T}$ of T to $\mathbf{Rel}$ (w.r.t. τ) is provided by the functions $\mathbf{Rel}(X, Y) \xrightarrow{\check{T} = \check{T}_{X,Y}} \mathbf{Rel}(TX, TY)$ (for every pair of sets X and Y) such that for every relation $X \overset{r}{\longrightarrow\!\!\!\!\!+\,} Y$, and every $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$, it follows that $\mathfrak{x}\,(\check{T}r)\,\mathfrak{y}$ iff $\mathfrak{x} \leqslant r^\tau(\mathfrak{y})$, which is equivalently described by a map $TY \xrightarrow{(\check{T}r)^\flat = \downarrow_{TX} \cdot r^\tau} PTX$, where $\downarrow_{TX}(\mathfrak{x}) = \{\mathfrak{z} \in TX \mid \mathfrak{z} \leqslant \mathfrak{x}\}$ (*lower set*). ▪

Example 34.

(1) Given the terminal power-enriched monad $(\mathbb{1}, !)$, the Kleisli extension of a relation $X \xrightarrow{r} Y$ is the relation $\{*\} \xrightarrow{\check{1}r} \{*\}$ such that $*\,(\check{1}r)\,*$.

(2) Given the powerset monad $(\mathbb{P} = (P, m, e), 1_{\mathbb{P}})$, the respective Kleisli extension can be described as follows. Given a relation $X \xrightarrow{r} Y$, for every $A \in PX$, $B \in PY$, it follows that $A \leqslant r^{1_{\mathbb{P}}}(B)$ iff $A \subseteq r^{1_{\mathbb{P}}}(B)$ iff $A \subseteq m_X \cdot P(1_X \cdot r^{\flat})(B)$ iff $A \subseteq \bigcup Pr^{\flat}(B) = \bigcup_{y \in B} r^{\flat}(y)$ iff for every $x \in A$, there exists $y \in B$ such that $x \in r^{\flat}(y)$ iff for every $x \in A$, there exists $y \in B$ such that $x\,r\,y$ iff $A \subseteq r^{\circ}(B)$, where $r^{\circ}(B) = \{x \in X \mid \text{there exists } y \in B \text{ such that } x\,r\,y\}$. As a consequence, $A\,\check{P}r\,B$ iff $A \subseteq r^{\circ}(B)$, i.e., one obtains the lax extension of the functor P from Lecture 1.

(3) Given the filter monad $(\mathbb{F} = (F, m, e), \tau)$, where $\mathbb{P} \xrightarrow{\tau} \mathbb{F}$ is the principal filter natural transformation, the respective Kleisli extension can be described as follows. Given a relation $X \xrightarrow{r} Y$, a subset $A \subseteq X$, and a filter $\mathfrak{y} \in FY$, it follows by Definition 1 that $A \in m_X \cdot F(\tau_X \cdot r^{\flat})(\mathfrak{y})$ iff $A^{\mathbb{F}} = \{\mathfrak{x} \in FX \mid A \in \mathfrak{x}\} \in F(\tau_X \cdot r^{\flat})(\mathfrak{y})$ iff $(\tau_X \cdot r^{\flat})^{-1}(A^{\mathbb{F}}) \in \mathfrak{y}$ iff $\{y \in Y \mid \tau_X \cdot r^{\flat}(y) \in A^{\mathbb{F}}\} \in \mathfrak{y}$ iff $\{y \in Y \mid A \in \tau_X \cdot r^{\flat}(y)\} \in \mathfrak{y}$ iff $\{y \in Y \mid r^{\flat}(y) \subseteq A\} \in \mathfrak{y}$ (since $\tau_X(B) = \{C \subseteq X \mid B \subseteq C\}$) iff there exists $B \in \mathfrak{y}$ such that $r^{\circ}(B) \subseteq A$. As a consequence, $r^{\tau}(\mathfrak{y}) = m_X \cdot F(\tau_X \cdot r^{\flat})(\mathfrak{y}) = \uparrow_{PX} \{r^{\circ}(B) \mid B \in \mathfrak{y}\}$, where given a partially ordered set $(Z, \leqslant)$ and a subset $S \subseteq Z$, $\uparrow_Z (S) = \{z \in Z \mid \text{there exists } s \in S \text{ such that } s \leqslant z\}$. Thus, given $\mathfrak{x} \in FX$ and $\mathfrak{y} \in FY$, it follows that $\mathfrak{x}\,(\check{F}r)\,\mathfrak{y}$ iff $\mathfrak{x} \leqslant r^{\tau}(\mathfrak{y})$ iff $\mathfrak{x} \supseteq r^{\tau}(\mathfrak{y})$ iff $\mathfrak{x} \supseteq r^{\circ}[\mathfrak{y}]$, where $r^{\circ}[\mathfrak{y}] = \{r^{\circ}(B) \mid B \in \mathfrak{y}\}$. Observe that the Kleisli extension of the filter monad coincides with the respective lax extension $\check{F}$. ■

Definition 35. A *lax functor* $\mathbf{C} \xrightarrow{F} \mathbf{D}$ of preordered categories (recall Lecture 4) is a pair of maps $\mathcal{O}_{\mathbf{C}} \xrightarrow{F_{\mathcal{O}}} \mathcal{O}_{\mathbf{D}}$, $\mathcal{M}_{\mathbf{C}} \xrightarrow{F_{\mathcal{M}}} \mathcal{M}_{\mathbf{D}}$ (both denoted F), which satisfy the following axioms:

(1) $F(X \xrightarrow{f} Y) = FX \xrightarrow{Ff} FY$ for every $\mathbf{C}$-morphism $X \xrightarrow{f} Y$;

(2) $Ff \leqslant Fg$ for every $\mathbf{C}$-morphisms $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ such that $f \leqslant g$;

(3) $Fg \cdot Ff \leqslant F(g \cdot f)$ for every $\mathbf{C}$-morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$;

(4) $1_{FC} \leqslant F1_C$ for every $\mathbf{C}$-object C. ■

Remark 36.

(1) Recall from Lecture 4 that there is a functor $(V\text{-}\mathbf{Cat})^{op} \xrightarrow{(-)^*} V\text{-}\mathbf{Mod}$ defined by $((X, a) \xrightarrow{f} (Y, b))^* = (Y, b) \xrightarrow{f^*} (X, a)$, where $f^* = f^{\circ} \cdot b$. In case of $V = 2$, this functor induces a functor $\mathbf{Prost} \xrightarrow{(-)^*} \mathbf{Mod}^{op}$ defined by $((X, \leqslant_X) \xrightarrow{f} (Y, \leqslant_Y))^* = (Y, \leqslant_Y) \xrightarrow{f^* = f^{\circ} \cdot (\leqslant_Y)} (X, \leqslant_X)$.

(2) There exists a lax functor $\mathbf{Mod} \xrightarrow{|-|_L} \mathbf{Rel}$ defined by $|(X, \leqslant_X) \xrightarrow{r} (Y, \leqslant_Y)|_L = X \xrightarrow{r} Y$, which preserves the composition, but given a preordered set $(X, \leqslant_X)$, $1_{|(X, \leqslant_X)|_L} = 1_X \leqslant (\leqslant_X) = |1_{(X, \leqslant_X)}|_L$. ■

Remark 37. In the definition of lax extension of a $\mathbf{Set}$-functor T to the category $V\text{-}\mathbf{Rel}$ (recall Lecture 1), the following statements are equivalent:

(1) $Tf \leqslant \hat{T}f$ and $(Tf)^{\circ} \leqslant \hat{T}(f^{\circ})$ for every map $X \xrightarrow{f} Y$;

(2) $(Tf)^{\circ} \leqslant \hat{T}(f^{\circ})$ and $\hat{T}(f^{\circ} \cdot r) = (Tf)^{\circ} \cdot \hat{T}r$ for every map $X \xrightarrow{f} Y$ and every relation $Z \xrightarrow{r} Y$. ■

Proposition 38. *Given a power-enriched monad $(\mathbb{T}, \tau)$, the Kleisli extension $\check{T}$ of T to* $\mathbf{Rel}$ *provides a lax extension $\check{\mathbb{T}} = (\check{T}, m, e)$ of $\mathbb{T} = (T, m, e)$ to* $\mathbf{Rel}$.

Proof. To show that $\mathbf{Rel} \xrightarrow{\check{T}} \mathbf{Rel}$ is a lax functor, one can express it as a composition of lax functors as follows. Observe first that given a relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\;\;} Y$, $TX \overset{\check{T}r}{\longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\;\;} TY$ can be expressed as $TX \overset{(r^{\tau})^{*}}{\longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\;\;} TY$ with the help of the functor $(-)^{*}$ of Remark 36 (1). Thus, the Kleisli extension $\check{T}^{op}$ can be written as the composition of functors $\mathbf{Rel}^{op} \xrightarrow{(-)^{\tau}} \mathbf{Sup} \xrightarrow{|-|} \mathbf{Prost} \xrightarrow{(-)^{*}} \mathbf{Mod}^{op}$, where $|-|$ is the forgetful functor. Notice second that the Kleisli extension $\check{T}^{op}$ can be expressed as the following composition

$$\mathbf{Rel}^{op} \xrightarrow{(-)^{\flat}} \mathbf{Set}_{\mathbb{P}} \xrightarrow{E} \mathbf{Set}_{\mathbb{T}} \xrightarrow{L} \mathbf{Sup} \xrightarrow{|-|} \mathbf{Prost} \xrightarrow{(-)^{*}} \mathbf{Mod}^{op} \xrightarrow{|-|_{L}} \mathbf{Rel}^{op},$$

where all the arrows (except the last one) are functors, and the last arrow is the lax functor of Remark 36 (2).

To show that $(Tf)^{\circ} \leqslant \check{T}(f^{\circ})$ for every map $X \xrightarrow{f} Y$, one can consider the following commutative (except for the down right part, where one should notice that given a monotone map $(X, \leqslant_X) \xrightarrow{f} (Y, \leqslant_Y)$, it follows that $f^{\circ} \leqslant f^{\circ} \cdot (\leqslant_Y)$, since $1_X \leqslant (\leqslant_Y)$) diagram:

$$\begin{tikzcd}
\mathbf{Rel}^{op} \arrow[r, "(-)^{\flat}"] \arrow[rrrr, bend left, "(-)^{\tau}"] \arrow[rrrrrr, bend left=30, "\check{T}^{op}"] & \mathbf{Set}_{\mathbb{P}} \arrow[r, "E"] & \mathbf{Set}_{\mathbb{T}} \arrow[r, "L"] \arrow[dr, "G_{\mathbb{T}}"] & \mathbf{Sup} \arrow[r, "|-|"] \arrow[d, "|-|"] & \mathbf{Prost} \arrow[r, "(-)^{*}"] \arrow[dl, "|-|"'] & \mathbf{Mod}^{op} \arrow[r, "|-|_{L}"] \arrow[d, phantom, "\leqslant"] & \mathbf{Rel}^{op} \arrow[d, "1_{\mathbf{Rel}^{op}}"] \\
& \mathbf{Set} \arrow[ul, "(-)^{\circ}"] \arrow[u, "F_{\mathbb{P}}"] \arrow[ur, "F_{\mathbb{T}}"] \arrow[rr, "T"'] & & \mathbf{Set} \arrow[rrr, "(-)^{\circ}"'] & & & \mathbf{Rel}^{op}.
\end{tikzcd} \tag{4.1}$$

The second condition of Definition 37 (2), i.e., $\check{T}(f^{\circ} \cdot r) = (Tf)^{\circ} \cdot \check{T}r$ for every map $X \xrightarrow{f} Y$ and every relation $Z \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\;\;} Y$ can be shown as follows. Given $\mathfrak{x} \in TX$ and $\mathfrak{z} \in TZ$, $\mathfrak{x}\,\check{T}(f^{\circ} \cdot r)\,\mathfrak{z}$ iff $\mathfrak{x} \leqslant (f^{\circ} \cdot r)^{\tau}(\mathfrak{z})$ iff $\mathfrak{x} \leqslant r^{\tau} \cdot (f^{\circ})^{\tau}(\mathfrak{z})$ (since $\mathbf{Rel}^{op} \xrightarrow{(-)^{\tau}} \mathbf{Prost}$ is a functor) iff $\mathfrak{x} \leqslant r^{\tau} \cdot Tf(\mathfrak{z})$ (by diagram (4.1)) iff $\mathfrak{x}\,((Tf)^{\circ} \cdot \check{T}r)\,\mathfrak{z}$.

Altogether, it follows that $\check{T}$ is a lax extension of the **Set**-functor T to the category **Rel**. To show that

$$\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[d, "r"', "\shortmid"{marking}] \arrow[dr, phantom, "\leqslant"] & TX \arrow[d, "\check{T}r", "\shortmid"{marking}] \\
Y \arrow[r, "e_Y"'] & TY
\end{tikzcd}$$

for every relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\;\;} Y$, notice first that the following commutative diagram

$$\begin{tikzcd}
PX \arrow[r, "Pd_X"] \arrow[dr, "1_{PX}"'] \arrow[rr, bend left, "Pe_X"] & PPX \arrow[r, "P\tau_X"] \arrow[d, "n_X"] & PTX \arrow[r, "\tau_{TX}"] & TTX \arrow[d, "m_X"] \\
& PX \arrow[rr, "\tau_X"'] & & TX
\end{tikzcd}$$

implies $\tau_X = m_X \cdot \tau_{TX} \cdot Pe_X = \bigvee_{TX} \cdot Pe_X$ (recall condition (2.1)). Observe second that given $x \in X$ and $y \in Y$ such that $x\,r\,y$, it follows that (recall Definition 31) $e_X(x) \leqslant \bigvee_{x' \in r^{\flat}(y)} e_X(x') = \bigvee_{TX} Pe_X(r^{\flat}(y)) = \tau_X \cdot r^{\flat}(y) \overset{(\dagger)}{=} m_X \cdot T\tau_X \cdot Tr^{\flat} \cdot e_Y(y) = m_X \cdot T(\tau_X \cdot r^{\flat}) \cdot e_Y(y) = (\tau_X \cdot r^{\flat})^{\mathbb{T}} \cdot e_Y(y) \overset{\text{Definition 32}}{=} r^{\tau} \cdot e_Y(y)$, where

(†) relies on commutativity of the following diagram:

$$
\begin{tikzcd}
Y \arrow[rr, "e_Y"] \arrow[d, "r^\flat"'] & & TY \arrow[d, "Tr^\flat"] & \\
PX \arrow[r, "d_{PX}"] \arrow[rr, bend left, "e_{PX}"] \arrow[dr, "1_{PX}"'] & PPX \arrow[r, "\tau_{PX}"] \arrow[d, "n_X"'] & PTX \arrow[r, "T\tau_X"] & TTX \arrow[d, "m_X"] \\
& PX \arrow[rr, "\tau_X"'] & & TX.
\end{tikzcd}
$$

As a consequence, one obtains that $e_X(x)\,(\check{T}r)\,e_Y(y)$.

Lastly, to show that

$$
\begin{tikzcd}
TTX \arrow[r, "m_X"] \arrow[d, "\check{T}\check{T}r"', "|" marking] \arrow[dr, phantom, "\leqslant"] & TX \arrow[d, "\check{T}r", "|" marking] \\
TTY \arrow[r, "m_Y"'] & TY
\end{tikzcd}
$$

for every relation $X \overset{r}{\relbar\joinrel\mapstochar\joinrel\rightarrow} Y$, observe first that $m_X \cdot \tau_{TX} \cdot {\downarrow_{TX}} = \bigvee_{TX} \cdot {\downarrow_{TX}} = 1_{TX}$, and notice second that $(r^\tau)^{\mathbb{T}} = (r^\tau \cdot 1_{TY})^{\mathbb{T}} \overset{\text{Definition 32}}{=} ((\tau_X \cdot r^\flat)^{\mathbb{T}} \cdot 1_{TY})^{\mathbb{T}} \overset{\text{Definition 10}}{=} (\tau_X \cdot r^\flat)^{\mathbb{T}} \cdot 1_{TY}^{\mathbb{T}} \overset{\text{Definition 32}}{=} r^\tau \cdot m_Y$. Therefore, given $\mathfrak{X} \in TTX$ and $\mathfrak{Y} \in TTY$ such that $\mathfrak{X}\,(\check{T}\check{T}r)\,\mathfrak{Y}$, it follows that $\mathfrak{X} \leqslant (\check{T}r)^\tau(\mathfrak{Y})$, which implies $m_X(\mathfrak{X}) \overset{\text{Proposition 22 (4)}}{\leqslant} m_X((\check{T}r)^\tau(\mathfrak{Y})) = m_X \cdot (\check{T}r)^\tau(\mathfrak{Y}) \overset{\text{Definition 32}}{=} m_X \cdot (\tau_{TX} \cdot (\check{T}r)^\flat)^{\mathbb{T}}(\mathfrak{Y}) \overset{\text{Definition 33}}{=} 1_{TX}^{\mathbb{T}} \cdot (\tau_{TX} \cdot {\downarrow_{TX}} \cdot r^\tau)^{\mathbb{T}}(\mathfrak{Y}) \overset{\text{Definition 10}}{=} (1_{TX}^{\mathbb{T}} \cdot \tau_{TX} \cdot {\downarrow_{TX}} \cdot r^\tau)^{\mathbb{T}}(\mathfrak{Y}) = (m_X \cdot \tau_{TX} \cdot {\downarrow_{TX}} \cdot r^\tau)^{\mathbb{T}}(\mathfrak{Y}) = (r^\tau)^{\mathbb{T}}(\mathfrak{Y}) = r^\tau \cdot m_Y(\mathfrak{Y})$. As a consequence, one arrives at $\mathfrak{X}\,(\check{T}r)\,\mathfrak{Y}$, which finishes the proof. $\square$

Proposition 39. *Given a monad* $\mathbb{T} = (T, m, e)$ *on* **Set**, *there exists a functor* $\mathbf{Set} \xrightarrow{\tilde{T}} (\mathbb{T}, 2)\text{-}\mathbf{Cat}$ *defined by* $\tilde{T}(X \xrightarrow{f} Y) = (TX, \tilde{m}_X) \xrightarrow{Tf} (TY, \tilde{m}_Y)$, *where* $\tilde{m}_X = \hat{T}1_X \cdot m_X$. *The functor makes the following triangle*

$$
\begin{tikzcd}
\mathbf{Set} \arrow[rr, "T"] \arrow[dr, "\tilde{T}"'] & & \mathbf{Set} \\
& (\mathbb{T}, 2)\text{-}\mathbf{Cat} \arrow[ur, "|-|"'] &
\end{tikzcd}
$$

commute ($|-|$ *is the forgetful functor). The preorder on* TX *induced by* $\tilde{m}_X$ *is given by* $\mathfrak{x} \leqslant \mathfrak{y}$ *iff* $\mathfrak{x}\,\hat{T}1_X\,\mathfrak{y}$.

Remark 40. Since the Kleisli extension provides a power-enriched monad $(\mathbb{T}, \tau)$ with a lax extension, there exists an induced preorder on TX associated with $\check{T}$ as in Proposition 39, i.e., $\mathfrak{x} \leqslant_{\text{ind}} \mathfrak{y}$ iff $\mathfrak{x}\,\check{T}1_X\,\mathfrak{y}$. There also exists a partial order on TX provided by the monad morphism $\mathbb{P} \xrightarrow{\tau} \mathbb{T}$ as in Remark 23 (1), i.e., $\mathfrak{x} \leqslant_\tau \mathfrak{y}$ iff $m_X \cdot \tau_{TX}(\{\mathfrak{x}, \mathfrak{y}\}) = \mathfrak{y}$. Following Definition 33, $\mathfrak{x}\,(\check{T}r)\,\mathfrak{y}$ iff $\mathfrak{x} \leqslant_\tau r^\tau(\mathfrak{y})$ for every relation $X \overset{r}{\relbar\joinrel\mapstochar\joinrel\rightarrow} X$. Thus, if $r = 1_X$, then $\mathfrak{x}\,(\check{T}1_X)\,\mathfrak{y}$ iff $\mathfrak{x} \leqslant_\tau (1_X)^\tau(\mathfrak{y})$ iff $\mathfrak{x} \leqslant_\tau \mathfrak{y}$, since $(-)^\tau$ is a functor. Thus, the induced preorder associated with the lax extension $\check{T}$ coincides with the partial order provided by the monad morphism τ. Also notice that $\check{T}$ fails to preserve identity relations unless $\mathbb{T} = \mathbb{1}$ is the terminal power-enriched monad. ▪

Theorem 41. *Given a power-enriched monad* $(\mathbb{T}, \tau)$ *equipped with its Kleisli extension* $\check{T}$, *there exists a concrete isomorphism* $(\mathbb{T}, 2)\text{-}\mathbf{Cat} \cong \mathbb{T}\text{-}\mathbf{Mon}$.

Proof. The proof relies on a lax algebraic generalization of the classical correspondence between convergence and neighborhoods in topological spaces. In particular, given a topological space X, a filter $\mathfrak{x}$ on X converges to some point $y \in X$ precisely when $\mathfrak{x}$ is finer than the neighborhood filter of y. This correspondence can be formalized via maps $\mathbf{Set}(X, FX) \xrightarrow{\mathtt{conv}} \mathbf{Rel}(FX, X)$ and $\mathbf{Rel}(FX, X) \xrightarrow{\mathtt{nbhd}} \mathbf{Set}(X, FX)$, replacing the filter monad $\mathbb{F}$ with a power-enriched monad $(\mathbb{T},\tau)$ and identifying $\mathbf{Rel}(TX,X)$ with $\mathbf{Set}(X,PTX)$, isomorphic as ordered sets. One thus defines $\mathtt{conv}(\nu) = \downarrow_{TX}\cdot\nu$ and $\mathtt{nbhd}(r) = \bigvee_{TX}\cdot r^\flat$ for every map $X \xrightarrow{\nu} TX$ and every relation $TX \xrightarrow{r} X$. In pointwise notation, these maps can be written as $\mathfrak{x}\,\mathtt{conv}(\nu)\,x$ iff $\mathfrak{x}\leqslant\nu(x)$ and $(\mathtt{nbhd}(r))(x) = \bigvee\{\mathfrak{y}\in TX \,|\, \mathfrak{y}\in r^\flat(x)\} = \bigvee\{\mathfrak{y}\in TX\,|\,\mathfrak{y}\,r\,x\}$ for every $\mathfrak{x}\in TX$ and every $x\in X$.

Lemma 42. *Given a $\bigvee$-semilattice A, there exists the adjunction $\bigvee\dashv\downarrow\colon A\to PA$, where PA is the powerset of A ordered by set inclusion, and $\downarrow(a)=\downarrow a=\{b\in A\,|\,b\leqslant a\}$, such that $\bigvee\cdot\downarrow = 1_A$.*

Proof. Given $a\in A$ and $S\subseteq A$, it follows that $\bigvee S\leqslant a$ iff $S\subseteq\downarrow a$, and, moreover, $\bigvee\downarrow a = a$. □

Proposition 43. *When $\mathbf{Set}(X,TX)$ and $\mathbf{Rel}(TX,X)$ are equipped with pointwise partial order, there exists an adjunction (recall Lecture 4) $\mathtt{nbhd}\dashv\mathtt{conv}:\mathbf{Set}(X,TX)\to\mathbf{Rel}(TX,X)$ for every set X. Additionally, the fixpoints of $\mathtt{conv}\cdot\mathtt{nbhd}$ are precisely the unitary relations (recall Lecture 6), and $\mathtt{nbhd}\cdot\mathtt{conv} = 1_{\mathbf{Set}(X,TX)}$, so that the fixpoints of $\mathtt{nbhd}\cdot\mathtt{conv}$ are the maps $X\xrightarrow{\nu}TX$.*

Proof. Notice that given a map $X\xrightarrow{\nu}TX$ and a relation $TX\xrightarrow{r}X$, it follows that $\mathtt{nbhd}(r)\leqslant\nu$ iff $(\mathtt{nbhd}(r))(x)\leqslant\nu(x)$ for every $x\in X$ iff $\bigvee\{\mathfrak{x}\in TX\,|\,\mathfrak{x}\,r\,x\}\leqslant\nu(x)$ for every $x\in X$ iff (Lemma 42) $\{\mathfrak{x}\in TX\,|\,\mathfrak{x}\,r\,x\}\subseteq\downarrow\nu(x)$ for every $x\in X$ iff $\mathfrak{x}\,r\,x$ implies $\mathfrak{x}\leqslant\nu(x)$ for every $\mathfrak{x}\in TX$ and every $x\in X$ iff $\mathfrak{x}\,r\,x$ implies $\mathfrak{x}\,\mathtt{conv}(\nu)\,x$ for every $\mathfrak{x}\in TX$ and every $x\in X$ iff $r\subseteq\mathtt{conv}(\nu)$ iff $r\leqslant\mathtt{conv}(\nu)$. As a consequence, $\mathtt{nbhd}(r)\leqslant\mathtt{nbhd}(r)$ implies $r\leqslant\mathtt{conv}\cdot\mathtt{nbhd}(r)$, and $\mathtt{conv}(\nu)\leqslant\mathtt{conv}(\nu)$ implies $\mathtt{nbhd}\cdot\mathtt{conv}(\nu)\leqslant\nu$, i.e., $1_{\mathbf{Rel}(TX,X)}\leqslant\mathtt{conv}\cdot\mathtt{nbhd}$ and $\mathtt{nbhd}\cdot\mathtt{conv}\leqslant 1_{\mathbf{Set}(X,TX)}$. Moreover, both $\mathtt{nbhd}$ and $\mathtt{conv}$ are monotone maps.

Given a map $X\xrightarrow{\nu}TX$, for every $x\in X$, it follows that $(\mathtt{nbhd}\cdot\mathtt{conv}(\nu))(x) = (\mathtt{nbhd}(\mathtt{conv}(\nu)))(x) = \bigvee\{\mathfrak{x}\in TX\,|\,\mathfrak{x}\,\mathtt{conv}(\nu)\,x\} = \bigvee\{\mathfrak{x}\in TX\,|\,\mathfrak{x}\leqslant\nu(x)\} = \bigvee\downarrow\nu(x) \overset{\text{Lemma 42}}{=} \nu(x)$, namely, $\mathtt{nbhd}\cdot\mathtt{conv}(\nu)=\nu$. As a result, one obtains that $\mathtt{nbhd}\cdot\mathtt{conv} = 1_{\mathbf{Set}(X,TX)}$, i.e., the fixpoints of $\mathtt{nbhd}\cdot\mathtt{conv}$ are the maps $X\xrightarrow{\nu}TX$.

The statement on unitary relations relies on a sequence of technical calculations. □

Moreover, the above adjoint maps $\mathtt{nbhd}$ and $\mathtt{conv}$ are monoid homomorphisms between $\mathbf{Set}_{\mathbb{T}}(X,X)$ and $(\mathbb{T},2)$-$\mathbf{URel}^{op}(X,X)$ (the set of unitary relations $TX\xrightarrow{r}X$), namely, they satisfy

$$\begin{array}{ll} \mathtt{nbhd}(s\circ r) = \mathtt{nbhd}(r)\circ\mathtt{nbhd}(s) & \qquad \mathtt{conv}(\mu)\circ\mathtt{conv}(\nu) = \mathtt{conv}(\nu\circ\mu) \\ \mathtt{nbhd}(1^\natural_X) = e_X & \qquad \mathtt{conv}(e_X) = 1^\natural_X \end{array}$$

for all unitary relations $TX \underset{s}{\overset{r}{\rightrightarrows}} X$, and all maps $X \underset{\nu}{\overset{\mu}{\rightrightarrows}} TX$, where $s\circ r = s\cdot\check{T}r\cdot m^\circ_X$ (*Kleisli convolution*) and $1^\natural_X = e^\circ_X\circ e^\circ_X$ (properties of power-enriched monads imply that the Kleisli convolution is associative).

Lemma 44. *For a set X, a relation $TX\xrightarrow{a}X$ provides a $(\mathbb{T},2)$-category (X,a) iff $a\circ a = a$ and $1^\natural_X\leqslant a$.*

Proof.

$\Rightarrow$: First, notice that given a $(\mathbb{T},2)$-category (X,a), it follows that $a\cdot\hat{T}a\leqslant a\cdot m_X$ and $1_X\leqslant a\cdot e_X$ (recall Lecture 1), which implies $a\circ a = a\cdot\hat{T}a\cdot m^\circ_X\leqslant a\cdot m_X\cdot m^\circ_X \overset{m_X\cdot m^\circ_X\leqslant 1_{TX}}{\leqslant} a$ and $e^\circ_X\leqslant a\cdot e_X\cdot e^\circ_X \overset{e_X\cdot e^\circ_X\leqslant 1_{TX}}{\leqslant} a$. Second, observe that the operation $\circ$ is monotone in both arguments by its very definition. Third, notice that given a lax extension $\hat{\mathbb{T}} = (\hat{T},m,e)$ to V-$\mathbf{Rel}$ of a monad $\mathbb{T}=(T,m,e)$ on $\mathbf{Set}$, it follows that $\hat{T}1_X = \hat{T}e^\circ_X\cdot m^\circ_X$, which implies $a\circ e^\circ_X = a\cdot\hat{T}e^\circ_X\cdot m^\circ_X = a\cdot\hat{T}1_X\geqslant a\cdot T1_X = a\cdot 1_{TX} = a$, since $\hat{T}$ is a lax extension of T. Thus, $a\leqslant a\circ e^\circ_X\leqslant a\circ a\leqslant a$ (i.e., $a\circ a = a$) and $1^\natural_X = e^\circ_X\circ e^\circ_X\leqslant a\circ a\leqslant a$ (i.e., $1^\natural_X\leqslant a$).

$\Leftarrow$: Observe that, first, $a\circ a = a$ implies $a\cdot\hat{T}a\cdot m^\circ_X = a\circ a\leqslant a$ implies $a\cdot\hat{T}a \overset{1_{TTX}\leqslant m^\circ_X\cdot m_X}{\leqslant} a\cdot\hat{T}a\cdot m^\circ_X\cdot m_X\leqslant a\cdot m_X$ (i.e., $a\cdot\hat{T}a\leqslant a\cdot m_X$), and, second, $1^\natural_X\leqslant a$ implies $e^\circ_X = e^\circ_X\cdot 1_{TX} = e^\circ_X\cdot T1_X \overset{\hat{T}\text{ is a lax extension of }T}{\leqslant} e^\circ_X\cdot\hat{T}1_X = e^\circ_X\cdot\hat{T}e^\circ_X\cdot m^\circ_X = e^\circ_X\circ e^\circ_X\leqslant a$ implies $1_X\leqslant e^\circ_X\cdot e_X\leqslant a\cdot e_X$ (i.e., $1_X\leqslant a\cdot e_X$). □

Given a $(\mathbb{T},2)$-algebra (X,r), it follows that $(X,\texttt{nbhd}(r))$ is a $\mathbb{T}$-monoid, and conversely if (X,ν) is a $\mathbb{T}$-monoid, then $(X,\texttt{conv}(\nu))$ is a $(\mathbb{T},2)$-algebra. Moreover, this one-to-one correspondence is functorial. $\square$

Corollary 45. *The category* **Top** *is concretely isomorphic to the category* $(\mathbb{F},2)$-**Cat***, whose objects are pairs* (X,a)*, where* $FX \overset{a}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} X$ *is a relation, which represents convergence and which satisfies (denoting "a" and "$\hat{F}a$" by "$\longrightarrow$")* $\mathfrak{X} \longrightarrow \mathfrak{y}$ *and* $\mathfrak{y} \longrightarrow z$ *imply* $\mathfrak{X} \longrightarrow z$*, and* $\dot{x} \longrightarrow x$ *for every* $\mathfrak{X} \in FFX$*,* $\mathfrak{y} \in FX$*,* $x,z \in X$ *(notice that* $\mathfrak{X} \longrightarrow \mathfrak{y}$ *iff* $\mathfrak{X} \supseteq a^{\circ}[\mathfrak{y}]$ *as in Example 34 (3)); and whose morphisms* $(X,a) \xrightarrow{f} (Y,b)$ *are convergence-preserving maps* $X \xrightarrow{f} Y$*, namely,* $\mathfrak{x} \longrightarrow z$ *implies* $Ff(\mathfrak{x}) \longrightarrow f(z)$ *for every* $\mathfrak{x} \in FX$*,* $z \in X$*.*

Proof. The statement follows from Theorems 17, 41. $\square$

Corollary 46. *Given the up-set monad* $\mathbb{U}$ *(for a set* X*,* $UX = \{\mathfrak{a} \subseteq PX \mid {\uparrow_{PX}}\ \mathfrak{a} = \mathfrak{a}\}$*) equipped with the Kleisli extension associated with the principal filter natural transformation, there is a concrete isomorphism* **Cls** $\cong (\mathbb{U},2)$-**Cat***, where* **Cls** *is the category of closure spaces and continuous maps (cf. Lecture 1).*

References

[1] J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and Concrete Categories: the Joy of Cats*, Repr. Theory Appl. Categ. **17** (2006), 1–507.

[2] D. Hofmann and G. J. Seal, *A cottage industry of lax extensions*, Categ. Gen. Algebr. Struct. Appl. **3** (2015), no. 1, 113–151.

[3] D. Hofmann, G. J. Seal, and W. Tholen (eds.), *Monoidal Topology: A Categorical Approach to Order, Metric and Topology*, Cambridge University Press, 2014.

[4] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., Springer-Verlag, 1998.

[5] J. MacDonald and M. Sobral, *Aspects of Monads*, Categorical Foundations: Special Topics in Order, Topology, Algebra, and Sheaf Theory (M. C. Pedicchio and W. Tholen, eds.), Cambridge University Press, 2004, pp. 213–268.

[6] G. J. Seal, *A Kleisli-based approach to lax algebras*, Appl. Categ. Structures **17** (2009), no. 1, 75–89.

[7] G. J. Seal, *Tensors, monads and actions*, Theory Appl. Categ. **28** (2013), 403–434.